Originally published as:


DOI: http://doi.org/10.1111/j.1365-246X.2007.03385.x
A fast converging and anti-aliasing algorithm for Green’s functions in terms of spherical or cylindrical harmonics

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Accepted 2007 February 6. Received 2007 February 5; in original form 2006 October 25

SUMMARY
Geophysical observables are generally related to earth structure and source parameters in a complicated non-linear way. Consequently, a large number of forward modelling processes are commonly necessary to obtain a satisfactory estimate of such parameters from observed data. The most time-consuming part of the forward modelling is the computation of the Green’s functions of the different earth models to be tested. In this study, we present a fast converging algorithm: the differential transform method for the computation of Green’s functions in terms of spherical or cylindrical harmonics. In this method, a deconvolutable high-pass filter is used to enhance the numerical significance of the far-field spectrum of Green’s functions. Compared with existing fast converging algorithms such as the Kummer’s transformation and the disc factor method, the differential transform method is more efficient except for the extremely near-source region. The new method can be used to suppress numerical phases (non-physical seismic signals) associated with the aliasing effect that may arise in synthetic seismograms when the latter are computed from a windowed wavenumber (or slowness) spectrum. The numerical efficiency of the new method is demonstrated via two representative tests.

Key words: filter techniques, Fourier transforms, Hankel transform, spherical harmonics, synthetic seismograms.

1 INTRODUCTION
With recent advances in computational capacity, many geophysical boundary-value problems can be solved for complicated 3-D models of the earth. In practice, however, the use of 3-D models is commonly restricted to certain scenario simulations (forward modelling). In most cases, particularly for inverse problems, simplified earth models such as a multilayered sphere or half-space are still widely used. These simple models satisfy the degree of accuracy required to make interpretations in a wide range of practical applications and can provide semi-analytical references for more complex 3-D models. Since geophysical observables are mostly related to earth structure and source parameters in a complicated and non-linear way, a straightforward inversion of these parameters from observed data is generally difficult if not impossible. Sometimes, millions of forward modelling processes are necessary to find a single model that provides a satisfactory fit to the data (e.g. see Lorenzo-Martín et al. 2006). The time-consuming part of the forward modelling is the computation of Green’s functions that represent the response of the models to any point source. Therefore, a stable and efficient algorithm for the Green’s functions is always of practical interest in many geophysical disciplines.

Green’s functions of a spherical earth can be expressed in terms of spherical surface harmonics; that is, by the Legendre series in the form

\[ f(\theta) = \sum_{n=0}^{\infty} F_n P_n^m(\cos \theta), \]  

where \( \theta \) is the angular distance to any point source and \( P_n^m(x) \) is the associated Legendre polynomial of degree \( n \) and order \( m \) (\( n \geq m > 0 \)). Note that the order \( m \) depends on the radiation pattern of the source and is normally limited to the lowest orders, for example, up to 2 in case of a single force or a point dislocation source. The coefficients \( F_n \) represent the discrete spatial spectrum of the Green’s function over degree \( n \). The discrete character of the spectrum reflects the natural periodicity of the angular distance.

If a local problem is considered, a flat-earth model such as a homogeneous or layered half-space is often used. The Green’s functions of the flat earth can be expressed by the inverse Hankel transform of any order \( m > 0 \), or simply the Bessel integral in the form

\[ f(r) = \int_0^{\infty} F(k) J_m(kr) k \, dk, \]  

where \( r \) is the horizontal distance to the source, \( k \) is the horizontal wavenumber, \( J_m(x) \) is the Bessel function of order \( m \) that is dependent on the radiation pattern of the source, and \( F(k) \) is the Hankel transformed function (also called the image function) of \( f(r) \). The
continuous wavenumber spectrum of the Green’s function is related to the unlimited distance coordinate \( r \).

The spatial spectra of the Green's functions, \( F_\omega \) or \( F(k) \), can be calculated using either the analytical propagator or the semi-analytical Runge-Kutta integration algorithm. As the classic Thomson-Haskell propagator algorithm (Thomson 1950; Haskell 1953) has been known to be unstable for high wavenumbers, a number of successful improvements have been proposed (Knopoff 1964; Dunkin 1965; Fuchs & Müller 1971; Kennett 1983; Wang 1999; Chapman 2003). In most cases, it can be supposed that the spatial spectra of the Green’s functions can be calculated with a sufficient degree of accuracy. An important issue for the present study is the fast converging of the infinite Legendre series in eq. (1) or the infinite Bessel integral in eq. (2).

### 1.1 Previous fast converging techniques for elastostatic Green’s functions

In the elastostatic case, it is known that the Hankel transformed Green’s function converges exponentially to zero at the large wavenumber limit, \( F(k) \sim e^{-k|\Delta z|} \to 0 \) for \( k \to \infty \), where \( |\Delta z| \) is the vertical distance from the observation point to the source. Accordingly, there is generally no convergence problem as long as \( |\Delta z| \) is sufficiently large; however, when \( |\Delta z| \to 0 \), that is, when the observation point (receiver) and the source are located at the same depth, it is possible that the Hankel transformed Green’s function does not converge to zero. Instead, it may converge to a non-zero constant or even diverges to infinity. In previous studies, the convergence problem in the latter case has been solved using the Kummer’s transformation or the disc factor technique (Farrell 1972).

To use the Kummer's transformation, the high-wavenumber asymptotic expression of the Hankel transformed Green’s function should have a simple form such that its inverse Hankel transform can be extracted from the numerical integration and calculated analytically. For example, if \( F(k) \) has a simple asymptotic form \( F_\omega(k) \) for \( k \to \infty \), eq. (2) can be reformulated to

\[
f(r) = \int_0^\infty F_\omega(k)J_\omega(kr)\,dk + \int_0^\infty [F(k) - F_\omega(k)]J_\omega(kr)\,dk
\]

\[
= f_\omega(r) + \int_0^K [F(k) - F_\omega(k)]J_\omega(kr)\,dk,
\]

where \( f_\omega(r) \) is the Bessel integral of \( F_\omega(k) \) and is considered to be known. Because the residual function \( F(k) - F_\omega(k) \) converges to zero for \( k \to \infty \), its infinite Bessel integral can be cut off at a sufficiently large wavenumber \( K \). This approach was used by Farrell (1972) to calculate Green’s functions given by a slow converging alternative series, and it can be used to estimate the residual integral over \( k \to k_c \). When \( k \gg r_c^{-1} \), \( |D(k)| \) decreases with \( k^{-3/2} \). Therefore, the disc factor is a kind of low-pass filter in the wavenumber domain and consequently accelerates the convergence of the Hankel transform at the large wavenumber limit. A more efficient disc factor can be generated with a Gaussian distribution of the source (Wang et al. 2003). Using \( r_c \), as the half-bandwidth of the source distribution, the disc factor becomes

\[
D(k) = e^{-\frac{1}{2}kr_c^2},
\]

that is, the Gaussian image distribution with the half-bandwidth \( r_c^{-1} \). In contrast to Kummer’s transformation, the use of the disc factor may affect the numerical results of the Green’s functions. In the elastostatic case, however, this effect decreases rapidly with decreasing disc radius and is negligible if the ratio \( r_c/r \) does not exceed a few per cent.

### 1.2 Previous anti-aliasing techniques for synthetic seismograms

In the elastodynamic case, the Green’s functions are composed of seismic waves and are, therefore, also termed synthetic seismograms. These synthetic seismograms are usually computed in two steps: the inverse Hankel transform yields the temporal spectrum which is then converted to the time-domain seismograms by, for example, the fast Fourier transform (FFT). Hence, the inverse Hankel transform is carried out in the frequency domain and eq. (2) is rewritten in the form

\[
f(r, \omega) = \int_0^\infty F(k, \omega)J_\omega(kr)\,dk,
\]

where \( \omega \) is the circular frequency. If the source is not directly placed at a material interface, the asymptotic form of the Hankel transformed Green’s function \( F(k, \omega) \) can be derived from the Stokes solutions for seismic waves associated with a point force in an infinite homogeneous medium (e.g. see Aki & Richards 1980). In this case, the Kummer’s transformation can be used to solve the convergence problem here as in the static case. In the general case, the disc factor technique can be used. In contrast to the static case, the disc radius of a seismic point source should be sufficiently smaller in size than the characteristic wavelength to be considered. Otherwise, the resolution of the arrival time of seismic wave impulses, also called seismic phases will be affected because the onset and duration of these phases are directly related to the source geometry. A small disc radius would represent a large cut-off wavenumber resulting in greater computational effort. This limitation means that the disc factor method is inefficient, especially when computing high-frequency seismograms for teleseismic distances. The same problem also affects the Peak-Trough Averaging Method (PTAM) that was recently proposed by Zhang et al. (2003); the authors found that when the cut-off wavenumber exceeds a certain critical value \( k_c \), the integral in eq. (6) yields oscillatory results with a monotonically and smoothly decaying amplitude. This critical wavenumber is generally several times larger than the wavenumber of any surface wave. The PTAM was based on the principle of the Repeated Averaging Method (PAM) (Dahlquist & Björck 1971) for calculating slowly convergent alternative series, and it can be used to estimate the residual integral over \( k > k_c \).

All of the converging methods described above are useful when the complete wave field (i.e. all body and surface waves and the static displacement) must be considered. In practice, however, it is not always necessary to model the complete wave field. More usually, only those waves within a selected slowness (or incidence

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angle) range are of interest, where the slowness represents the inverse of the apparent velocity, \( p = k/\omega \). In seismology, the slowness is a more frequently used parameter than the wavenumber, and the wavenumber integral is usually transformed to the slowness integral to ensure that the integration range is independent of the frequency. In addition, the space-domain aliasing problem is better solved by windowing a constant slowness range than by windowing a constant wavenumber range (Fuchs 1968).

Synthetic seismograms are commonly used to identify different seismic phases. When only body waves below a certain incidence angle (i.e. with a certain penetration depth) are of interest, an upper cut-off slowness \( p_{\text{cut}} \) can be used,

\[
f(r, \omega) = \omega^2 \int_0^{p_{\text{cut}}} F(p, \omega) J_m(\omega r) p \, dp.
\]  

In practice, the cut-off slowness \( p_{\text{cut}} \) is often selected to be much smaller than the slowness of the horizontally propagating S wave, that is, \( p_{\text{cut}} \ll V_s^{-1} \), where \( V_s \) is the S wave velocity of the medium at the source site. This highly restricted slowness window is used not only to reduce the computational effort, but also to increase the numerical resolution of the deep body wave phases that have small slowness. Especially in the case of shallow sources, the deep phases may be smaller than the shallow body waves or surface waves by several orders of magnitude. If the full-wave integration approach is employed, the small deep phases may lose their numerical significance and even become invisible in synthetic seismograms. However, any artificial truncation of the slowness integral leads to non-physical wave signals, also called numerical phases, in synthetic seismograms (Kind 1979). The numerical phases appear to propagate with the cut-off slowness in both forward and backward directions. They can be reduced to some extent by using a linear or non-linear taper such as the cosine function from 0 to \( \pi/2 \) over a small range before the cut-off slowness. We know of no other method that can be used to avoid the numerical phases in a more efficient manner.

In this study, we propose a differential transform method that, when combined with the tapering technique, provides remarkable suppression of the numerical phases. Moreover, this method solves the convergence problem of the Green’s functions in a more general and efficient way than that achieved by existing methods.

### 2 Differential Transform Method

It is well known that near-field seismic waves are represented by a smooth wavenumber spectrum, while far-field seismic waves are represented by an oscillatory wavenumber spectrum. The smooth near-field content becomes dominant when the Green’s function is calculated for a similar depth to that of the source. In this case, the artificial slowness cut-off brings non-physical spectral information into the system and leads to numerical phases in synthetic seismograms. The sharper the cut-off, the stronger the numerical phases. For this reason, a numerical taper function is usually used to smooth the cut-off. Further suppression of the numerical phases should be possible if the near-field waves are attenuated. To achieve this, we derive the differential transform method.

Using the Bessel equation in the form

\[
J_m(kr) = \frac{1}{r^m} \left\{ \frac{m^2}{k^2} J_m(kr) - \frac{1}{k} \frac{\partial}{\partial k} \left[ k \frac{\partial J_m(kr)}{\partial k} \right] \right\}, \quad (r > 0),
\]  

the inverse Hankel transform eq. (2) can be expressed equivalently by

\[
f(r) = \frac{1}{r^2} \int_0^\infty F(k) \left\{ \frac{m^2}{k^2} J_m(kr) - \frac{1}{k} \frac{\partial}{\partial k} \left[ k \frac{\partial J_m(kr)}{\partial k} \right] \right\} k \, dk.
\] (9)

Integration by parts yields

\[
f(r) = \frac{k}{r^2} \left\{ \frac{dF(k)}{dk} J_m(kr) - F(k) \frac{\partial J_m(kr)}{\partial k} \right\}^\infty_0 + \frac{1}{r^2} \int_0^\infty \left\{ \frac{m^2}{k^2} F(k) - \frac{1}{k} \frac{d}{dk} \left[ k \frac{dF(k)}{dk} \right] \right\} J_m(kr) k \, dk.
\] (10)

Here, we suppose that the image function \( F(k) \) is differentiable for all wavenumbers \( k \). We can also suppose that the two constants of integration are zero; that is, the following conditions are satisfied:

\[
\lim_{k \to \infty} \left\{ \frac{dF(k)}{dk} J_m(kr) - F(k) \frac{\partial J_m(kr)}{\partial k} \right\} = 0,
\] (11)

and

\[
\lim_{k \to 0} \left\{ \frac{dF(k)}{dk} J_m(kr) - F(k) \frac{\partial J_m(kr)}{\partial k} \right\} = 0.
\] (12)

Note that \( J_m(x) \sim \frac{x^m}{2^{m+1}} e^x \) for \( x \to 0 \) and \( |J_m(x)| \leq \sqrt{\frac{2}{\pi x}} \) for \( x \to \infty \). The condition eq. (11) is satisfied because the factor \( e^{-k|\Delta z|} \) is included in \( F(k) \), where \( |\Delta z| \) is the vertical distance between the receiver and the source. It is possible that eq. (11) is not satisfied when only \( |\Delta z| = 0 \), that is, when the receiver is placed at the same depth as the source; however, this special case only occurs in a mathematical sense. From a physical viewpoint, we can always replace \( |\Delta z| = 0 \) by a sufficient small \( |\Delta z| = \delta > 0 \). As long as \( \delta \ll r \), no practical difference can occur in the numerical results for the Green’s function. The only problem is the condition eq. (12), which is not satisfied in the general case. For example, the Hankel transformed Boussinesq solutions for \( m = 0 \) and 1 are proportional to \( k^{-1} \) for \( k \to 0 \). In such cases, pre-processing is necessary. For example, the following transformation can be made before eq. (10) is used:

\[
f(r) = \int_0^\infty F(k) J_m(kr) k \, dk
\]

\[
= \int_0^\infty F(k) \left( 1 - e^{-k^2/k_0^2} \right) J_m(kr) k \, dk
\]

\[
+ \int_0^\infty F(k) e^{-k^2/k_0^2} J_m(kr) k \, dk,
\] (13)

where \( k_0 (> 0) \) is a small constant wavenumber. The second integral on the right-hand side of eq. (13) can be truncated when \( k \) is several times larger than \( k_0 \). The new image function \( F(k)(1 - e^{-k^2/k_0^2}) \) converges to \( F(k)k^2/k_0^2 \) when \( k \to 0 \); this satisfies the condition eq. (12).

In summary, the Green’s function can be alternatively calculated by

\[
f(r) = \frac{1}{r^2} \int_0^\infty G(k) J_m(kr) k \, dk,
\] (14)

where

\[
G(k) = \frac{m^2}{k^2} F(k) - \frac{1}{k} \frac{d}{dk} \left[ k \frac{dF(k)}{dk} \right],
\] (15)

and is called the differential transform of \( F(k) \). It is interesting to note that \( G(k) \) expresses the image function of the product \( r^2 f(r) \).
The factor $r^2$ represents a high-pass filter for $f(r)$ in the spatial domain, and the Bessel integral in eq. (15) represents the convolution of $r^2$ and $f(r)$ in the wavenumber domain. Use of the differential transform means that the smooth near-field spectrum is suppressed, while the oscillatory far-field spectrum is enhanced. Theoretically, the same result for the Green’s function can be produced either from the original spectrum $F(k)$ or from its differential transform $G(k)$ when the wavenumber integral is carried out over the full wavenumber range; however, the latter converges faster than the former by the factor $k^{-2}$. In particular, we will show that the use of the differential transform can efficiently suppress the numerical phases in synthetic seismograms, when the latter are computed from a windowed wavenumber (or slowness) spectrum.

To increase the intensity of the high-pass filtering, a higher order differential transform can be used. This is obtained when the operation in eq. (15), which can be defined as the first-order differential transform, is made recursively; however, the high-order transform generally requires a higher sampling rate of the image function, and it reduces the numerical accuracy of the Green’s function for small epicentral distances because of the enhanced filtering of the near-field. In practice, it is generally sufficient to use the differential transform of order 1 or 2.

Based on the same principle, a similar differential transform can be constructed for the discrete spectrum, that is, the expansion coefficients of the Legendre series. For this purpose, we reformulate the recursive formula

$$
(2n + 1) x P_m^n(x) = (n + m) P_m^{n-1}(x) + (n - m + 1) P_m^{n+1}(x)
$$

(16)

to the form

$$
P_m^n(x) = \frac{1}{1 - x} \left[ P_m^n(x) - \frac{n + m}{2n + 1} P_m^{n-1}(x) \right] - \frac{n - m + 1}{2n + 1} P_m^{n+1}(x),
$$

(17)

Substituting eq. (17) in eq. (1) and rearranging the summation order, we obtain

$$
f(\theta) = \sum_{n=m}^{\infty} F_n P_m^n(\cos \theta)
$$

$$
= \frac{1}{2 \sin^2(\theta/2)} \sum_{n=m}^{\infty} G_n P_m^n(\cos \theta), \quad (\theta > 0),
$$

(18)

where

$$
G_n = F_n - \frac{n + m + 1}{2n + 3} F_{n+1} - \frac{n - m}{2n - 1} F_{n-1}.
$$

(19)

Note that the terms $F_n$ for $n < m$ have been defined to be zero. Therefore, $G_n$ expresses the corresponding expansion coefficients of the product of $2 \sin^2(\theta/2)$ and $f(\theta)$. For large $n$,

$$
G_n \sim F_n - \frac{1}{2} F_{n+1} - \frac{1}{2} F_{n-1}
$$

$$
= -\frac{1}{2} \left[ (F_{n+1} - F_n) - (F_n - F_{n-1}) \right]
$$

$$
= -\frac{1}{2} \Delta^2 F_{n-1},
$$

$$
= -\frac{1}{2} \nabla^2 F_{n+1},
$$

(20)

where $\Delta$ and $\nabla$ represent the finite forward and backward difference operators, respectively. Therefore, eq. (19) represents the discrete analogue of the first-order differential transform defined by eq. (15).

The deconvolution factor $2 \sin^2(\theta/2)$ has the high-pass filter function as $r^2$ for the cylindrical geometry.

In the following two sections, some numerical tests are undertaken to demonstrate the efficiency of the new method in terms of suppressing the numerical phases in synthetic seismograms and in terms of accelerating the convergence of Green’s functions for an elastostatic problem.

### 3 Suppression of Numerical Phases in Synthetic Seismograms

In this test, synthetic seismograms are computed for the vertical surface displacement induced by an impulsive dip-slip double-couple buried at 1 km in a layered viscoelastic half-space model. The medium parameters of the half-space are adopted from the seismic reference earth’s model IASP91 (Kennett & Engdahl 1991). The earth’s flattening effect was taken into account using the flat-earth transformation proposed by Müller (1977). To suppress the time-domain aliasing, the complex-frequency method (Kind & Seidl 1982) was used. The imaginary part $f_i$ of the frequency is determined by the suppression factor $\alpha$ (Wang et al. 2006):

$$
f_i = \frac{\ln \alpha}{2 \pi T},
$$

(21)

where $T$ is the time window used. A uniform constant value of 0.01 was chosen for $\alpha$, implying that the alias phases will appear with a maximum of 1 per cent of their initial amplitude. The Hankel transformed Green’s functions in the frequency-wavenumber domain are calculated using the orthonormalized Thomson-Haskell propagator algorithm (Wang 1999; Wang & Kümpel 2003).

Fig. 1 shows the complete slowness spectrum of the displacement at 0.5 Hz. It is clear that this spectrum is characterized by a smooth curve, except for a simple pole at the slowness of about 0.32 s km$^{-1}$ that represents the strong Rayleigh wave with a velocity of about 3.1 km s$^{-1}$. Deep body wave phases with a slowness of less than 0.1 s km$^{-1}$ cannot be identified at all because their amplitude is several orders of magnitude smaller than that of the surface wave. Therefore, when only these phases are of interest, a strongly truncated slowness integral should be used to minimize the computational effort and protect the numerical significance from the strong surface wave.

Fig. 2(a) shows the Hankel transformed Green’s function (the real part only) for the slowness range below 0.12 s km$^{-1}$. The right-hand 20 per cent of the spectrum from 0.10 to 0.12 s km$^{-1}$ has been multiplied by a cosine taper function. The smooth spectrum curve indicates the dominance of the near-field body waves because of the shallow source depth.

Fig. 2(b) shows the first-order differential transform of the Hankel transformed Green’s function, truncated by the same cosine taper. Due to the high-pass filter property of the differential transform, the smooth near-field content has been suppressed, whereas the oscillatory far-field content has been enhanced. The effect is more pronounced when the second-order differential transform is used (Fig. 2c).

Synthetic seismograms are calculated for 31 equidistant distances, $d = 60^\circ, 61^\circ, \ldots$ and $90^\circ$, for the uniform time window of $T = 300$ s. Note that the timescale was reduced by the waiting time (time until the first arrival), which was roughly estimated to be $t_0 = 570 s + d \cdot 6.5$ s deg$^{-1}$. The time window consists of 1024 time samples. According to the FFT rule, 512 discrete temporal spectra are needed, each of which was computed from the original slowness
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Figure 1. Example of Hankel transformed Green's functions calculated for the seismic reference model IASP91.

Figure 2. Truncated spectrum of the spectral function shown in Fig. 1 (real part only). (a) Original spectral function, (b) first-order differential transform and (c) second-order differential transform of the spectral function. All spectra are multiplied by the factor \( \cos \left( \frac{p - 0.10}{2 \cdot 0.02} \right) \) for the taper range from \( p = 0.10 \) to \( 0.12 \text{ s km}^{-1} \).

spectrum (Fig. 2a), the first-order differential transform (Fig. 2b) and the second-order differential transform (Fig. 2c). Corresponding numerical results for the synthetic seismograms are compared in Figs 3–5.

Fig. 3 shows that when using the original slowness spectrum, the synthetic seismograms are dominated by numerical phases that propagate with the cut-off slowness in the forward and backward directions. These numerical phases are so strong that almost no physical phase is visible at all. In contrast, Fig. 4 shows that when using the first-order differential transform, the numerical phases are strongly suppressed and all major body phases such as \( P, PcP \) and \( PP \) and the shallow multiples related to \( P \) are clearly identifiable. The best results are obtained when using the second-order transform (Fig. 5). In this case, the numerical phases are invisible.

4 FAST CONVERGENCE OF GREEN'S FUNCTIONS IN TERMS OF SPHERICAL HARMONICS

The next example is a comparison of the differential transform method with the Kummer's transformation and the disc factor method for fast convergence of the infinite Legendre series. It concerns the static-displacement Green's functions of an elastic earth with a spherical symmetry due to a unit point mass load at the surface. The vertical and horizontal components of the displacement are expressed, respectively, in the forms

\[
u_r(\theta) = \frac{a}{m_c} \sum_{n=0}^{\infty} h_n P_n(\cos \theta),
\]

(22)
Figure 3. Synthetic seismograms calculated using the slowness spectrum shown in Fig. 2(a). Each trace is normalized independently to ensure the same peak amplitude.

Figure 4. As for Fig. 3, but calculated using the slowness spectrum shown in Fig. 2(b).

and

\[ u_\circ (\theta) = \frac{a}{m_e} \sum_{n=1}^{\infty} \frac{l_n}{l_n - l_\infty} \frac{\partial P_n(\cos \theta)}{\partial \theta}, \quad (23) \]

where \( a \) is the radius and \( m_e \) the mass of the earth, \( \theta \) is the angular distance to the point mass load, and \( h_n \) and \( l_n \) are the dimensionless mass-load Love numbers of order \( n \) (Farrell 1972).

4.1 Kummer’s transformation method

Farrell (1972) derived the asymptotic expressions for the Love numbers in the form,

\[ h_n \rightarrow \frac{1}{2} n \ln n, \quad (24) \]

\[ l_n \rightarrow \frac{1}{n} \ln n, \quad (25) \]

where \( h_\infty \) and \( l_\infty \) are constants that can be derived from the Boussinesq solutions for the homogeneous half-space with the elastic parameters of the uppermost layer of the spherical earth. For the PREM model Dziewonski & Anderson (1981), \( h_\infty = -6.2091 \) and \( l_\infty = 1.8901 \); these values differ from the computed values at \( n > 5000 \) by less than \( 10^{-3} \). The asymptotic values are used to calculate the displacement Green’s functions by the Kummer’s transformation, as performed by Farrell:

\[ u_\circ (\theta) = \frac{a h_\infty}{m_e} \sum_{n=0}^{\infty} P_n(\cos \theta) + \frac{a}{m_e} \sum_{n=0}^{\infty} (h_n - h_\infty) P_n(\cos \theta) \]

\[ = \frac{a h_\infty}{2 m_e \sin (\theta/2)} + \frac{a}{m_e} \sum_{n=0}^{\infty} (h_n - h_\infty) P_n(\cos \theta), \quad (26) \]

\[ u_\theta (\theta) = \frac{a l_\infty}{m_e} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial P_n(\cos \theta)}{\partial \theta} + \frac{a}{m_e} \sum_{n=1}^{\infty} (n l_n - l_\infty) \frac{1}{n} \frac{\partial P_n(\cos \theta)}{\partial \theta} \]

\[ = \frac{a l_\infty \cos (\theta/2) [1 + \sin (\theta/2)]}{2 m_e \sin (\theta/2)} + \frac{a}{m_e} \sum_{n=1}^{\infty} (n l_n - l_\infty) \frac{1}{n} \frac{\partial P_n(\cos \theta)}{\partial \theta}. \quad (27) \]

The Green’s functions are computed by selecting \( N = 10, 100, 500, 1000, 5000 \) and 10,000, as shown in Fig. 6. It is evident from the comparison that the cut-off degree should be larger than 5000 to attain a stable convergent result for both components at all distances under consideration.

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4.2 Disc factor method

If the unit point mass is replaced by a unit mass that is distributed uniformly within the radius $\alpha$, the Green’s functions are expressed by

$$u_r(\theta) = \frac{a}{m_e} \sum_{n=0}^{N} \Gamma_n(\alpha) h_n P_n(\cos \theta), \quad (28)$$

$$u_\theta(\theta) = \frac{a}{m_e} \sum_{n=1}^{N} \Gamma_n(\alpha) h_n \frac{\partial P_n(\cos \theta)}{\partial \theta}, \quad (29)$$

where $\Gamma_n(\alpha)$ is the disc factor (Farrell 1972):

$$\Gamma_n(\alpha) = \frac{1}{n(n+1) \sin \alpha} \frac{\partial P_n(\cos \alpha)}{\partial \alpha}. \quad (30)$$

From a physical viewpoint, it makes sense to choose a source radius that is proportional to the observation distance from the source. The results derived from the disc factor method are obtained for $\alpha = \theta/20$ (Fig. 7). A larger disc radius, that is, $\alpha = \theta/10$, was also tested, but this did not result in a significant change in the numerical results. The disc factor method appears to be inefficient for calculating the near-field displacements; however, it requires a smaller cut-off degree for large distances than the Kummer’s transformation method. For the
horizontal component, for example, it is sufficient to use $N = 1000$ for $\theta > 4^\circ$, $N = 500$ for $\theta > 10^\circ$, or $N = 100$ for $\theta > 40^\circ$. A similar rule applies to the vertical component, but the corresponding cut-off degrees are slightly higher.

4.3 Differential transform method

To use the differential transform defined by eq. (18), we replace the derivative of the Legendre polynomial \( \frac{\partial P_n(\cos \theta)}{\partial \theta} \) in eq. (23) with $-P_n^1(\cos \theta)$. For large $n$, the asymptotic relations are easily derived:

\[
\begin{align*}
\hat{h}_n^{(1)} & \to -\frac{h_\infty}{2n^2}, \\
n_{l_n}^{(1)} & \to \frac{l_\infty}{2n^2},
\end{align*}
\]

where $\hat{h}_n^{(1)}$ and $n_{l_n}^{(1)}$ denote the first-order differential transform of the Love numbers $h_n$ and $l_n$, respectively. Therefore, the Legendre series in eqs (26) and (27) converge faster by factor $n^{-2}$ when
using the transformed Love numbers. The corresponding numerical results of the Green’s functions are shown in Fig. 8. In comparison with the two previous methods outlined above, the new method is more efficient except for the vertical component at distances very close to the source. For distances greater than several tenths of a degree, the necessary cut-off degree is at least an order of magnitude smaller than that required by the disc factor method. For example, the convergent values are already attained when \( N = 10 \) for both the vertical and horizontal components at all \( \theta > 10^\circ \). The second-order differential transform was also considered for the present test. An improvement for the vertical component at near-source distances can be achieved by using the first-order differential transform combined with the Kummer’s transformation (Fig. 9).

The advantage of this special high-pass filter is that it is simply convolutable, that is, the differential transform is directly derived from the original spectral function and, therefore, does not result in a significant increase in computational effort. In fact, the computation times required for the synthetic seismograms shown in Figs 3–5 are nearly identical. We also note that no information can be theoretically lost in using the differential transform. This is the reason that we use the term ‘transform’ rather than ‘filter’.

As shown in Fig. 8, the differential transform method is not useful for computing Green’s functions at extremely near-source distances. This limitation is overcome by combining the new method with the Kummer’s transformation. In a wider sense, the Kummer’s transformation is equivalent to the difference model approach. In practice, it is always advisable to check whether simplified analytical model is available before a more complicated model is considered. The numerical computation will only be more accurate and efficient if it is solely applied to the difference between the complicated and simplified models.

## 5 DISCUSSION AND CONCLUSIONS

In general, Green’s functions are composed of near-field and far-field terms that are classified on the basis of their spatial attenuation. Most seismic data only include far-field information, but fault slips and permanent deformation of the medium surrounding the fault zone reflect the near-field effect. Without considering intrinsic inelastic damping, the near-field body waves are attenuated by the geometric spreading factor \( r^{-3} \) or \( r^{-2} \), compared with \( r^{-1} \) for far-field body waves and \( r^{-1/2} \) for surface waves. Accordingly, any near-field phase can be considered as a strong impulse within a small neighbourhood around the origin of the 4-D time–space coordinates. Such an impulse has a slowly varying spectrum over a broad frequency–wavenumber band. For shallow sources, the smooth spectrum associated with the near-field terms is clearly dominant over that associated with the far-field terms, but it has no contribution to the Green’s functions at large distances. The key technique of the differential transform method is the use of the space-domain high-pass filter \( r^2 \) (or \( 2 \sin^2 \frac{\theta}{2} \) for spherical geometry) to enhance the numerical significance of the far-field terms over the undesired near-field terms (see Figs 2b and c).

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## ACKNOWLEDGMENTS

H. Wang was supported by the National Natural Science Foundation of China (Grant No. 40574010). Part of this work was conducted during H. Wang’s visit to the GeoForschungsZentrum Potsdam that was funded by the Bureau of Personnel and Education of the Chinese Academy of Sciences. Comments from Prof. M. Korn and two anonymous reviewers are gratefully acknowledged.

## REFERENCES


