DC resistivity FE modelling and inversion in view of a parallelised Multi-EM inversion approach

Julia Weißflog¹, Felix Eckhofer², Ralph-Uwe Börner¹, Michael Eiermann², Oliver G. Ernst², Klaus Spitzer¹

¹Institut für Geophysik und Geoinformatik
²Institut für Numerische Mathematik und Optimierung
TU Bergakademie Freiberg (Germany)

Summary

We present current research associated with a common project of the TU Bergakademie Freiberg and the Deutsches Geoforschungszentrum Potsdam (GFZ). The aim of this project is to exploit the distinct sensitivity patterns of different EM methods to enhance the resolution. Here, we focus on combining the direct current and the transient electromagnetic method. Further methods to be integrated are magnetotellurics and controlled-source EM. For this purpose, we have properly reimplemented our finite element secondary field DC forward modelling approach on unstructured grids and proved its theoretical convergence rate of $O(h^2)$. We derive expressions for the Jacobian with respect to their efficient calculation in an inversion process and examine the spatial sensitivity pattern in order to design a coherent DC/TEM set-up.

Keywords: DC resistivity, finite element method, secondary field approach, sensitivity matrix

1 Introduction

The project Three-dimensional Multi-Scale and Multi-Method Inversion to Determine the Electrical Conductivity Distribution of the Subsurface Using Parallel Computing Architectures (Multi-EM) addresses the combination of different electromagnetic methods in a joint inversion approach to exploit their individual advantages. In Freiberg, we focus on the transient electromagnetic (TEM) and the DC resistivity method.

A prerequisite for any inversion strategy is an efficient forward modelling algorithm as outlined, e.g., by Rücker et al. (2006). Therefore, we have implemented a new DC resistivity forward operator in MATLAB using finite elements on unstructured tetrahedral grids that can easily be combined with our already existing TEM software (Afanasjew et al., 2010). This code enables us to deal with even complex topography and to extract the derivatives, which are crucial for the inversion while retaining full control over the assembly process of the system matrix.

To avoid degradation of the rate of convergence due to singularities in the source terms we deploy a secondary field approach.
For simplicity, we will apply a regularised Gauss-Newton method in view of a combination of different electromagnetic methods in one inversion algorithm.

2 DC Resistivity Modelling

DC resistivity modelling requires the discretisation of the equation of continuity:

$$- \nabla \cdot (\sigma \nabla \phi) = I \delta(x - x_0),$$  

(1)

with $\phi$ as the electric potential, a given distribution of conductivity $\sigma(x)$ (cf. Figure 1) and a point source (electrode) $A$ of strength $I$ located at $x_0 \in \mathbb{R}^3$.

![Figure 1: Used model with the source electrode A located along a borehole at $x = 40$ m](image)

Finite elements enable us to deal with complex structures and permit extraction of the the derivatives with respect to the model parameters from the system matrix $A(\sigma)$ in a straightforward manner. Obtaining all entries of $A(\sigma)$ requires the evaluation of the following integral for each tetrahedron $K$ that belongs to the grid:

$$a_K(\phi_j, \phi_i) = \sigma_K \int_K \nabla \phi_j(x) \cdot \nabla \phi_i(x) \, dx,$$

with the basis functions $\phi_i$ and $\phi_j$ and the conductivity $\sigma_K$ of the tetrahedron $K$.

Within the assembly process, $a_K(\phi_j, \phi_i)$ has to be transformed onto the reference element $\hat{K}$. The transformation reads

$$\xi = B_K^{-1}(x - b_k),$$
with

\[
B_K = \begin{bmatrix}
x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\
y_1 - y_4 & y_2 - y_4 & y_3 - y_4 \\
z_1 - z_4 & z_2 - z_4 & z_3 - z_4
\end{bmatrix}
\quad \text{and} \quad
b_K = \begin{bmatrix}
x_4 \\
y_4 \\
z_4
\end{bmatrix},
\]

which leads to

\[
a_K(\phi_j, \phi_i) = \sigma_K \int_K \left( B_K^{-T} \nabla \hat{\phi}_j(\xi) \right) \cdot \left( B_K^{-T} \nabla \hat{\phi}_i(\xi) \right) |\det B_K| \, d\xi.
\]

### 3 Secondary Field Approach

Singularities in the source terms decrease the convergence rate of the finite element approximation. To avoid this, we apply a secondary field approach. The decomposition of the total potential $\phi$ into primary ($\phi_p$) and secondary potential ($\phi_s$) yields the following equation:

\[
-\nabla \cdot (\sigma \nabla \phi_s) = \nabla \cdot (\sigma_s \nabla \phi_p)
\]

\[
= \nabla \cdot (\sigma \nabla \phi_p) - \nabla \cdot (\sigma_0 \nabla \phi_p)
\]

with an anomalous conductivity $\sigma_s(x) = \sigma(x) - \sigma_0$. The discrete representation of equation (2) reads

\[
A(\sigma) u_s = -A(\sigma) u_p + \sigma_0 A(1) u_p,
\]

where, in addition to $A(\sigma)$, only the matrix $A(1)$ has to be assembled. For multiple sources there is no need to reassemble any matrix. The primary potential is calculated analytically for a homogeneous halfspace:

\[
u_p = u_p(x_i) = \frac{I}{2\pi \sigma_0 ||x_i - x_0||}.
\]

In Table 1 on page 3, we illustrate the efficiency of the secondary field approach for regular grid

<table>
<thead>
<tr>
<th>DOFs</th>
<th>$L_2$-error $e$</th>
<th>$\alpha$ - Rate of Convergence</th>
<th>$L_2$-error $e$</th>
<th>$\alpha$ - Rate of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Total field</td>
<td>Secondary field approach</td>
<td></td>
</tr>
<tr>
<td>427</td>
<td>1.073763e0</td>
<td>-</td>
<td>4.745951e-1</td>
<td>-</td>
</tr>
<tr>
<td>3017</td>
<td>3.570459e-1</td>
<td>1.59</td>
<td>1.322951e-1</td>
<td>1.84</td>
</tr>
<tr>
<td>2268</td>
<td>1.657142e-1</td>
<td>1.11</td>
<td>3.382799e-2</td>
<td>1.97</td>
</tr>
<tr>
<td>17589</td>
<td>1.061691e-1</td>
<td>0.64</td>
<td>8.465897e-3</td>
<td>2.00</td>
</tr>
</tbody>
</table>

Table 1: $L_2$-error and convergence ($e(h) = o(h^\alpha)$) with and without secondary field approach
refinements in terms of an improved convergence rate of an error measure \( e(h) = \mathcal{O}(h^\alpha) \) where \( h \) is the maximum element diameter of the given mesh.

4 Calculating Sensitivities

To apply any inversion strategy we need to calculate the sensitivity or Jacobian matrix \( J = \frac{\partial \mathbf{u}}{\partial \sigma} \) which contains the derivatives of the discrete potential \( \mathbf{u} \) with respect to all entries in \( \sigma \).

Considering the secondary potential, we differentiate (3) with respect to the model parameters using the product rule:

\[
\frac{\partial A(\sigma)}{\partial \sigma_i} \mathbf{u}_s + A(\sigma) \frac{\partial \mathbf{u}_s}{\partial \sigma_i} = \ldots \\
\ldots - \frac{\partial A(\sigma)}{\partial \sigma_i} \mathbf{u}_p - A(\sigma) \frac{\partial \mathbf{u}_p}{\partial \sigma_i} \\
+ \frac{\partial \sigma_0 A(1)}{\partial \sigma_i} \mathbf{u}_p + \sigma_0 A(1) \frac{\partial \mathbf{u}_p}{\partial \sigma_i}.
\]

\[ (5) \]

\( \frac{\partial A(\sigma)}{\partial \sigma} \) is a three-way tensor, where the number of slices equals the number of parameters in \( \sigma \) and slice \( i \) belongs to the derivative with respect to \( \sigma_i \). For example the first slice of \( A(\sigma) \) belongs to the partial derivative with respect to \( \sigma_1 \) (Figure 2).

![Figure 2: Slices of tensor \( \frac{\partial A(\sigma)}{\partial \sigma_1} \)](image)

Depending on whether \( \sigma_i \) is equal to the conductivity \( \sigma_0 \) around the source or not, several terms vanish and the derivatives differ only in an additional term \( \frac{1}{\sigma_0} \mathbf{u}_p \), which is added if \( \sigma_i = \sigma_0 \):

\[
\frac{\partial \mathbf{u}_s}{\partial \sigma_i} = -A(\sigma)^{-1} \left( \frac{\partial A(\sigma)}{\partial \sigma_i} \cdot (\mathbf{u}_p + \mathbf{u}_s) \right) \\
+ \frac{1}{\sigma_0} \mathbf{u}_p.
\]

\[ (6) \]

The derivative of the primary potential \( \mathbf{u}_p \) is equal to zero for \( \sigma_i = \sigma_0 \), otherwise we derive (4) and arrive at

\[
\frac{\partial \mathbf{u}_p}{\partial \sigma_0} = -\frac{1}{\sigma_0} \mathbf{u}_p.
\]

\[ (7) \]

Hence, if we add the sensitivities for the primary and secondary potential, we get the following expression regarding the total potential \( \mathbf{u} \):
\[ J = -A(\sigma)^{-1} \left( \frac{\partial A(\sigma)}{\partial \sigma} \times_2 (u_p + u_s) \right). \]  

(8)

The \( i \)-th (\( i = 1 \ldots n_{\text{parameters}} \)) column of \( J \) contains the sensitivities with respect to the \( i \)-th parameter and the \( k \)-th (\( k = 1 \ldots n_{\text{DOFs}} \)) row contains the sensitivities for the \( k \)-th electrode position M.

5 Numerical Experiments

To show the influence of embedded conductive or resistive bodies in a homogeneous halfspace on the resolution of the pole-pole configuration, we have carried out sensitivity studies. The current source and sink electrode, A respectively, is located at \( x = 40 \) m and \( z = 95 \) m, whereas the potential electrode M is located at \( x = 120 \) m and \( z = 0 \) m.

![Figure 3: Sensitivities for a homogeneous halfspace](image)

For the homogeneous halfspace we see the well-known shape of negative sensitivity between A and M (Spitzer, 1998). In Figure 4, the body causes serious distortions of the sensitivity pattern.

6 Conclusion

We have successfully reproduced the expected sensitivities using our new 3D DC forward operator. First inversions using the reference model (cf. Figure 1) appear to be working well and will be reported in detail later. Preliminary tests using the MATLAB Parallel Computing Toolbox yield promising results regarding future plans for a parallel code. With this re-development we have created a perfect basis to join the MATLAB-based forward modelling codes for both DC and TEM.
Figure 4: Sensitivities for the model from Figure 1

7 Acknowledgements

This project is funded by the German Ministry of Education and Research (BMBF) and the German Research Foundation (DFG) under the Geotechnologien Programme, grant 03G0746A,B.

References

