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Key Points:
- Time scales associated with bounce scattering can be as short as days or even hours
- High-order scattering can significantly contribute to the bounce scattering
- Bounce scattering rates show high sensitivity to the assumed wave normal angle

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Estimation of bounce resonant scattering by fast magnetosonic waves

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Abstract In this study we present calculation of bounce resonant scattering of near equatorially mirroring electrons by fast magnetosonic waves. We first explore the sensitivity of the scattering rates and estimated time scales to a different number of resonances included into calculations. We then explore the sensitivity of calculated rates to the assumed wave normal angle. Our results show that at large radial distances bounce resonant scattering is capable of providing local isotropization of nearly equatorially mirroring electrons on the time scales as short as hours or even days depending on the assumed properties of waves. We also present analysis of the bounce frequencies of electrons and protons at various energies and radial distances and discuss the wave modes that are capable of bounce resonance with these particles. In particular, the importance of bounce resonance as a potential mechanism for the radiation belt remediation is discussed.

1. Introduction

Particles trapped by the Earth's magnetic field undergo three types of periodic motion: gyration around the field line, bounce motion due to the mirror force and slow azimuthal drift due to the curvature and gradient of the magnetic field. Adiabatic theory provides a first approximation of the kinematics of particles in the radiation belts. Adiabatic invariants associated with each type of the periodic motion is approximately conserved as long as the forces in the system and parameters of the system vary on the time scales longer than the time scale associated with the periodic motion. While slow variations of the magnetic fields will result in approximately reversible changes of particle's semiperiodic trajectory in phase space, rapidly changing magnetic field associated with various plasma waves may violate adiabatic invariants and result in a nonreversible change in energy and pitch angle. In recent years violation of the first [e.g., *Horne and Thorne*, 1998; *Summers et al.*, 1998] and third invariants [e.g., *Kellog*, 1959] received much attention and was quantified in a number of studies (see reviews by *Shprits et al.* [2008a, 2008b], *Thorne et al.* [2010], *Millan and Baker* [2012]), while the violation of the second invariant received relatively little attention.

Shprits [2009] considered gyroresonant scattering rates produced by dayside and nightside chorus waves outside of the plasmasphere. The results showed the absence of scattering for the near equatorially mirroring electrons for a range of energies and radial distances. Such gaps in scattering would leave nearly equatorially mirroring electrons unaffected by VLF waves and would produce highly peaked distributions [*Lyons and Williams*, 1975] for a relatively broad range of energies. While measuring particles mirroring very close to the equator is a challenging task, if such distributions were present in the radiation belts, they would have been likely observed by recent missions that spend significant time near the geomagnetic equator. While the pitch angle resolution of the instruments may be higher 15°, the unaffected by scattering nearly equatorially mirroring particles may significantly effect the highest pitch angle bin on the pitch angle distribution at the time when satellite traverses the geomagnetic equator. The goal of this study is to investigate whether bounce resonant scattering may be able to fill such gaps and feed nearly equatorially mirroring electrons to lower pitch angles where these particles will be scattered by VLF waves.

Roberts and Schulz [1968] considered bounce resonance with fast-mode Alfven waves and concluded that these waves may provide a feeding mechanism of equatorially mirroring electrons to the pitch angle scattering by chorus at lower pitch angles [*Roberts and Schulz*, 1968; *Roberts*, 1968; *Schulz and Lanzerotti*, 1974].

However, at that time, detailed knowledge of wave distributions of this and other modes were not available, and the authors were not able to quantify bounce scattering rates based on wave observations. The Orbiting Geophysical Observatory (OGO 3) detected plasma waves that were very closely confined to the equatorial plane with frequencies between the proton gyrofrequency and lower hybrid frequency [*Russell et al.*, 1970]. *Russell et al.* [1970] noticed that these emissions may be in resonance with the harmonics of electron bounce motion and thus may potentially produce electron scattering. *Perraut et al.* [1982] and *Horne et al.* [2000] suggested that ring-like distributions in velocity space (distributions with variable gradient of phase space density with respect to proton velocity) may potentially lead to wave generation through instabilities. Recent study of *Balikhin et al.* [2015] provided definitive experimental evidence for this generation mechanism.

A number of recent statistical studies provided empirical descriptions of the amplitudes and properties of fast magnetosonic waves [e.g., *Santolik et al.*, 2004; *Shprits et al.*, 2013; *Tsurutani et al.*, 2014]. These waves, also often referred to as fast magnetosonic waves or equatorial noise, are one of the most frequently observed emissions in space. These emissions are measured on approximately 60% of satellite equatorial traversals [*Santolik et al.*, 2004].

Shprits [2009] showed that fast magnetosonic waves can be in first-order resonance with radiation belt electrons only at high L shells, while consideration of high-order resonances may allow magnetosonic waves to be in bounce resonance with electrons closer to the Earth. In this study we quantify the scattering by magnetosonic waves show sensitivity simulations and discuss other models that are capable of bounce resonant scattering particles in the inner magnetosphere.

2. Estimation of the Bounce Resonance Scattering Rates

With the assumption that waves move slower in comparison with the bounce motion, the bounce resonance diffusion coefficient D_{xx} can be calculated following [*Schulz and Lanzerotti*, 1974, p. 64]

$$D_{xx} = (y^6/2mMB_{\text{eq}}) \cdot \sum_{l=1}^{l_{\max}} (l/z_l)^2 \cdot J_l^2(z_l)\mathcal{F}_{||}(l\omega_b/2\pi) \tag{1}$$

where we implied the summation over resonance order l from 1 to the highest order for which waves and particles can stay coherent $l_{\max}$, y is the sin of the pitch angle, x is the cos of pitch angle, m is the electron rest mass which is independent of the energy of the particle and should not be confused with the often used relativistic mass, B_{eq} is the equatorial value of the magnetic field on the field line of interest, M is the first adiabatic invariant, J_l is the ordinary Bessel function of the order l, $\mathcal{F}$ is the power spectral density of an external fluctuating force acting on an oscillating particle, ω_b is the bounce angular frequency, argument of the Bessel function z_l is defined as $v_{\|}k_{\|}/\omega_b = pxk_{\|}/(\gamma m\omega_b)$, $k_{\|}$ and $v_{\|}$ are the projection of the wave vector and velocity vector, respectively, on the direction of the magnetic filed line corresponding to the multiple of the bounce frequency $\omega = \omega_b l$, and p is the relativistic momentum. The derivation of equation (1) following *Schulz and Lanzerotti* [1974, p. 64] is given in Appendix A.

For this initial study, we assume the power spectral density following observations of a strong magnetosonic wave event on Cluster satellite [*Horne et al.*, 2007]. Wave spectral density is assumed to be approximated as

$$\mathcal{B}(\omega) \equiv (\mathrm{d}B^2/\mathrm{d}\omega) = CB_w^2 \exp(-((\omega-\omega_m)/\delta\omega)^2) \tag{2}$$

where

$$C = (1/\delta\omega)(2/\sqrt{(\pi)})[\mathrm{erf}((\omega_m-\omega_{lc})/\delta\omega) + \mathrm{erf}((\omega_{uc}-\omega_m)/\delta\omega)]^{(-1)} \tag{3}$$

Following the observations of *Horne et al.* [2007], we assume that $B_w = 200$ pT; $\delta\omega = 30$ rad/s; ω_m=20 rad/s; ω_{lc}=1 rad/s; ω_{uc}=100 rad/s. The choice of parameters and amplitudes is discussed in more detail in section 3.

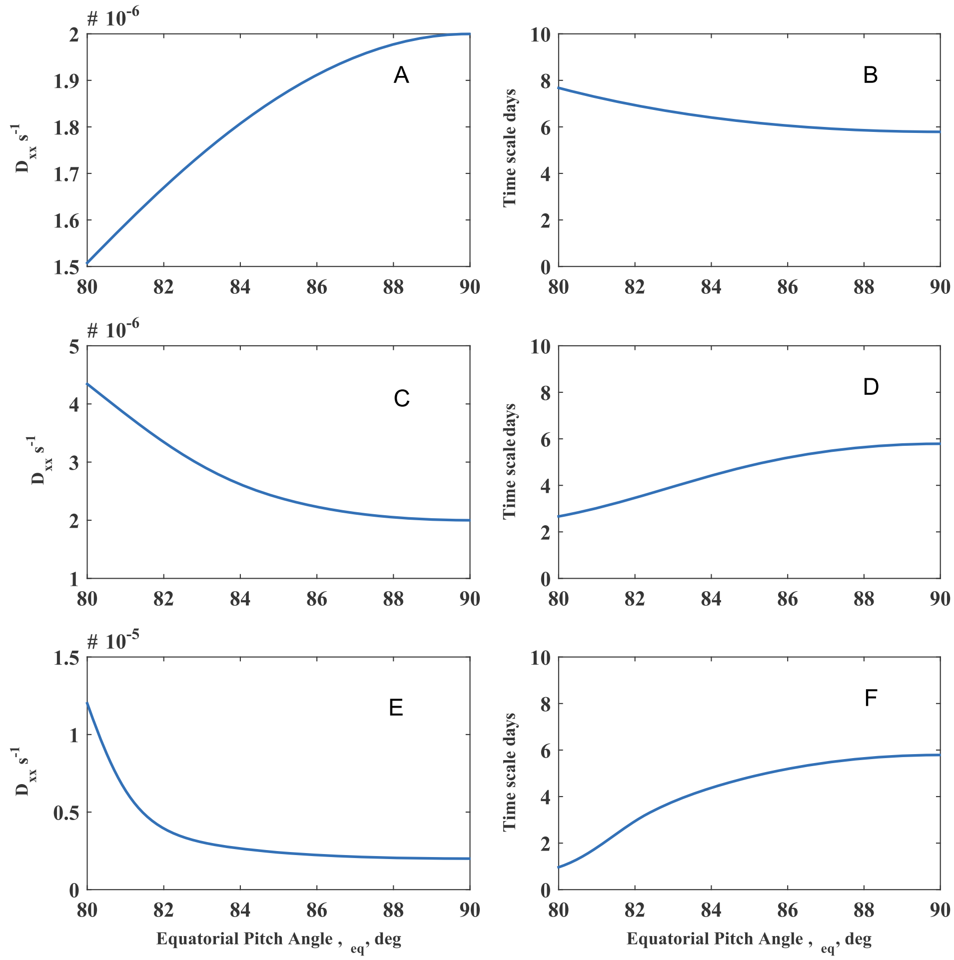

Figure 1. Bounce scattering diffusion coefficients and time scales for (a) $E = 1$ MeV, $\theta = 89°$, $L = 7$, (b) $l_{max} = 1$, (c and d) $l_{max} = 5$, (e and f) $l_{max} = 20$.

The results of scattering by the first-order resonance are shown in Figures 1a and 1b. For the chosen spectrum of wave bounce resonance can result in scattering time scales that are computed as $\tau = D_{xx}^{-1}$ of approximately 6 days. The inclusion of higher resonances (up to the fifth) produces additional scattering (Figures 1c and 1d), which mostly affects pitch angles of approximately 86°. Note that while the diffusion coefficient in equation (1) is explicitly proportional to the square of the order of the resonance, there is also a dependence of the resonance in the order of the Bessel function and in the argument of the Bessel function. Calculation of scattering rates up to the twentieth harmonic resonance (Figures 1e and 1f) show very small increase in scattering around 86°as compared to the case due to the first five harmonics.

The contribution of the higher harmonics decrease with increasing harmonic number and becomes negligible above 100. The high harmonics above 20 can contribute to scattering but are not considered here since such high harmonic interactions are not likely to be physically meaningful for the considered type of wave. Figure 2 shows the sensitivity calculations to the assumed value of the wave normal angle (θ) and energy (E). Decrease in the wave normal angle to 86° increases the parallel component of the mirror force, which significantly increases scattering rates that produce effective scattering on the scale of a day for the near equatorially mirroring electrons (Figures 2c and 2d). The scattering rates for lower energy electrons (E=0.5 MeV; Figures 2e–2h) decrease as compared to the scattering rates at 1 MeV (Figures 2a–2d) but are still on the scale of a day for less oblique waves and on the scale of 5–10 days for the highly oblique waves. Such relatively fast scattering can help fill the gap in gyroresonant scattering [*Shprits*, 2009] for high pitch angles at E=0.5 MeV.

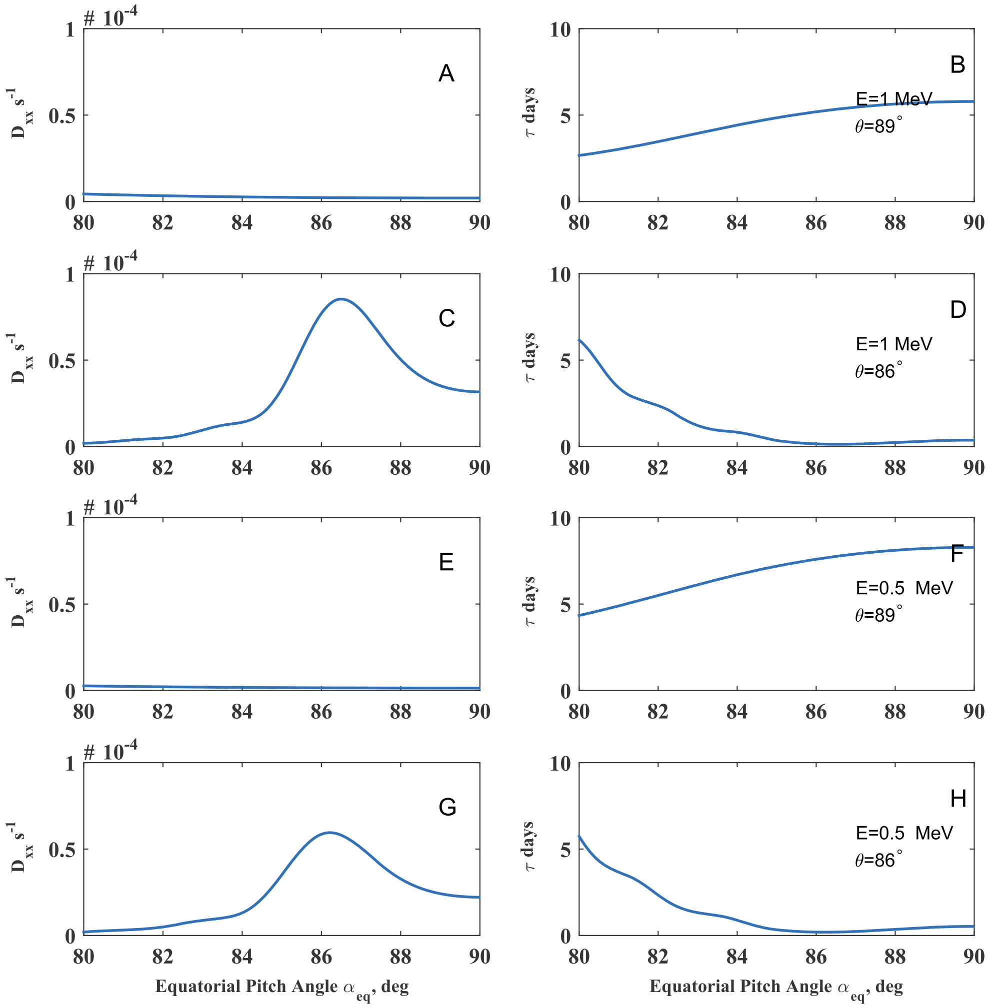

Figure 2. Bounce scattering diffusion coefficients and time scales computed for (a and b) 1 MeV electrons and wave normal angle $\theta = 89°$, (c and d) 1 MeV electrons and wave normal angle $\theta = 86°$, (e and f) 0.5 MeV electrons and wave normal angle $\theta = 89°$, and (g and h) 0.5 MeV electrons and wave normal angle $\theta = 86°$. Resonances up to fifth are included.

3. Implications for Radiation Belt Remediation

The magnetosonic waves are usually present between the ion gyrofrequency and lower hybrid frequency, while electromagnetic ion cyclotron (EMIC) waves are observed below local ion frequencies. The comparison of bounce frequency and fundamental plasma frequencies can show radial distances at which different types of waves can dominate scattering. Frequency analysis shown in Figure 3 indicates that electrons will not be effectively scattered by the bounce resonance with magnetosonic waves at high L shells where magnetosonic waves can be in resonance with low harmonics of bounce motion. As previously noticed by *Shprits* [2009], bounce resonance with EMIC waves may also potentially contribute to scattering at lower L shell. It remains unclear if the waves can stay coherent with particles over the time scales of many bounces. However, the large amplitudes of the observed EMIC waves (on the scale of 1–10 pT) may result in significant scattering on time scales of a few bounced or even less than one bounce which will not be described by the formalism discussed in this study and should require additional particle tracing modeling.

Figure 3 also shows that anthropologically produced EMIC waves in the inner belt may produce scattering of the near 90° particles. Such scattering may be considered for the radiation belt remediation as gyroresonant scattering generated by satellites EMIC waves does not significantly affect near equatorially mirroring electrons [*de Soria-Santacruz et al.*, 2014]. The combined scattering by the bounce and gyroresonance will be a

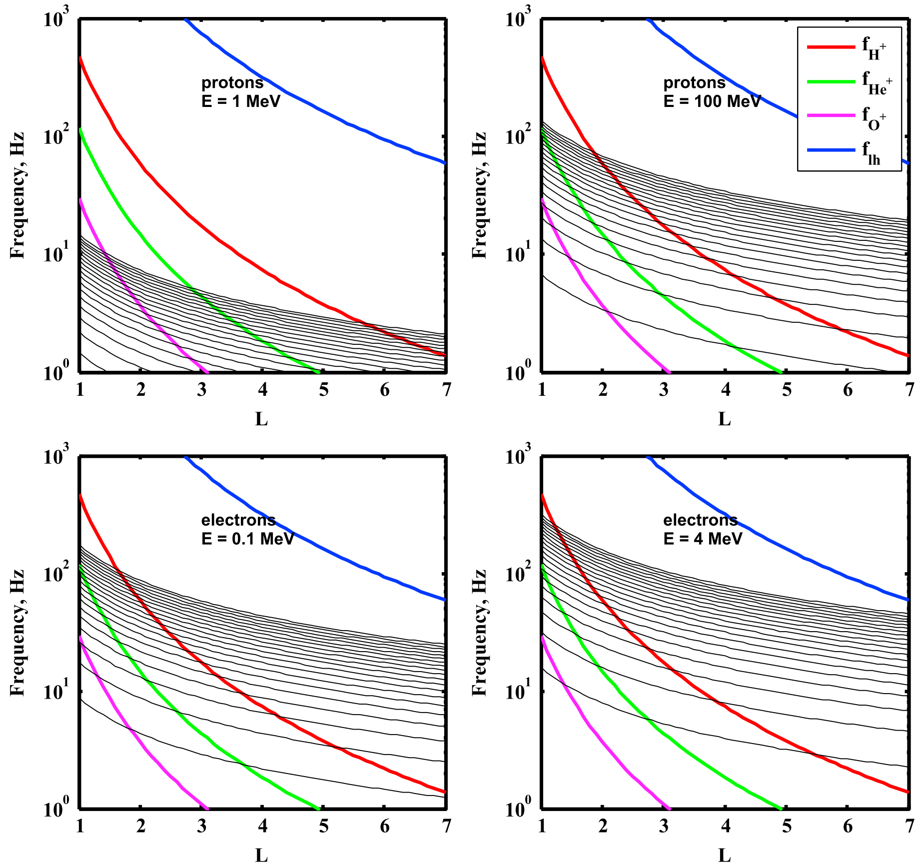

Figure 3. Black lines show harmonics of the electron bounce resonance frequencies as a function of L shell. Colored bold lines show hydrogen, helium, and oxygen gyrofrequencies. Characteristic frequencies and bounce frequencies are shown for (a) 1 MeV protons, (b) 100 MeV protons, (c) 0.1 MeV electrons, and (d) 4 MeV electrons. f_{lh} is the lower hybrid frequency.

subject of future research and may result in the development of a viable method for precipitating particles in the most hazardous for satellites, region of the near-Earth environment.

4. Summary and Conclusions

Bounce resonance scattering by magnetosonic waves strongly depends on the amplitude of waves, spectral distribution of waves, and wave normal distribution. In this study we have assumed waves with relatively high total amplitudes. A recent study of *Tsurutani et al.* [2014] showed that wave amplitudes may reach values up to 10 times higher than what was reported by *Horne et al.* [2007]. For the presented calculations the critical parameter is the power spectral density at the dominant first resonance. If the bounce frequency matches the frequency where power spectral density maximizes, the corresponding diffusion coefficients may be much higher than estimated using the assumed in this study frequency distribution. However, statistically, waves are likely to be weaker [e.g., *Shprits et al.*, 2013] which may result in a lower average rates than estimated using the wave power from *Horne et al.* [2007] that is used in this study and correspond to very disturbed conditions. Electrons will be most efficiently scattered when the bounce resonance is close of the peak of the frequency distribution. If most of the wave power is close to the low harmonics of the bounce resonance, the scattering may be more efficient than presented in this study. While further quantitative studies combining statistical wave properties and calculations will be required to carefully evaluate bounce scattering, the combination of large waves and not the most preferential for scattering distribution of frequency may provide realistic results.

For a given wave amplitude, the scattering is stronger when waves are less oblique. Calculations for a realistic range of values of wave normal angles shows that scattering rates can be on a scale of a few days, which indicates that bounce resonance scattering may assist chorus waves at high L shells. Such local isotropization of the pitch angle distributions at high pitch angles may result in the eventual loss of electrons into the loss cone, or alternatively, transport of electrons locally accelerated at around 80° toward higher pitch angles.

In general, the scattering time scales presented in this study should be treated only as a rough estimate of the scale of isotropization of the nearly equatorially mirroring particles. The inverse diffusion coefficient only provides an indication of how efficient is scattering. The time scale of diffusion of 90° electrons toward the region in pitch angle where gyroresonant scattering by chorus is important will depend on the steepness of the gradient in pitch angle distribution. In the cases when chorus and hiss provide no scattering at high pitch angles even relatively slow bounce scattering may significantly modify the pitch angle distribution.

Several previous studies [*Albert*, 2012; *Ripoll et al.*, 2014] noticed that waves generated by antennas in space will predominantly generate oblique waves that cannot effectively scatter high pitch angle particles during resonance-gyroresonant interactions. Consideration of the bounce resonance by waves may provide an efficient way to scatter higher pitch angle electrons which may significantly enhance the scattering efficiency by waves generated from antennas in space.

Appendix A

This appendix describes in detail derivation of the bounce resonance scattering rates presented in *Schulz and Lanzerotti* [1974].

We consider a motion of a particle in a magnetic field in the presence of a force f which has a component parallel to the magnetic field [*Northrop*, 1963]

$$\frac{\mathrm{d}p_{\|}}{\mathrm{d}t} + \frac{M}{\gamma}\frac{\partial B}{\partial s} = f_{\|}(s,t) \tag{A1}$$

where M is the first adiabatic invariant $M = p^2y^2/2mB = p_{\perp}^2/2mB$, B is the background magnetic field, s is the distance along the field line from the equatorial plane, γ is the relativistic factor, and m is the electron rest mass.

Multiplying the equation above by $p_{\|} = \gamma m v_{\|} = \gamma m \partial s/\partial t$, we obtain

$$p_{\|}\frac{\mathrm{d}p_{\|}}{\mathrm{d}t} + \frac{Mm}{\gamma}\frac{\partial s\gamma}{\partial t}\frac{\partial B}{\partial s} = f_{\|}(s,t)p_{\|} \tag{A2}$$

where m is the electron rest mass.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_{\|}^2}{2} + mMB\right) = f_{\|}(s,t)p_{\|} \tag{A3}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = f_{\|}(s,t)p_{\|}/m \tag{A4}$$

where $w = p^2/(2m)/(2m) = p_{\|}^2/(2m) + p_{\perp}^2/(2m)$

To represent the force $f_{\|}$ as a sum of the Fourier time series, we consider a fixed but relatively long time interval τ.

$$f_{\|}(s,t) = \sum_{n=1}^{\infty} f_n \cos(k_{n\|}s - \omega_n t + \psi_n);\ \omega_n = \frac{2\pi n}{\tau} \tag{A5}$$

$$< |f_{\|}(s,t)|^2 > = \sum_{n=1}^{\infty} \int_0^{2\pi/\omega_n} f_n \cos^2(k_{n\|}s - \omega_n t + \psi_n)\mathrm{d}t = \sum_{n=1}^{\infty} f_n^2/2 \tag{A6}$$

where $k_{n\|}$ is the parallel wave number corresponding to the frequency ω_n.

Contribution of the nth component lies in the angular frequency range of $\Delta\omega = \omega_{n+1} - \omega_n = (2\pi(n+1) - 2\pi n)/\tau = 2\pi/\tau$.

Now we introduce the power spectral density $\mathcal{F}(\omega)$, which gives the power per unit circular frequency, and power spectral density $\mathcal{F}(\nu)$ in terms of frequency $\nu = \omega/2\pi$

$$\mathcal{F}(\omega_n) = (f_n^2/2)/(2\pi/\tau) \tag{A7}$$

$$\mathcal{F}(\nu_n) = (f_n^2\tau/2) \tag{A8}$$

The bounce motion of small oscillations for nearly equatorially mirroring electrons may be represented as a harmonic oscillation, so that the distance s along the field line may be written as follows:

$$s(t) = A\sin(\omega_b t + \phi_2) \tag{A9}$$

$$v_{\|}(t) = A\omega_b\cos(\omega_b t + \phi)\ A = \frac{px}{\gamma m\omega_b} \tag{A10}$$

where x is the cos of the equatorial pitch angle, ω_b is the cyclic frequency associated with the bounce motion, and ϕ is the phase of the bounce motion which will be later considered as a random variable. For the change of energy over the considered time period, we obtain the following:

$$\Delta w = \int_0^\tau f(s,t)(p_{\|}/m)\mathrm{d}t \approx \int_0^\tau \sum_{n=1}^{\infty} f_n\cos(k_n s - \omega_n t + \psi_n)(px/m)\cos(\omega_b t + \phi)\mathrm{d}t \tag{A11}$$

where we used the expression for the harmonic bounce motion. For simplicity we omitted the || subscript for k, and p is the total particle's momentum. If we define $\wr\theta = \omega_b t + \phi\wr$, $\alpha = \frac{pxf_n}{2m}$ and $z_n = k_n\frac{px}{\gamma m\omega_b}$ and use a well-known expression $\exp(iz\sin(\theta)) = \sum_{l=-\infty}^{+\infty} e^{il\theta}J_l(z_n)$, the change in energy may be written as follows:

$$\begin{aligned}\Delta w &\approx \int_0^\tau \sum_{n=1}^{\infty}\frac{pxf_n}{m}\frac{1}{2}(e^{i\theta}+e^{-i\theta})\ Re(\exp(i(k_n s - \omega_n t + \psi_n)))\mathrm{d}t\\ &= \int_0^\tau \sum_{n=1}^{\infty}\alpha Re(e^{i\theta}+e^{-i\theta})(\exp(i[z_n\sin\theta - \omega_n t + \psi_n]))\mathrm{d}t\\ &= \int_0^\tau \sum_{n=1}^{\infty}\alpha Re(e^{i\theta}+e^{-i\theta})e^{-i\omega_n t + i\psi_n}\sum_{l=-\infty}^{+\infty} e^{il\theta}J_l(z_n)\mathrm{d}t\\ &= \int_0^\tau \sum_{n=1}^{\infty}\alpha Re[\sum_{l=-\infty}^{+\infty}(e^{i\theta - i\omega_n t + i\psi_n + il\theta})J_l(z_n) + \sum_{l=-\infty}^{+\infty}(e^{-i\theta - i\omega_n t + i\psi_n + il\theta})J_l(z_n)]\mathrm{d}t\end{aligned} \tag{A12}$$

If we assume that interactions are weak and that scattering over one bounce is small, any time dependence inside the argument of the exponent will lead to random oscillations that will phase mix and will be averaged to zero. Only if there is no time dependence in each of the terms in the equation above, there will be net change in w for the considered ensemble of particles. For the first term that leads to a condition

$$\omega_b - \omega_n + l\omega_b = 0 \tag{A13}$$

$$\omega_n = \omega_b(1+l) = 2\pi n/\tau. \tag{A14}$$

We can choose τ relatively large so that it equals large integer number of bounce periods $\tau = N\tau_b$. By assuming that τ should be an integer of the bounce frequency, we are not restricting our considerations as we already assumed that bounce scattering over one bounce period is negligible.

$$\tau = \tau_b * N = \frac{2\pi N}{\omega_b} \tag{A15}$$

$$\omega_b(l+1) = \frac{2\pi n}{2\pi N}\omega_b \tag{A16}$$

We get that in two summations in the first term of the equation above only terms related as $n = (l+1)N$ will produce nonnegligible contributions to the integral. Similarly, for the second term, the condition becomes $n = (l-1)N$. Thus, only two terms will be left in the summation over n.

$$\Delta w = \sum_{l=-\infty}^{+\infty}\frac{px}{2m}Re[f_{(l+1)N}\exp(i(\phi + l\phi + \psi_{N(l+1)}))J_l(z_{N(l+1)}) \tag{A17}$$

$$+ f_{(l-1)N}\exp(i(-\phi + l\phi + \psi_{N(l-1)}))J_l(z_{N(l-1)})]\tau \tag{A18}$$

Since the summation is from $-\infty$ to $+\infty$, we can shift summations by 1 and substitute $l+1$ to l in the first term and $l-1$ to l in the second term.

$$\Delta w = \sum_{l=-\infty}^{+\infty} \frac{px}{2m} Re[f_{lN} \exp(i(\phi + (l-1)\phi + \psi_{N(l)}))J_{l-1}(z_{Nl}) \tag{A19}$$

$$+ f_{lN} \exp(i(-\phi + (l+1)\phi + \psi_{Nl}))J_{l+1}(z_{Nl})]\tau \tag{A20}$$

$$= \sum_{l=-\infty}^{+\infty} \frac{px}{2m} f_{lN} \cos((l\phi + \psi_{Nl}))[J_{l-1}(z_{Nl}) + J_{l+1}(z_{Nl})]\tau \tag{A21}$$

By noting that $[J_{l-1}(z_{lN}) + J_{l+1}(z_{lN})] = 2lJ_l(z_{lN})/z_{lN}$, we obtain

$$\Delta w = \sum_{l=-\infty}^{+\infty} \frac{px}{2m} f_{lN} \cos((l\phi) + \psi_{lN(l1)})(2lJ_l(z_{lN})/z_{lN})\tau \tag{A22}$$

For any integer k, $\omega_k = \frac{2\pi k}{\tau}$. We have already assumed that $\tau = N\tau_{\text{bounce}}$, where τ_{bounce} is the bounce time.

If $k = N \times l \Rightarrow \omega_{Nl} = \frac{2\pi l N \omega_b}{2\pi N} = l\omega_b$, which means that the frequency in equation (A22) corresponds to multiples of the bounce frequency. Recall that$\mathcal{F}_{\|k}(v) = (f_k^2 \tau)/2$.

As we consider ensemble of particles with different initial conditions and are interested in the diffusive processes on the time scales longer than the bounce motion, the equation for the change of energy needs to be averaged over the phase of the bounce motion and the initial phase.

$$<< \cos^2(l\phi + \psi_{lN}) >> = (1/4\pi^2) \int_0^{2\pi} \int_0^{2\pi} \cos^2(l\phi + \psi_{lN}) \mathrm{d}\phi \mathrm{d}\psi_{lN} = (1/2\pi) \int_0^{2\pi} 1/2 \mathrm{d}\psi_{lN} \tag{A23}$$

The diffusion coefficient may be then written as

$$D_{ww} = \frac{< \Delta w \Delta w >}{2\tau} = \frac{1}{2} (\frac{px}{m})^2 \frac{1}{2} \tau^2 \sum_{l=-\infty}^{+\infty} \frac{2\mathcal{F}_{\|}(\omega_b l/(2\pi))}{\tau^2} \frac{l^2 J_l^2(z_l)}{z_l^2} \tag{A24}$$

$$= (\frac{px}{m})^2 \sum_{l=1}^{+\infty} \mathcal{F}_{\|}(\omega_b l/(2\pi)) l^2 J_l^2(z_l)/z_l^2 \tag{A25}$$

Note that in equation (A24) there is a factor of 1/2 that comes from the averaging of the cos function and in equation (A25) there is a factor of 2 that comes from the change of the limits of the summation.

To convert to a more traditional diffusion coefficient in pitch angle recall that

$$D_{ww} = D_{xx} (\partial w / \partial x)_{M,L}^{-2} \tag{A26}$$

It is convenient to compute the derivative $(\partial w/\partial x)_{M,L}$ for an equatorially mirroring particle

$$(\partial w/\partial x)_{M,L} = \frac{\partial}{\partial x}|_{M,L} \frac{p^2 x^2}{2m} \tag{A27}$$

$$= \frac{\partial}{\partial x}|_{M,L} \frac{p^2 y^2 x^2 B_{\text{eq}}}{2m B_{\text{eq}} y^2}_{M,L} = \frac{\partial}{\partial x}|_{M,L} (M \frac{x^2}{1 - x^2}) \tag{A28}$$

$$= M B_{\text{eq}} \frac{2x}{y^4} \tag{A29}$$

Substituting equation (A29) into equation (A26), we obtain equation (2.34) from [*Schulz and Lanzerotti*, 1974, p. 64]

$$D_{xx} = (y^6 / 2mMB_{\text{eq}}) \cdot (l/z_l)^2 \cdot J_l^2(z_l) \mathcal{F}_{||}(l\omega_b/2\pi) \tag{A30}$$

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