Originally published as:


DOI: http://doi.org/10.1093/gji/ggt026
Compressible viscoelastodynamics of a spherical body at long timescales and its isostatic equilibrium

G. Cambiotti, V. Klemann and R. Sabadini

1 Department of Earth Sciences, University of Milan, Milan, Italy. E-mail: gabriele.cambiotti@unimi.it
2 GFZ German Research Centre for Geosciences, Earth System Modelling, Potsdam, Germany

1 INTRODUCTION

The basic problem when considering material compressibility in glacial-isostatic adjustment (GIA) is that it is often neglected in the definition of the initial state. This means, that a perturbation theory is applied to a non-consistently specified initial state. This problem is repeatedly discussed in literature when a general agreement came up that compressibility should be considered in GIA studies. The main aspects of this discussion are: the identification of instabilities to be physically related to convective instability (Plag & Jüttner, 1995), the discussion of this behaviour by means of instability modes in the spectral Laplace domain (Vermeersen et al. 1996; Hanyk et al. 1999; Klemann et al. 2003) and the discussion of its relevance for realistic Earth-like structures (Vermeersen & Mitrovica 2000). The result of this intellectual process was: the instabilities exist, but they are not relevant for the Earth because of the fact that the characteristic times for these instabilities are much larger than the age of the planet, when a realistic Earth structure like Preliminary Reference Earth Model (PREM) (Dziewonski & Anderson 1981) is considered.

Attempts to establish a consistent theory (Wolf & Kaufmann 2000; Martinec et al. 2001; Wolf & Li 2002) have so far failed also because of the fact that the elastic structure, compressibility and density have to be prescribed from theoretical aspects and deviate from the PREM structure. Recently, the problem of compressible viscoelasticity was revisited by Cambiotti & Sabadini (2010)—further on abbreviated as C&S10—where, starting from an adiabatic and chemically homogeneous initial state, deviations are parametrized similar to the Brunt–Väisälä frequency in dynamics. The authors distinguished between compressional (adiabatic and chemically homogeneous) and compositional (non-adiabatic and chemically heterogeneous) stratifications, where the latter were shown to result in a new class of spectral modes, named compositional modes. For unstable compositional stratifications, this class describes the instabilities discussed above. For stable compositional stratifications, instead, it controls the long timescale deformations towards the isostatic equilibrium with surface loading and they are responsible for divergent tangential displacement at the Earth surface (C&S10).

Whereas the consequences of compositional stratifications were discussed in C&S10 for the spectral behaviour and deformations at the Earth surface, herein we also focus on deformations within the mantle to give a full description of the perturbed state of the loaded Earth. The fluid limit, which is comprised in Maxwell viscoelasticity, justifies to discuss the deformational behaviour in view of the Longman (1963) paradox. This paradox treats the boundary condition for a compressible fluid core which is considered when determining the static (quasi-static) elastic (viscoelastic) deformations of the mantle because of surface loading (Longman, 1963). After outlining the paradox, we present the analytical representation of the compositional modes, discuss their influence for describing the isostatic equilibrium and conclude with the physical meaning and consequences of compositional stratifications.
2 LONGMAN (1963) PARADOX

The problem of a viscoelastic planet loaded at its surface for an infinite time is expected to end in the isostatic compensation of the load. The elastic stress determined by the load relaxes accordingly to the viscoelastic rheology and, at large timescales (in the limit of time \( t \to \infty \), the so-called fluid limit), when the transition from the elastic to the Newtonian fluid behaviours occurs, the planet achieves a static equilibrium with the surface load where there is no flow that generates shear stresses. In this respect, at large timescales, although governed by a Newtonian rheology, the planet is usually regarded as an inviscid body (Wu & Peltier 1982).

To characterize the isostatic equilibrium, we thus look for a static state of an inviscid planet (indistinguishable from a Newtonian body without flow) that is able to support surface loading. To this purpose, usually an Eulerian approach is adopted and the planet is assumed to be in hydrostatic equilibrium before perturbations. Then, after spherical harmonic expansion, the momentum and Poisson equations are solved in terms of local-incremental fields as discussed in Appendix A. This description of isostatic compensation is however incomplete when used to estimate the fluid limit of viscoelastic planets to surface loading. Indeed, although it constrains topography perturbations and local-incremental pressure, density and gravitational-potential, eqs (A15) and (A21)–(A24), we have no knowledge about displacements of material particles within the inviscid layers, as well as tangential displacements at the boundaries. Although this is expected for an inviscid planet dealt with an Eulerian formalism, the perturbed state of a viscoelastic planet due to surface loading should be completely determined at any time, including the fluid limit.

Before considering the fluid limit of the viscoelastic response to surface loading, that will be the topic of the following sections, it is noteworthy to first investigate what kind of description (and paradox, as we are going to show) of the isostatic equilibrium we can get adopting a Lagrangian point of view and specifying the constitutive equation for the degree-\( \ell \) material-incremental pressure \( p^\delta \) for isentropic perturbation

\[
p^\delta = -\kappa \chi, \tag{1}
\]

where \( \kappa \) and \( \chi \) are the adiabatic bulk modulus and the degree-\( \ell \) volume expansion \( \chi \) (the indexing for harmonic degree \( \ell \) is omitted for brevity). We only consider isentropic perturbation because it is usually assumed in most of GIA studies.

A similar problem was already dealt with by Longman (1963) to specify proper core–mantle boundary (CMB) conditions between an inviscid core and an elastic (viscoelastic) solid mantle for static (quasi-static) perturbations. Rather than following the derivation of Longman (1963), we begin our derivation by translating the results obtained in Appendix A into the Lagrangian formalism.

Let us consider the degree-\( \ell \) radial, \( U \), and tangential, \( V \), displacements of a material particle with respect to its position at the initial state of hydrostatic equilibrium. Their relations to the degree-\( \ell \) volume variation, \( \chi \), and material-incremental density, \( \rho^\delta \), are

\[
\chi = \frac{\partial U}{r} + \frac{\ell(\ell + 1)}{r} V, \tag{2}
\]

\[
\rho^\delta = -\rho_0 \chi, \tag{3}
\]

where \( r \) and \( \rho_0 \) are the radial distance from the Earth centre and the initial density. We also express the local-incremental pressure, \( p^\Delta \), and density, \( \rho^\Delta \), in terms of the material-incremental pressure, \( p^\delta \), and density, \( \rho^\delta \), and the respective advective terms (Wolf, 1991)

\[
p^\Delta = p^\delta + \rho_0 g U, \tag{4}
\]

\[
\rho^\Delta = \rho^\delta - U \partial_\rho \rho_0, \tag{5}
\]

where \( g \) is the initial gravity. Note that eq. (4) holds everywhere in the inviscid planet, although eq. (A6) makes sense only at the boundaries within the Eulerian approach (Dahlen 1974).

From the fact that local-incremental density and pressure must be zero in isostatic equilibrium, eqs (A15) and (A22), from eqs (1) and (3)–(5) we obtain two distinct equations relating volume variations and radial displacements

\[
- \rho_0 \chi - U \partial_\rho \rho_0 = 0, \tag{6}
\]

\[
- \kappa \chi + g \rho_0 U = 0. \tag{7}
\]

As noted by Longman (1963), they can be arranged as follows

\[
\gamma \chi = 0, \tag{8}
\]

\[
\gamma U = 0, \tag{9}
\]

where \( \gamma \) is the so-called compositional coefficient entering the generalized Williamson–Adams equation (Wolf & Kaufmann 2000),

\[
\partial_\rho + \frac{\rho_0^2 g}{\kappa} = \gamma. \tag{10}
\]

In this respect, the compositional coefficient, \( \gamma \), describes departure from the adiabatic and chemically homogeneous stratification. So, if \( \gamma \) is zero, the stratification is adiabatic and chemically homogeneous (hereafter compositional stratification), and, if \( \gamma \) differs from zero, the stratification is non-adiabatic and chemically heterogeneous (hereafter compositional stratification).

2.1 Compressional stratifications

For compressional stratifications (\( \gamma = 0 \)), eqs (8) and (9) are identically satisfied. This means that eqs (6) and (7) are linearly dependent and they actually express only one relation between volume variations and radial displacements. Thus, although related to each other, they remain unconstrained within the inviscid layers and only at the boundaries the radial displacement is constrained by eqs (A23) and (A24). With respect to the Eulerian description of isostatic equilibrium, there is no additional information but for the trivial information about volume variations which follows from the assumed constitutive equation for the material-incremental pressure, eq. (1):

\[
\chi(r_j) = 0 \quad \forall \ j < N, \tag{11}
\]

\[
\chi(a) = -\frac{g(a)}{\kappa_0(a)} \sigma_L, \tag{12}
\]

where \( N \) is the number of layers of the Earth model, \( r_j \) is the radius of the \( j \)-th boundary, with \( r_j < r_{j+1} \) and \( a = r_N \) being the Earth’s radius, and the subscript 0 to the bulk modulus denotes that it satisfies the Williamson–Adams equation, eq. (10) with \( \gamma = 0 \). Further information about deformations within the layers, as well as tangential displacements at the boundaries, shall be obtained by evaluating the fluid limit of the viscoelastic response.

2.2 Compositional stratifications

For compositional stratifications (\( \gamma \neq 0 \)), eqs (8) and (9) are satisfied only when both, volume variations and radial displacements,
are zero everywhere. From eq. (2), also tangential displacements yield zero. In this respect, material particles are constrained to remain at their initial positions, suggesting that the only static equilibrium of a compositional inviscid planet can be the initial state of hydrostatic equilibrium. This circumstance, however, leads to the paradox that isostatic compensation of surface loading is impossible. Indeed, if radial displacements are zero everywhere, the eq. (A24) is violated. In other words, for compositional inviscid planets there cannot be volume variations and, thus, there is no material-incremental pressure that supports the load. This problem, once referred to isostatic compensation of the solid mantle lumping into the inviscid core, was named Longman (1963) paradox and debated in the 1970s by many authors (e.g. Smylie & Mansinha, 1971; Chinnery, 1975; Crossley & Gubbins, 1975; Dahlen & Fels, 1978). In that case, indeed, the compositional inviscid core would not be able to compensate radial stresses due to the static perturbed states of the mantle.

To solve this problem, Smylie & Mansinha (1971) and Chinnery (1975) suggested an ad hoc discontinuity of the radial displacement at the CMB (which should be actually interpreted as a gap between radial displacement and geoid perturbations just below the mantle) resulting in a non-zero material-incremental pressure, eq. (A12) to allow isostatic compensation. Crossley & Gubbins (1975) arrived at similar conclusions by considering a thin boundary layer between the solid mantle and the inviscid core where the shear modulus drops to zero without discontinuity. In this way, they did not impose any ad hoc discontinuity in the radial displacement, but they explained it as the drop of the radial displacement within this thin boundary layer (resulting in the gap between radial displacement and geoid perturbation).

In the end, Dahlen & Fels (1978) showed that difficulties in obtaining static perturbations within the core were essentially due to the unrealistic assumption of an inviscid core where there is no form of dissipation. In view of the very low viscosity of the core, they considered the static response of the inviscid core as the limit for time going to infinity of its dynamic response composed of internal gravity modes. However, because of analytical and numerical complexities, they faced a simplified version of the problem: a 2-D box with rigid walls filled with a compositional fluid in hydrostatic equilibrium (the core) and a sudden deformation of the upper boundary (the solid mantle lumping into the core) that displaces the fluid from its original position. Then, by determining the internal gravity modes in the frequency domain and introducing an ad hoc form of dissipation, they investigated the transient dynamic response composed of the whole set of internal gravity modes and evaluated its limit for time $t \rightarrow \infty$ to obtain the final static response to the deformation of the upper boundary of the box. Within this scheme, they showed that particles come back to their initial position (as expected for compositional stratifications) but for a thin boundary layer just below the upper boundary that adapts to the imposed deformation. Particularly, the size of the thin boundary layer where deformations are localized decreases with the magnitude of the dissipation. Apart from these investigations of the way in which isostatic compensation at the CMB is possible also for compositional stratifications of the fluid core, no problem arises adopting the Eulerian description as long as the core is simply assumed to remain in hydrostatic equilibrium. This aspect is independent from any constitutive equations (Dahlen 1974; Crossley & Gubbins 1975; Dahlen & Fels 1978), like eq. (1) from which the Longman (1963) paradox originates. Although a detailed description of the deformation within the inviscid core is not interesting when applying it to the earth—we can determine the static (quasi-static) perturbation within the elastic (viscoelastic) mantle and at the Earth surface—such a description is necessary for this discussion where we consider a viscoelastic planet loaded at its surface. Indeed, information that we would loose adopting the Eulerian approach refers to the perturbed state of the mantle and the Earth surface: one for all, we would not be able to quantify tangential displacements at the Earth surface in the fluid limit. Furthermore, although the necessity of assuming a specific constitutive equation for the inviscid core could be questioned (Dahlen 1974), it cannot in the present case just because a constitutive equation, the Maxwell rheology, is the basic ingredient of viscoelastodynamics.

Thus, to shed light on the consequences of the Longman (1963) paradox in viscoelastodynamics and the behaviour of viscoelastic planets at large timescales, in the following sections we will discuss the effects of compositional stratifications on viscoelastic perturbations and, then, on the final isostatic equilibrium of surface loading.

3 COMPOSITIONAL MODES

C&S10 found the analytical solution of viscoelastic perturbations in the Laplace s-domain for a specific self-gravitating compressible Maxwell Earth model, called ‘homogeneous self-compressed compressible sphere’. This model is composed of an incompressible inviscid core and a compressible Maxwell mantle with constant shear modulus, $\mu$, bulk modulus, $K$, and viscosity, $\nu$. To account for the self-compression of the mantle at the initial state of hydrostatic equilibrium, the initial density profile varies with radial distance from the Earth centre, $r$, according to

$$\rho_0(r) = \begin{cases} \frac{a}{r} & 0 \leq r \leq b \\ \frac{a}{r^2} & b < r \leq a. \end{cases}$$

(13)

Here, $a$ and $b$ are the Earth radius and the core radius, respectively, and $a$ is a constant related to the total Earth mass by $M_E = 2 \pi a^2$. This choice of the initial density profile fixes the initial gravity acceleration within the mantle to

$$g = 2 \pi G a,$$

(14)

where $G$ is the gravitational constant. Considering the generalized Williamson–Adams eq. (10) with the compositional coefficient $\gamma = 0$, we obtain that compressional stratifications are characterized by a constant bulk modulus

$$K_0 = g a = 2 \pi G a^2$$

(15)

to which we will refer as compressional bulk modulus, $K_0$. Departure of the bulk modulus, $K$, from this value results in a compositional stratification. In particular, it is stable if $K > K_0$ and unstable if $K < K_0$ (C&S10). In light of this, the analytical solution for the ‘homogeneous self-compressed compressible sphere’ enables us to investigate the effects of both compressional and compositional stratifications in viscoelastodynamics. Previous analytical solutions were obtained assuming material or local incompressibility (Sabadini et al. 1982; Wu & Peltier 1982; Martinec et al. 2001), or compressibility only during the perturbations as in the case of the ‘homogeneous compressible sphere’ of Gilbert & Backus (1968) that describes an unstable compositional stratification because the initial density is constant from the centre to the surface of the Earth (Plag & Jüttner 1995).

The study of the analytical solution of the ‘homogeneous self-compressed compressible sphere’ in the Laplace s-domain points out that radial, $h(s)$, tangential, $l(s)$ and gravitational, $k(s)$,
viscoelastic Love numbers can be recast by a spectrum of relaxation modes
\[ \hat{k}(s) = \begin{pmatrix} \hat{h}(s) \\ \hat{\ell}(s) \\ -1 - \hat{k}(s) \end{pmatrix} = k_E + \sum_{j=1}^{\infty} \frac{k_j}{s - s_j}, \quad (16) \]

where \( k_E = (h_E, l_E, -1 - k_E) \) consists of the elastic Love numbers, \( k_j = (h_j, l_j, k_j) \) contains the residues of the \( j \)th relaxation mode and \( s_j \) is the corresponding pole. Here, \( T \) denotes the whole set of relaxation modes, which is denumerable but infinite (C&S10). The relaxation modes are split into two types:
\[ T = S \cup C. \quad (17) \]

The set \( S \) of fundamental modes appears both for compositional and compositional stratifications: the M0 and C0 buoyancy modes, the infinite and denumerable set of dilational modes, \( D_m \), with \( m = 1, \ldots, \infty \), and the pair of transient compressible modes, \( Z_- \) and \( Z_+ \). The buoyancy modes, M0 and C0, appear also for incompressible models (Sabadini et al. 1982; Wu & Peltier 1982) and are associated with the Earth’s surface and CMB, respectively. The dilational modes, \( D_m \) (Han & Wahr 1995; Vermeersen et al. 1996), and transient compressible modes, \( Z_- \) and \( Z_+ \) (Cambiotti et al. 2009; Cambiotti & Sabadini 2010) are associated with the compressible viscoelastic layers (in the present case only the mantle) and have characteristic relaxation times of the order of the compressional relaxation time \( \tau_C \):
\[ \tau_C = \tau_M \frac{\kappa + \frac{4}{3} \mu}{\kappa}, \quad (18) \]

where \( \tau_M = \nu/\mu \) is the Maxwell time. Particularly, in the Laplace \( s \)-domain, \( s = -1/\tau_C \) is the cluster point of the poles of the denumerable and infinite set of dilational modes (Vermeersen et al. 1996).

The set \( C \) of compositional modes, \( C_m \), with \( m = 1, \ldots, \infty \), is again denumerable and infinite, but is triggered only in the case of compositional stratifications. Their analytical approximation given in C&S10 (eq. 38) is
\[ s_{C_m} = -\ell(\ell + 1) \frac{k_0}{v} \left( \log \left( \frac{\omega}{\pi m} \right) \right)^3 + O(m^{-6}), \quad (19) \]

with
\[ \epsilon = \frac{k - k_0}{k}. \quad (20) \]

Note that the origin of the Laplace \( s \)-domain is the cluster point of the poles \( s_{C_m} \) for \( m \to \infty \) because they converge to zero as \( m^{-4} \) and, in this respect, compositional modes describe long timescale perturbations beyond the relaxation of the fundamental modes.

The fundamental modes describe the transition from the elastic to the Newtonian-fluid behaviour, whereas the compositional modes control the long timescale perturbations towards the isostatic equilibrium specified by the inviscid fluid. Accordingly, we split the perturbations due to a point-like surface load of unit mass and Heaviside time history into contributions describing the elastic response, the transition to the Newtonian fluid and the final transition towards the isostatic equilibrium:
\[ K(t) = \begin{pmatrix} U(t) \\ V(t) \\ \phi^\ell(t) \end{pmatrix} = k_E - \sum_{j=1}^{\infty} k_j \left( 1 - e^{s_j t} \right) - \sum_{m=1}^{\infty} k_{C_m} \left( 1 - e^{s_{C_m} t} \right), \quad (21) \]

where \( U \) and \( V \) are the degree-\( \ell \) non-dimensional radial and tangential displacements (normalized by \( a/M_0 \)), and \( \phi^\ell \) is the degree-\( \ell \) non-dimensional gravitational-potential perturbation (normalized by \( a g(a)/M_0 \)). Because of the fact that the poles \( s_j \) of the fundamental modes are negative and that their characteristic relaxation times, \( |1/s_j| \), are shorter than those of the compositional modes, \( |1/s_{C_m}| \), we can write the final transition towards the isostatic equilibrium as
\[ K(t) = k_S + K_C(t), \quad (22) \]

where \( k_S \) is the secular perturbation due to the elastic response and the relaxation of the fundamental modes
\[ k_S = \begin{pmatrix} h_S \\ l_S \\ -1 - k_S \end{pmatrix} = k_E - \sum_{j=1}^{\infty} \frac{k_j}{s_j}, \quad (23) \]

and where \( K_C(t) \) is the perturbation only due to the compositional modes
\[ K_C(t) = \begin{pmatrix} U_C(t) \\ V_C(t) \\ \phi_C^\ell(t) \end{pmatrix} = -\sum_{m=1}^{\infty} \frac{k_{C_m}}{s_{C_m}} \left( 1 - e^{s_{C_m} t} \right). \quad (24) \]

For compositional stratifications, which have no compositional modes, the viscoelastic Love number \( \hat{k}(s) \) is an analytic function in a neighbourhood of the origin of the Laplace domain, \( s = 0 \). Thus, \( \hat{k}(s = 0) \) exists and is finite and, from eqs (16) and (23), we obtain the following identity:
\[ k_S = \hat{k}(s = 0). \quad (25) \]

This implies that the summation over the strengths \( k_j/s_j \) of the fundamental modes entering eq. (23) converges to a finite value. Furthermore, the secular perturbations describe the isostatic equilibrium to surface loading (Wu & Peltier 1982). In this respect, from eqs (A21) and (A23) and (A24), the secular gravitational-potential perturbations are zero,
\[ \lim_{t \to \infty} \phi^\ell(t) = -1 - k_S = 0, \quad (26) \]

as well as the CMB topography perturbation
\[ \lim_{t \to \infty} U(b, t) = h_S(b) = 0, \quad (27) \]

although the Earth surface topography perturbation (due to the point-like load and normalized by \( a/M_0 \)) yields
\[ \lim_{t \to \infty} U(a, t) = h_S(a) = -\frac{2 \ell + 1}{2}. \quad (28) \]

Deformations inside the Earth, as well as the tangential displacement at the CMB and Earth surface, are instead unconstrained due to the indeterminateness of static perturbations of the inviscid body discussed in Section 2 and Appendix A. They must be obtained solving the whole viscoelastic problem using eq. (23).

For compositional stratifications, because of the fact that the origin of the Laplace domain is the cluster point of compositional
modes, eq. (19), the viscoelastic Love number \( \tilde{k}(s) \) is not analytic at \( s = 0 \) and, so, \( \tilde{k}(s = 0) \) is not well defined. To interpret the final transition towards the isostatic equilibrium, we have to consider the perturbation due to compositional modes. For unstable stratifications (i.e. when \( \kappa < \kappa_0 \) and \( \epsilon < 0 \)), their poles, \( s_c, \) are positive and eq. (24) describes Rayleigh–Taylor instabilities (Plag & Jüttner, 1995) resulting in divergent displacement and divergent gravitational-potential in the limit of time \( t \to \infty \). In contrast, for stable stratifications (i.e. when \( \kappa > \kappa_0 \) and \( \epsilon > 0 \)), the poles are negative and eq. (24) describes a slow relaxation process. From numerical evaluation of the strengths \( k_c / s_c \) of the first few compositional modes (at most to \( m = 20 \) for the harmonic degree \( \ell = 2 \)), C&S10 concluded that tangential strengths at the Earth surface converge to a non-zero value, say \( N \), in the limit for \( m \to \infty \),

\[
\lim_{m \to \infty} \frac{I_c}{s_c} = N.
\]

Using the representation of eq. (24), and eqs (70) and (72) of C&S10, tangential displacements \( V_C \) at the Earth surface due to stable compositional modes at large timescale can be approximated by

\[
V_C(a, t) = - \sum_{m=1}^\infty \frac{I_c}{s_c} \left( 1 - e^{-s_c t} \right)
\]

\[
\approx N \left( \ell(\ell + 1) \frac{\epsilon k_0}{v} \right)^{1/4} \log \left( \frac{D}{\ell} \right) \frac{\Gamma[3/4]}{\pi},
\]

with \( \Gamma \) being the Gamma function. This tangential displacement of material away from the load, named ‘long-period tangential flux of material’, diverges as \( \ell^{-1/4} \) in the limit of \( \ell \to \infty \). The mathematical proof of eq. (29) and a physical interpretation of this peculiar flux will be discussed in the following section.

### 4 Isostatic Equilibrium of Viscoelastic Planets

To investigate the final transition of the viscoelastic planet towards the isostatic equilibrium, eq. (22), we have to take into account the whole infinite and denumerable set of compositional modes in the case of a stable compositional stratification. Because of numerical complexities in computing the summation over compositional modes entering eq. (24) without a too low cut-off, we are interested in an analytical formulation for the strengths \( k_c / s_c \) of compositional modes, in addition to the analytical expression for their poles \( s_c \), eq. (19). Furthermore, to get deeper insight into the final isostatic equilibrium, we will also consider the time evolution of degree-\( \ell \) volume strengths at the Earth surface (normalized by \( 1 / M_E \))

\[
\chi(t) = s_x - \sum_{m=1}^\infty \frac{x_c}{s_c} \left( 1 - e^{-s_c t} \right).
\]

where \( s_x \) is the secular volume variation, and \( x_c / s_c \) are the volume strengths of compositional modes.

Analytical expressions for the strengths can be obtained by neglecting self-gravitation because this greatly facilitates to handle the analytical expression of the viscoelastic Love number \( \tilde{k}(s) \) in the Laplace \( s \)-domain. This simplification does not alter the main issue of the Longman (1963) paradox or the appearance of instability (Klemann et al. 2003). Indeed, for compositional stratifications, eqs (8) and (9) hold also in the simpler gravitating case. The only main alteration concerns the fact that the material-incremental pressure is no longer due to the gap between radial displacement and geoid perturbation as in eq. (A6), but only to radial displacement

\[
p^s = -\rho_0 g U,
\]

because geoid perturbation is zero by definition in the gravitating case. On the other hand, in the self-gravitating case, geoid perturbations converge to zero in the isostatic equilibrium, eq. (A21), and, thus, the final isostatic equilibrium is the same for gravitating and self-gravitating Earth models.

In Appendix B, we obtain the analytical solution of the viscoelastic Love number \( \tilde{k}(s) \) in the Laplace \( s \)-domain for the ‘homogeneous self-compressed compressible sphere’ by neglecting self-gravitation, eq. (B32). In Appendix C, we use this solution to obtain analytical approximations for poles and strengths of compositional modes, eqs (C9–C11) and (C15). Note that the dominant terms of the analytical approximations for the poles are the same, if accounting for self-gravitation, eq. (19), or not accounting for, eq. (C9). This confirms the small influence of self-gravitation on the final transition from Newtonian-fluid behaviour to isostatic equilibrium.

We consider a ‘homogeneous self-compressed compressible sphere’ with shear modulus \( \mu = 1.45 \times 10^{11} \) Pa, bulk modulus \( \kappa = 2.71 \times 10^{11} \) Pa and viscosity \( \nu = 10^2 \) Pa s. The core radius and Earth radius are \( b = 3480 \) km and \( a = 6371 \) km, respectively, and the constant \( a_0 \), which characterizes the initial density profile, eq. (13), yields \( a_0 = 2.34 \times 10^4 \) kg m\(^{-3}\). Note that we chose the bulk modulus, \( \kappa \), to be 15 per cent larger than the compressional bulk modulus \( \kappa_0 = 2.30 \times 10^{11} \) Pa, eq. (15), to obtain a stable compositional stratification and to take into account that, for the Earth, the contribution of the compositional stratification does not amount to more than 10–20 per cent of that of the compositional stratification (Birch 1952, 1964; Wolf & Kaufmann 2000).

Fig. 1 shows for degrees \( \ell = 2, \ldots, 30 \) the characteristic relaxation times \( t_c = |1 / s_c| \) of the first 10 compositional modes obtained numerically without any approximation and using their analytical approximation, eq. (C9). The comparison shows that the analytical approximation fits quite well and only deviates for the first compositional modes and increasing harmonic degrees. The shortest exact characteristic relaxation time is 105 kyr, corresponding to the first compositional mode at degree \( \ell = 5 \) marked by the open circle in Fig. 1. This short timescale is due to the simplified model; more realistic Earth models based on PREM predict much larger characteristic relaxation times of order 1–100 Myr (Plag & Jüttner 1995; Vermeersen & Mitrovica 2000). This is because of the fact that only the upper mantle of PREM has a compositional stratification, whereas its lower mantle is mainly characterized by a compositional stratification.

From eqs (C10) and (C11), we obtain the radial, tangential and volume strengths at the Earth surface

\[
\frac{h_c(a)}{s_c} = -\epsilon \left( \frac{2 \ell + 1}{\pi m^2} \right) \log \left( \frac{D}{\ell} \right),
\]

\[
\frac{l_c(a)}{s_c} = -\epsilon \left( \frac{2 \ell + 1}{\ell(\ell + 1)} \right) \log \left( \frac{D}{\ell} \right),
\]

\[
\frac{x_c(a)}{s_c} = (\epsilon - 1) \epsilon \left( \frac{2 \ell + 1}{\ell(\ell + 1)} \right) \log \left( \frac{D}{\ell} \right).
\]

Here, the tangential strength at the Earth surface converges to a non-zero value in the limit of \( m \to \infty \). This finding proves mathematically eq. (29), that C&S10 obtained numerically in the case of self-gravitation, and provides the analytical expression for the
Figure 1. Characteristic relaxation times, $\tau_{\text{CM}} = |1/s_{\text{CM}}|$, of the first 10 compositional modes obtained numerically (solid lines) and using their analytical approximation, eq. (C9) (dashed lines). The open circle indicates the shortest characteristic relaxation time, which appears for the first compositional mode at degree $\ell = 5$.  

We find a proportionality to the fourth root of $\epsilon$, which describes deviation from compressional stratification, eq. (20), and an inversely proportionality to the fourth root of viscosity, $\nu$. These dependences confirm the findings of C&S10, that the ‘long-period tangential flux of material’ is a specific process of compositional stratifications when the material behaves like a Newtonian fluid, that is after relaxation of the initial elastic stress due to loading.

The radial and volume strengths at the Earth surface, eqs (33) and (35), decay to zero as $m^{-2}$ and $m^{-4}$, respectively. This assures the existence of the limits for time $t \to \infty$ of radial displacement and volume variation at the Earth surface

$$\lim_{t \to \infty} U(a, t) = h_S(a) - \sum_{m=1}^{\infty} \frac{h_{C_m}(a)}{s_{C_m}},$$

$$\lim_{t \to \infty} \chi(a, t) = x_S(a) - \sum_{m=1}^{\infty} \frac{x_{C_m}(a)}{s_{C_m}}.$$  

The summation over $m$ is obtained using the analytical approximations for the strengths of compositional modes, eqs (33) and (35). There, the numerical estimates of the first modes are considered to avoid inaccuracy of eqs (33) and (35) at low $m$ and the analytical approximation is used only for $m > 20$. In this way, we confirmed the isostatic equilibrium,

$$\lim_{t \to \infty} U(a, t) \simeq -\frac{2\ell + 1}{2},$$

$$\lim_{t \to \infty} \chi(a, t) \simeq -\frac{\rho(a) g(a)}{\kappa(a)} \frac{2\ell + 1}{\kappa} = -\frac{\kappa_0}{\kappa} \frac{2\ell + 1}{\kappa},$$

for degrees $\ell = 2, \ldots , 30$ with an accuracy of $10^{-5}$ attributable to numerical precision of the algorithm. Note that eq. (40) is consistent with the constitutive equation for the inviscid fluid, eq. (1), and the requirement that material-incremental pressure at the Earth surface equals the pressure due to the point-like surface load of unit mass. The second equality in eq. (40) holds in the case of the ‘homogeneous self-compressed compressible sphere’, using eq. (15).

In contrast to the compressional stratification, the isostatic equilibrium is achieved here only after relaxation of compositional modes. This is shown in Fig. 2 for both radial displacement and volume variation for degrees $\ell = 2, \ldots , 30$ by plotting the difference between the secular perturbations at the Earth surface, $h_S(a)$ and $x_S(a)$, and the respective values of the final isostatic equilibrium, eqs (39) and (40). For the considered parametrization, the differences start at 0.076 and 0.014 at degree 2 and increase to 0.191 and 0.066 at degrees larger than 15 and 10, for radial displacement and volume variation, respectively.

Fig. 3 shows the radial variation from the CMB to the surface for the degree-2 secular radial, $h_S$, and tangential, $l_S$, displacements and volume variation, $x_S$, of the considered compressional and compositional stratifications. The secular radial displacements and volume variations of the compositional stratification differs from the
compressional stratification by about 0.1 and 0.4, respectively. In particular, the secular radial displacement and volume variation of the compositional stratification are non-zero at the CMB, whereas they vanish for the compressional stratification as it describes consistently an isostatic equilibrium. Secular tangential displacements differ mainly in the upper part of mantle by about 0.1, where the compositional stratification predicts smaller values than compressional stratification.

The transition from the secular deformation to the isostatic equilibrium for the case of compositional stratification is shown in Fig. 4, at increasing times $t = 10^5, 10^6, 10^7, 10^8, 10^{10}$ kyr. According to eqs (8) and (9), radial and tangential displacements and volume variations converge to zero within the viscoelastic mantle in the limit of time $t \to \infty$. Thus, compositional modes account for buoyancy forces that bring particles from the deformed secular state described by the fundamental modes back to the initial state of hydrostatic equilibrium. This process starts at the CMB and proceeds towards the Earth surface generating a superficial layer that thins with increasing time and where deformations still occur. There, the tangential material flow, which goes towards and away from the load in the lower and upper parts of this layer, diverges according to eq. (36), although the radial displacement and volume variation at the Earth surface converge to the non-zero values $-2.5$ and $-2.12$, satisfying the isostatic conditions, eqs (39) and (40) with $\ell = 2$.

Another important remark is that flow velocities go to zero everywhere, even if the ‘long-period tangential flux of material’ is triggered in the thin, superficial layer. Indeed, in view of the fact that tangential displacement at the Earth surface diverges as $t^{1/4}$ in the limit of $t \to \infty$, eq. (36), the velocity of this flux goes to zero as $t^{-3/4}$. This means that shear stresses vanish in the limit for time $t \to \infty$, and the viscoelastic planet behaves as an inviscid body, consistently with the fact that radial displacement at the Earth surface satisfies the condition derived from the Eulerian description of the final isostatic equilibrium, eq. (39), and that volume variation at the Earth surface converge to the same value necessary for an inviscid fluid to equal the pressure due to point-like surface load of unit mass, eq. (40).

### 5 Conclusion

Extending the analysis of the perturbations of the ‘homogeneous self-compressed compressible sphere’ (Cambiotti & Sabadini 2010), we have shown that both, compressional and stable compositional stratifications, achieve the isostatic equilibrium with surface loading, although in very different ways. For compressional stratifications, the final isostatic equilibrium of surface loading is achieved after the relaxation of elastic stresses by means of mass rearrangement within the whole mantle described by the fundamental modes of the Earth model: the $M_0$ and $C_0$ buoyancy modes, the dilatational modes and the transient compositional modes. Differently, for stable compositional stratifications, the new mass rearrangement achieved after relaxation of elastic stresses cannot be an equilibrium state as pointed out by the Longman (1963) paradox. Thus, buoyancy forces within the mantle, described by the denumerable and infinite set compositional modes, bring material particles back to their positions at the initial state of hydrostatic equilibrium and the isostatic equilibrium of surface loading is achieved by means of mass rearrangement confined in a thin, superficial layer. Particularly, differently from compositional stratification where shear stresses vanish during the transition from the elastic to the Newtonian fluid behaviours, for compositional stratification Newtonian shear stresses still act after the relaxation of elastic stress due to the flow of material particles that come back to their initial position. These residual shear stresses, however, vanish in the fluid limit, for time $t \to \infty$, and topography perturbation and volume variation at the Earth surface, eqs (39) and (40), are consistent with the Eulerian description of isostatic equilibrium and the constitutive equation for the inviscid fluid, eq. (1), and the requirement that material-incremental pressure at the Earth surface equals the pressure due to the point-like surface load of unit mass.

Although isostatic equilibrium of surface loading for a high viscous viscoelastic mantle and isostatic compensation of the solid mantle bumping into the inviscid core are different physical problems, it is noteworthy that for both cases (and when the stratification is compositional) the final static equilibrium involves only perturbations within a thin layer just below the respective outer boundaries, the Earth surface (the present work) or the CMB (Dahlen & Fels, 1978). This reflects the similarity of the mathematical formulations
Figure 4. Radial displacement, $U(r, t)$, tangential displacement, $V(r, t)$ and volume variation, $\chi(r, t)$ for degree 2 (from left to right) at increasing times $t = 10^2, 10^3, 10^4, 10^5, 10^6, 10^8$ kyr (from top to bottom).

in the fluid limit for quasi-static viscoelastic perturbation of the viscoelastic mantle and in the static limit of the dynamic response of the inviscid core. Indeed, after Laplace or Fourier transforms, and in the limit of the Laplace variable $s$ or frequency $\omega$ going to zero, for both problems the momentum equation converges to the same form that, for compositional stratification, leads to the paradox discussed by Longman (1963) for which the volume variations, as well as the material-incremental pressure, must be zero within the layers. However, as pointed out in the present work and in Dahlen & Fels (1978), this constrain does not hold at the boundaries, where instead volume variations and material incremental pressure are correctly dictated by the Earth surface and CMB boundary conditions, and isostatic compensation is thus possible.

**Acknowledgements**

Part of this work was supported by COST Action ES0701 ‘Improved constraints on models of Glacial Isostatic Adjustment’ where the first author acknowledges funding by an SRTM for his stay at GFZ Potsdam. The work of VK has been supported by the DFG Priority Program SPP1257 (KL2284/1-3).

**References**


\[ \nabla^2 \phi^\Delta = 4 \pi G \rho^\Delta. \]  

Here, \( \nabla^2 \) is the radial part of the Laplacian operator
\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\ell (\ell + 1)}{r^2},
\]
with \( r \) being the radial distance from the Earth centre, \( \rho_0 \), \( g \) and \( G \) are the initial density, gravity and gravitational constant, and \( p^\Delta, \rho^\Delta \) and \( \phi^\Delta \) are the degree-\( \ell \) local-incremental pressure, density and gravitational-potential (the indexing for harmonic degree \( \ell \) is omitted for brevity). Note that \( \phi^\Delta \) includes both perturbations due to Earth mass rearrangements and the direct effect of the load.

We consider the unperturbed Earth model composed of \( N \) spherical shells with boundaries where the initial density profile has step-like discontinuities
\[ [\rho_0(r_j)]^+_\ell \neq 0. \]  

Here, \( r_j \) is the radius of the \( j \)th boundary, with \( r_j < r_{j+1} \) and \( a = r_0 \) being the Earth’s radius, and \( [f(r_j)]^+_\ell \) denotes the discontinuity of the field \( f \) at the \( j \)th boundary. To define proper boundary conditions, it is customarily assumed that deformed boundaries consist of the same material particles present at the initial unperturbed state (Dahlen, 1974). This assumption allows to consider the material-incremental pressure, \( p^\ell \), just below, \( r^-_j \) and above, \( r^+_j \), the boundary
\[ p^\ell(r^-_j) = p^\ell(r^+_j) = \rho_0(r^-_j) g(r_j) U(r_j), \]  

where \( U(r_j) \) is the degree-\( \ell \) radial displacement of the boundary (i.e. the topography perturbation). Then, by requiring the continuity of the normal stresses acting on deformed internal boundaries, \([p^\ell(r_j)]^+_\ell = 0\), we obtain the following condition for the local-incremental pressure
\[ [p^\ell(r_j)]^+_\ell = [\rho_0(r_j)]^+_\ell g(r_j) U(r_j) \quad \forall j < N. \]  

At the Earth surface, instead, by requiring that material-incremental pressure equals the pressure due to surface loading, \( p^\ell(a^-) = g(a) \sigma_\ell \), we obtain
\[ p^\ell(a^-) - \rho_0(a) g(a) U(a) = g(a) \sigma_\ell, \]  

where \( \sigma_\ell \) is the surface density describing the load. We also require the continuity of the local-incremental gravitational-potential and the discontinuity of its radial derivative
\[ [\phi^\Delta(r_j)]^+_\ell = 0 \quad \forall j \leq N, \]  

\[ [\partial_\ell \phi^\Delta(r_j)]^+_\ell = 4 \pi G [\rho_0(r_j)]^+_\ell U(r_j) \quad \forall j < N, \]  

\[ [\partial_\ell \phi^\Delta(a^-)]^+_\ell = 4 \pi G \rho_0(a^-) U + 4 \pi G \sigma_\ell. \]  

Local-incremental pressure and geoid perturbations are simply related by the tangential component of the momentum equation, eq. (A2) and it is noteworthy to use this relation to eliminate the local-incremental pressure from the above equations. By recalling also the continuity of the local-incremental potential, eq. (A9), from eq. (A6) we obtain
\[ p^\ell(r^-_j) = -\rho_0(r^-_j) g(r_j) \left[ U(r_j) + \frac{\phi^\Delta(r_j)}{g(r_j)} \right]. \]

It shows that the material-incremental pressure \( p^\ell \) is due to the gap between topography, \( U \), and geoid, \(-\phi^\Delta/g\), perturbations. Also, the boundary conditions for the local-incremental pressure, eqs (A7) and (A8), become relations between topography and geoid perturbations.
\[ U(r_j) = -\frac{\phi^\lambda(r_j)}{g(r_j)} \quad \forall \ j < N, \]  
(A13)

\[ U(a) = -\frac{\sigma_\ell}{\rho_0(a^-)} - \frac{\phi^\lambda(a)}{g(a)}. \]  
(A14)

Note that internal topography perturbations coincide with geoid perturbations, although there is a gap between them at the Earth surface. Particularly, in view of eq. (A12), this means that the material-incremental pressure is zero at internal boundaries, while it differs from zero at the Earth surface to support the load. In the end, from the radial component of the momentum equation, eq. (A1), the local-incremental density yields zero

\[ \rho^\lambda = 0. \]  
(A15)

In view of the above consideration, to solve the problem, we only need to obtain the local-incremental gravitational-potential \( \phi^\lambda \) from the Poisson equation and the remaining boundary conditions. In view of eqs (A9) and (A13)–(A15), eq. (A3) becomes the Laplace equation

\[ \nabla^2 \phi^\lambda = 0, \]  
(A16)

and the boundary conditions given by eqs (A10) and (A11) become

\[ \left[ \partial_r \phi^\lambda(r_j) \right]^+ = -\frac{4\pi G \rho_0}{g(r_j)} \phi^\lambda(r_j) \quad \forall \ j < N, \]  
(A17)

\[ \left[ \partial_r \phi^\lambda(a) \right]^+ = -\frac{4\pi G \rho_0(a^-)}{g(a)} \phi^\lambda(a). \]  
(A18)

Then, using eq. (A9) and requiring the condition of regularity at the Earth centre and at infinity

\[ \phi^\lambda(r) \propto r^\ell \quad \text{for} \quad r \to 0, \]  
(A19)

\[ \phi^\lambda(r) \propto r^{-\ell-1} \quad \text{for} \quad r \to \infty, \]  
(A20)

it is straightforward to show that the only admissible solution of the Laplace equation, eq. (A16), is the trivial one, that is, the local-incremental gravitational-potential is zero

\[ \phi^\lambda = 0. \]  
(A21)

In view of eqs (A2) and (A13) and (A14), this means that also the local-incremental pressure is zero and that there are no internal topography perturbations

\[ p^\lambda = 0, \]  
(A22)

\[ U(r_j) = 0 \quad \forall \ j < N, \]  
(A23)

while the Earth surface topography perturbation yields

\[ U(a) = -\frac{\sigma_\ell}{\rho_0(a)}, \]  
(A24)

which is the simple statement of the Archimedes principle generalized to the case of an inviscid self-gravitating planet.

**APPENDIX B: ANALYTICAL SOLUTION**

Here, we derive the analytical solution of the momentum equation for the gravitating ‘homogeneous self-compressed compressible sphere’.

After expansion in spherical harmonics and Laplace transform, the radial and tangential components of the momentum equation within the mantle can be cast as follows,
\[ a_1 = -2 \zeta \mu, \quad (B18) \]
\[ a_2 = \kappa_0 + \zeta \mu. \quad (B19) \]

Denoting the two roots of eq. (B16) by \( X_1 \) and \( X_2 \),
\[ X_1 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}, \quad (B20) \]
\[ X_2 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}, \quad (B21) \]
we obtain four distinct expressions for \( x_j \) from the definition of \( X_k \), eq. (B14):
\[ x_j = -2 - \frac{1}{2} \sqrt{1 + 4 \left( L + X_j \right)}, \quad (B22) \]
\[ x_{j+2} = -2 + \frac{1}{2} \sqrt{1 + 4 \left( L + X_j \right)}, \quad (B23) \]
with \( j = 1, 2 \). In this way, we have obtained the analytical expressions for both scalars \( v_j \) and \( x_j \) which enter in the expressions for the four independent solutions of the radial and tangential components of the momentum equation, eqs (B10) and (B11).

To solve the surface loading problem, we consider the spheroidal vector solution \( \hat{y} \) in the Laplace domain
\[ \hat{y} = \begin{pmatrix} \hat{U} \\ \hat{V} \\ \kappa \hat{h} + 4/3 \hat{\mu} \partial_s \hat{U} \\ \hat{\mu} \partial_s \hat{V} + \hat{\mu} (\hat{U} - \hat{V})/r \end{pmatrix}, \quad (B24) \]
where the third and fourth components represent the degree-\( \ell \) radial and tangential stresses. Substituting eqs (B8)–(B11) into eq. (B24), we obtain the analytical expression for the spheroidal vector solution
\[ \hat{y}(r, s) = Y(r, s) C, \quad (B25) \]
where \( Y \) is the fundamental matrix and \( C \) is the vector consisting of four constants of integration
\[ Y = (y_1, y_2, y_3, y_4), \quad (B26) \]
\[ C = (C_1, C_2, C_3, C_4), \quad (B27) \]
with \( T \) standing for the transpose. Here, \( y_j \) are the four independent spheroidal vector solutions that we obtain by substituting \( U_j \) and \( V_j \), eqs (B10) and (B11), for \( \hat{U} \) and \( \hat{V} \) into eq. (B24).

The four constants of integration \( C \) must be determined imposing boundary conditions at the CMB \( (r = b) \) and the Earth surface \( (r = a) \). The CMB condition for the gravitating problem are given by
\[ \hat{y}(b, s) = I_c D, \quad (B28) \]
where \( I_c \) is the following matrix which describes isostatic compensation (first column) and free-slip (second column)
\[ I_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \rho_0(b^2) g(b) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 \pi G a^3/b & 0 \\ 0 & 0 \end{pmatrix}, \quad (B29) \]

and \( D \) consists of two constants of integration
\[ D = (D_1, D_2). \quad (B30) \]
The boundary condition for loading at the Earth surface takes the following form
\[ P_1 \hat{y}(a, s) = b = \begin{pmatrix} -(2 \ell + 1) g/(4 \pi a^2) \\ 0 \end{pmatrix}, \quad (B31) \]
where \( P_1 \) is the projector for the third and forth components. By eliminating the constants of integration \( C \) from eq. (B25) making use of the above boundary conditions, eqs (B28) and (B31), we obtain the analytical solution of the viscoelastic Love numbers \( \hat{k} \) in the Laplace domain
\[ \hat{k}(r, s) = \begin{pmatrix} \hat{h}(r, s) \\ \hat{l}(r, s) \end{pmatrix} = \frac{a}{M_E} \frac{[P_2 \Pi(r, b, s)][P_1 \Pi(a, b, s)]^\dagger b}{\Delta(s)}, \quad (B32) \]
where \( \dagger \) stands for the matrix of minors, \( \hat{h} \) and \( \hat{l} \) are the radial and tangential Love numbers, \( a/M_E \) is the normalizing factor and \( \Delta \) is the secular determinant:
\[ \Delta(s) = \det [P_1 \Pi(a, b, s)]. \quad (B33) \]

with \( P_2 \) being the projector for the first and second components and
\[ \Pi(r, b, s) = Y(r, s) Y^{-1}(b, s) I_c. \quad (B34) \]

**APPENDIX C: ANALYTICAL APPROXIMATIONS**

By definition, the poles of the compositional modes, \( s_{c_n} \), are the roots of the secular determinant
\[ \Delta(s_{c_n}) = 0, \quad (C1) \]
and, from the residue theorem, the residues \( k_{c_n} \) are given by
\[ k_{c_n}(r) = \begin{pmatrix} h_{c_n}(r) \\ l_{c_n}(r) \end{pmatrix} = \lim_{s \rightarrow s_{c_n}} (s - s_{c_n}) \hat{k}(r, s) \]
\[ = \frac{a}{M_E} \frac{[P_2 \Pi(r, b, s)][P_1 \Pi(a, b, s)]^\dagger b}{\partial \Delta(s)} \bigg|_{s = s_{c_n}}. \quad (C2) \]

As discussed in the main text, the origin of the Laplace domain, \( s = 0 \), is the cluster point of the compositional modes. To obtain the approximated analytical expression for the poles, \( s_{c_n} \), and the residues, \( k_{c_n} \), we thus need to investigate the behaviour of the independent solutions, eqs (B10) and (B11), in the limit of \( s \rightarrow 0 \).

We first consider the Taylor series of the exponents \( \beta \) for both scalars \( \mu \) and \( \delta \) at the origin
\[ \delta(s) = -L \epsilon \kappa_0 \delta^4(s). \quad (C3) \]

With respect to this variable, which goes to zero for \( s \rightarrow 0 \), the Taylor series take the following forms:
\[ x_1 = \delta^{-1} - \frac{1}{2} + i F_1 \delta + O(\delta^3). \quad (C4) \]
\[ x_2 = -\delta^{-1} - \frac{1}{2} - F_1 \delta + O(\delta^2), \quad (C5) \]
\[ x_3 = -x_1 - 1, \quad (C6) \]
\[ x_4 = -x_2 - 1, \quad (C7) \]
\[
F_1 = \frac{1 + 4 \left( L + \epsilon \right)}{8}.
\] (C8)

Note that the leading terms of the above Taylor series diverge in the limit for \( \delta \to \infty \). Particularly, \( x_1 \) and \( x_3 \) go to \(-i \infty\) and \(+i \infty\), respectively, and they are responsible for the oscillating pattern of the radial variation of the deformations through the mantle shown in Fig. 4. Instead, \( x_2 \) and \( x_4 \) go to \(-\infty\) and \(+\infty\), respectively.

We then determine the analytical expression of the poles \( s_{C_{\ell}} \) by expanding the secular equation (C1) in Taylor series with respect to the variable \( \delta \). After much straightforward algebra, we thus obtain

\[
s_{C_{\ell}} = -L \frac{\epsilon \kappa_0}{\nu} \left( \log \left( \frac{\epsilon}{L} \right) \right)^4
\]
\[
\left\{ 1 - 2 \log \left( \frac{\epsilon}{L} \right) \frac{[4 F_1 \log \left( \frac{\epsilon}{L} \right) - 3 \epsilon]}{(\pi \nu)^2} \right\} + O(m^{-5}).
\] (C9)

Similarly, from eq. (C2), we obtain the analytical expressions for the radial, \( h_{C_{\ell}} / s_{C_{\ell}} \), and tangential, \( l_{C_{\ell}} / s_{C_{\ell}} \), strengths

\[
h_{C_{\ell}}(r) = \frac{(-1)^n \left( 2 \ell + 1 \right)}{\pi \nu} \frac{a}{r} \left( \sin \theta_n \right)
\]
\[
\left[ \cos \theta_n - \frac{1 + \epsilon \log \left( \frac{\epsilon}{L} \right) - 3 \epsilon \log \left( \frac{\epsilon}{L} \right)}{2 (\pi \nu)^2} \right]
\]
\[
\left[ \log \left( \frac{\epsilon}{L} \right) \frac{(-1)^n e^{-\alpha \theta_n} + e^{-\alpha \theta_n} (1 - 3 \epsilon)}{2 (\pi \nu)^2} \right] + O(m^{-3}),
\] (C10)
\[
\frac{l_{C_{\ell}}(r)}{s_{C_{\ell}}} = \frac{(-1)^n \left( 2 \ell + 1 \right)}{L \log \left( \frac{\epsilon}{L} \right)} \sqrt{\frac{a}{r}} \left( -\cos \theta_n \right)
\]
\[
\left( -\log \left( \frac{\epsilon}{L} \right) \cos \theta_n + \left[ (2 + 3 \epsilon \log \left( \frac{\epsilon}{L} \right) - 3 \epsilon \log \left( \frac{\epsilon}{L} \right) \right] \sin \theta_n \right)
\]
\[
\frac{2}{2 \pi m} \left( \log \left( \frac{\epsilon}{L} \right) \left[ (-1)^n e^{-\alpha \theta_n} + e^{-\alpha \theta_n} (3 \epsilon - 1) \right] \right) + O(m^{-3})
\] (C11)

which yields

\[
x_{C_{\ell}} = \frac{\partial_r h_{C_{\ell}}}{r} + \frac{2}{r} h_{C_{\ell}} - \frac{\ell (\ell + 1)}{r} l_{C_{\ell}},
\] (C14)

Here, we have reported only the first two dominant terms of the Taylor series because of the lengthy expressions of the following terms, although we consider terms up to \( O(m^{-10}) \) for the poles and \( O(m^{-3}) \) for the strengths. Note also that we have taken into account the terms \( e^{-\alpha \theta_n} \) and \( e^{-\alpha \theta_n} \) in eqs (C10) and (C11) and (C15) because they yield 1 for \( r = a \) and \( r = b \), respectively. However, they may be neglected for different radial distances from the Earth radius and the core radius, \( r \neq a, b \), because they go to zero faster than \( m^{-\gamma} \), for \( m \to \infty \) and \( n > 0 \).