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Time-domain approach to linearized rotational response of a three-dimensional viscoelastic earth model induced by glacial-isostatic adjustment: I. Inertia-tensor perturbations

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SUMMARY

For a spherically symmetric viscoelastic earth model, the movement of the rotation vector due to surface and internal mass redistribution during the Pleistocene glaciation cycle has conventionally been computed in the Laplace-transform domain. The method involves multiplication of the Laplace transforms of the second-degree surface-load and tidal-load Love numbers with the time evolution of the surface load followed by inverse Laplace transformation into the time domain. The recently developed spectral finite-element method solves the field equations governing glacial-isostatic adjustment (GIA) directly in the time domain and, thus, eliminates the need of applying the Laplace-domain method. The new method offers the possibility to model the GIA-induced rotational response of the Earth by time integration of the linearized Liouville equation. The theory presented here derives the temporal perturbation of the inertia tensor, required to be specified in the Liouville equation, from time variations of the second-degree gravitational-potential coefficients by the MacCullagh's formulae. This extends the conventional approach based on the second-degree load Love numbers to general 3-D viscoelastic earth models. The verification of the theory of the GIA-induced rotational response of the Earth is performed by using two alternative approaches of computing the perturbation of the inertia tensor: a direct numerical integration and the Laplace-domain method. The time-domain solution of both the GIA and the induced rotational response of the Earth is readily combined with a time-domain solution of the sea level equation with a time-varying shoreline geometry. In a follow-up paper, we derive the theory for the case when GIA-induced perturbations in the centrifugal force affect not only the distribution of sea water, but also deformations and gravitational-potential perturbations of the Earth.

Key words: Earth's rotation, glacial-isostatic adjustment, Love numbers, second-degree geopotential coefficients, tensor spherical harmonics.

1 INTRODUCTION

Changes in climate influence the distribution of ice and water over the Earth's surface, which, in turn influence the climate itself. Ice accumulation or ablation followed by changes in sea level induce glacial-isostatic adjustment (GIA) of the solid Earth. Conversely, the solid-Earth deformation influences a rise and fall of sea level. Moreover, the redistribution of ice and water and changes in the mass distribution in the Earth's interior are capable to induce perturbations in the rotation of the Earth, both in direction and magnitude of the rotation vector. A wander of the rotation axis, in turn, induces variations in the centrifugal potential and, subsequently, variations in the sea level. All this means that the determination of sea level variations coupled with polar wander due to changes in ice–water mass load is a complex geophysical and mathematical problem.

A linear viscoelastic model has been employed to model the deformation and perturbations in stresses and gravitational potential of the solid Earth as a function of time after a surface-mass load is applied. The associated system of linearized differential equations and boundary condition has conventionally been solved in the Laplace-transform domain (Peltier 1974; Sabadini *et al.* 1982; Wu & Peltier 1982; Spada *et al.*

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1992; Vermeersen & Sabadini 1997). The method involves the time and spatial convolution of the surface-load Love numbers with a surface-load function. The determination of sea level variations using a time-dependent ocean representation by the Laplace-domain method usually requires complicated numerical implementation. This is because the computations of the spatial convolution of the surface-load Love numbers with the applied surface load for a large number of time steps is numerically time consuming.

The forward modelling of GIA of a spherical earth's model with 2-D and 3-D structure of mantle viscosity has recently being under active development. The *3-D finite-element method* (e.g. Gasperini & Sabadini 1989, 1990; Gasperini *et al.* 1990; Giunchi & Spada 2000; Kaufmann *et al.* 1997; Kaufmann & Wu 1998; Kaufmann *et al.* 2000; Wu *et al.* 1998; Wu 2004; Zhong *et al.* 2003; Forno *et al.* 2005) obtains the viscoelastic response of a linear or non-linear viscoelastic earth model with arbitrary 2-D and 3-D viscosity structure applying the finite-element method. The *perturbation approach in Cartesian geometry* (Kaufmann & Wolf 1999) models the lateral variations of mantle material parameters as small perturbations about a zeroth-order radial profile. The *semi-analytical approach* (D'Agostino *et al.* 1997) converts the viscoelastic problem with a 3-D viscosity structure to an iterative series of viscoelastic problems with a 1-D viscosity and with an additional coupling term in the linear momentum equation. The *normal-mode theory* (Tromp & Mitrovica 1999) is based on the eigenfunction-expansion formalism for computing the response of a 3-D, self-gravitating, linear viscoelastic earth model to an arbitrary surface load. The theory makes use of a surface-load representation theorem with the Green's tensor expanded as a series of the eigenfunctions associated with a homogeneous problem. The *spectral finite-difference method* (Martinec 1999) converts the partial differential equations for viscoelastic perturbations to a system of simultaneous ordinary differential equations in the radial variable and solves this system by numerical integration.

The recently developed time-domain, spectral finite-element method (Martinec 2000) solves the field equations governing GIA directly in the time domain and thus eliminates the need of applying the Laplace-domain method. The new method generalizes the initial-value approach by Hanyk *et al.* (1995, 1996) by implementing the sea-level equation and modelling the redistribution of glacial melt water in the oceans and the movement of the coastlines (Hagedoorn *et al.* 2003). In addition, the new method computes the rotational response of the Earth to surface-mass loading by direct numerical time integration of the linearized Liouville equation. The method of computing the time perturbation of the inertia tensor and the relative angular-momentum vector for a 3-D viscoelastic earth model, required to be specified in this equation, will be the subject of this paper.

The theory of the rotational response to surface loading has been refined several times over the past two and half decades and the literature dealing with this subject is quite extensive, for example, Nakiboglu & Lambeck (1980, 1981), Sabadini & Peltier (1981), Sabadini *et al.* (1982), Wu & Peltier (1984), Spada *et al.* (1992), Ricard *et al.* (1993), Vermeersen & Sabadini (1996), Milne & Mitrovica (1998), Mitrovica & Milne (1998), Johnston & Lambeck (1999), Vermeersen & Sabadini (1999), Nakada (2000) and Sabadini & Vermeersen (2002). The rotational theory in all these publications is based on the solution of the linearized Liouville equation in the Laplace-transform domain and the analysis of the normal modes of a self-gravitating, spherically symmetric earth model with a Maxwell-viscoelastic rheology. Differences exist in the way of solving the differential equations, whether the model is compressible or incompressible, whether it is forced by a surface load or by an internal load, or in the number of layers used to discretize the Earth. The common feature of all these solutions is that the inertia tensor associated with the surface-mass load and the inertia tensor associated with the mass redistribution in the Earth induced by the surface load are related by the second-degree surface-load Love number, either by multiplication in the Laplace-transform domain or by convolution in the time domain.

In this paper, we allow that the viscoelastic properties of the Earth vary in both radial and lateral directions. For such a generalized viscoelastic earth model, the concept of the Love numbers must be either generalized (e.g. Martinec 1992) or replaced by a concept based on the viscoelastic field variables. We will choose the latter approach and develop a theory, which enables us to express changes in the inertia tensor induced by a surface-mass load in terms of changes in the gravitational potential. This concept is fully compatible with the time-domain, spectral finite-element method used for GIA modelling and can easily be coupled with it. The main objective of the paper is to present the theory of the linearized rotational response to surface-mass loads in a transparent way.

The paper will proceed as follows. We begin by presenting the theory of the linearized response of a rotating deformable earth model to surface-mass changes (Section 2). This is followed by the description of the method used to determine the inertia-tensor perturbations from external gravitational-potential changes (Section 3). After this, we introduce two approximations conventionally used in GIA modelling to compute the inertia-tensor perturbations (Section 4). We then apply the theory of spherical harmonics to represent inertia tensors in terms of tensor spherical harmonics (Section 5). Finally, we verify the approach presented in Section 3 by a direct numerical integration (Section 6).

2 LINEARIZED RESPONSE OF A ROTATING DEFORMABLE EARTH TO SURFACE-MASS LOAD CHANGES

2.1 Reference and instantaneous inertia tensors

We consider the Earth as a self-gravitating deformable body, which is composed of a fluid core and a viscoelastic solid mantle. Let the Earth be in mechanical equilibrium at the time $t = 0$ and rotate about its centre of mass O with the uniform angular velocity $\vec{\Omega}_0$ (see Fig. 1). We will use this initial equilibrium configuration κ_0 as the reference configuration for the description of the rotational motion of the deformed Earth. We choose a Cartesian coordinate system $O(x_1, x_2, x_3)$ co-rotating with the Earth, such that the coordinate axes x_1, x_2, x_3 coincide with the principal axes of inertia of the configuration κ_0. Let A, B and C $(A = B < C)$ be the corresponding principal moments of inertia.

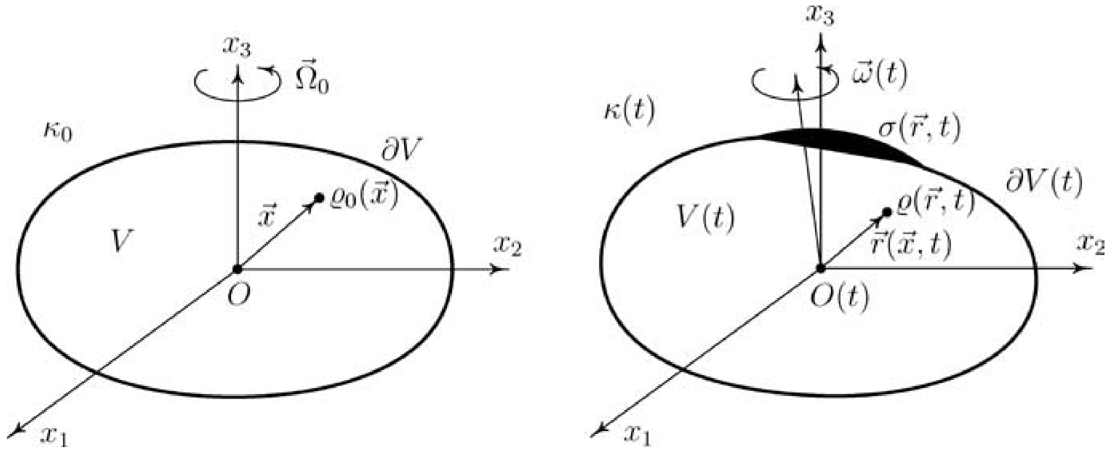

Figure 1. The initial equilibrium configuration κ_0 and the instantaneous configuration $\kappa(t)$.

We suppose that the axis of uniform rotation of the equilibrium configuration κ_0 coincides with the axis of the largest principal moment of inertia, so that $\vec{\Omega}_0 = \Omega_0\,\vec{e}_3$, where $\vec{e}_3$ is the Cartesian unit base vector in the x_3 direction. A material particle in κ_0 is assigned by its position vector $\vec{x}$, measured in the uniformly rotating reference frame $O(x_1, x_2, x_3)$. Let V be the volume of the configuration κ_0 and $\varrho_0(\vec{x})$ be the volume–mass density in V. The inertia tensor of the configuration κ_0, as viewed in the co-rotating frame $O(x_1, x_2, x_3)$, may be written as the volume integral

$$\boldsymbol{C}_0 = \int_V \varrho_0(\vec{x})[(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}]\,dV(\vec{x}), \tag{1}$$

where the dot and cross denote the scalar and dyadic product of vectors, respectively, and $\boldsymbol{I}$ is the second-order identity tensor.

Let a time- and space-dependent force be applied to the Earth at $t > 0$ and deform the initial configuration κ_0 into the time-dependent instantaneous configuration $\kappa(t)$. Let $O(t)$ be the centre of mass of the configuration $\kappa(t)$. We will require that the position of $O(t)$ with respect to the rotating frame $O(x_1, x_2, x_3)$ does not change in time and coincides with the centre of mass O of the configuration κ_0, that is $O(t) \equiv O$ at any time $t > 0$. Let $\vec{\omega}(t)$ denote the instantaneous angular velocity of the configuration $\kappa(t)$, as viewed in the rotating frame $O(x_1, x_2, x_3)$. This rotating frame will also be used to describe the position of particles in the configuration $\kappa(t)$. By $\vec{r}(\vec{x}, t)$, we denote the instantaneous position of a particle in the configuration $\kappa(t)$ initially located at the position $\vec{x}$ in the configuration κ_0. Furthermore, we denote the instantaneous volume of $\kappa(t)$ by $V(t)$ and the volume–mass density in $V(t)$ by $\varrho(\vec{r}, t)$. The instantaneous inertia tensor $\boldsymbol{C}^R(t)$ of the configuration $\kappa(t)$, as viewed in the rotating frame $O(x_1, x_2, x_3)$, can be written as

$$\boldsymbol{C}^R(t) = \int_{V(t)} \varrho(\vec{r}, t)[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\,dV(\vec{r}, t), \tag{2}$$

where the superscript R stands for ‘Response’. Taking into account the relation between the volume element $dV(\vec{r}, t)$ of the instantaneous volume $V(t)$ and the volume element $dV(\vec{x})$ of the initial volume V (e.g. Eringen 1980),

$$dV(\vec{r}(\vec{x}, t), t) = J(\vec{x}, t)\,dV(\vec{x}), \tag{3}$$

and the principle of mass conservation expressed in Lagrangian variables,

$$\varrho_0(\vec{x}) = J(\vec{x}, t)\varrho(\vec{r}(\vec{x}, t), t), \tag{4}$$

where $J(\vec{x}, t)$ is the Jacobian of the transformation between the instantaneous configuration $\kappa(t)$ and the initial configuration κ_0, the Eulerian variables in $\boldsymbol{C}^R(t)$ can be changed to Lagrangian variables as

$$\boldsymbol{C}^R(t) = \int_V \varrho_0(\vec{x})[(\vec{r}(\vec{x}, t)\cdot\vec{r}(\vec{x}, t))\boldsymbol{I} - \vec{r}(\vec{x}, t)\otimes\vec{r}(\vec{x}, t)\,]dV(\vec{x}). \tag{5}$$

2.2 Surface-mass load redistribution

We assume that the deformation of the Earth is induced by the redistribution of surface masses on the external boundary $\partial V(t)$ of $V(t)$ (Fig. 1). The instantaneous inertia tensor associated with a time-varying surface-mass load is

$$\boldsymbol{c}^L(t) = \int_{\partial V(t)} \sigma(\vec{r}, t)[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]dS(\vec{r}, t), \tag{6}$$

where $\sigma(\vec{r}, t)$, $\vec{r} \in \partial V(t)$, is the surface-mass density of the load at the time $t > 0$, $dS(\vec{r}, t)$ is the surface element of $\partial V(t)$ and the superscript L stands for ‘Load’. We will not formulate the principle of local mass conservation for $\sigma(\vec{r}, t)$, similar to the principle (4) for $\varrho(\vec{r}, t)$, since many factors, for example, the topography and bathymetry of the Earth, the deformation of ocean floor or time undulations of gravity field, influence this principle. To change Eulerian variables to Lagrangian variables in the integral on the right-hand side of eq. (6), we introduce, in addition to the surface-mass density $\sigma(\vec{r},t)$ as a measure of mass per deformed unit area $dS(\vec{r},t)$, the surface-mass density $\sigma^L(\vec{x},t)$ as a measure of mass per undeformed unit area $dS(\vec{x})$:

$$\sigma(\vec{r},t)\,dS(\vec{r},t) = \sigma^L(\vec{x},t)\,dS(\vec{x}). \tag{7}$$

Moreover, taking into account the relation between the surface element $dS(\vec{r},t)$ of the surface $\partial V(t)$ and the surface element $dS(\vec{x})$ of the surface ∂V (e.g. Eringen 1980),

$$dS(\vec{r}(\vec{x},t),t) = J(\vec{x},t)\sqrt{\vec{n}(\vec{x})\cdot \boldsymbol{B}(\vec{x},t)\cdot \vec{n}(\vec{x})}\,dS(\vec{x}), \tag{8}$$

where $\vec{n}(\vec{x})$ is the outward unit normal with respect to ∂V, $\boldsymbol{B}(\vec{x})$ is the Piola deformation tensor,

$$\boldsymbol{B}(\vec{x},t) := \boldsymbol{F}^{-1}\cdot(\boldsymbol{F}^{-1})^T, \tag{9}$$

and $\boldsymbol{F}^{-1}$ is the inverse deformation tensor, the Eulerian variables in $\boldsymbol{c}^L(t)$ can be changed to Lagrangian variables by

$$\boldsymbol{c}^L(t) = \int_{\partial V} \sigma^L(\vec{x},t)[(\vec{r}(\vec{x},t)\cdot\vec{r}(\vec{x},t))\boldsymbol{I} - \vec{r}(\vec{x},t)\otimes\vec{r}(\vec{x},t)]\,dS(\vec{x}). \tag{10}$$

The deformation induced by the redistribution of surface masses also causes a shift of the rotation axis from its equilibrium position (x_3 axis in Fig. 1). This, in turn, generates a perturbation of the centrifugal force causing an additional deformation of the Earth, called the rotational deformation (Munk & MacDonald 1960, Section 5.2; Moritz & Mueller 1987, Section 3.2). This type of deformation for a 3-D viscoelastic earth will be treated separately in the follow-up paper. Here, we only consider the effect of time-varying rotation on the redistribution of sea water.

2.3 Geometrical linearization

We now assume that the deformation of the Earth, caused by surface-mass load redistribution, is infinitesimal and described by displacement $\vec{u}(\vec{x},t)$ of the particle $\vec{x}$ from its equilibrium position. The instantaneous position $\vec{r}(\vec{x},t)$ of the particle $\vec{x}$ is then given by

$$\vec{r}(\vec{x},t) = \vec{x} + \vec{u}(\vec{x},t). \tag{11}$$

We will use a linearized Lagrangian description of the deformation both in the fluid core and in the viscoelastic mantle since a purely static (zero-frequency) deformation (Dahlen 1974) is not considered here. Because of the assumption of infinitesimal deformation, $\vec{u}$ is a small quantity and we can apply the principle of geometrical linearization to the instantaneous inertia tensor $\boldsymbol{C}^R(t)$ and express it as the sum of the initial inertia tensor $\boldsymbol{C}_0$ and a small time-dependent increment. In view of the transformation (11), the tensor integral kernel in eq. (5) can be linearized as

$$(\vec{r}(\vec{x},t)\cdot\vec{r}(\vec{x},t))\boldsymbol{I} - \vec{r}(\vec{x},t)\otimes\vec{r}(\vec{x},t) = (\vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{u})\boldsymbol{I} - \vec{x}\otimes\vec{x} - \vec{x}\otimes\vec{u} - \vec{u}\otimes\vec{x}, \tag{12}$$

which is correct to first order in $\|\vec{u}\|$. Substituting eq. (12) into eq. (5) leads to the first-order decomposition of $\boldsymbol{C}^R(t)$:

$$\boldsymbol{C}^R(t) = \boldsymbol{C}_0 + \boldsymbol{c}^R(t), \tag{13}$$

where the inertia-tensor increment $\boldsymbol{c}^R(t)$ can be expressed in the form

$$\boldsymbol{c}^R(t) := \int_V \varrho_0(\vec{x})\big[2\big(\vec{x}\cdot\vec{u}(\vec{x},t)\big)\boldsymbol{I} - \vec{x}\otimes\vec{u}(\vec{x},t) - \vec{u}(\vec{x},t)\otimes\vec{x}\big]\,dV(\vec{x}). \tag{14}$$

To approximate the expression for the inertia tensor $\boldsymbol{c}^L(t)$, the magnitude of the surface-mass load must be specified. Because of the isostatic principle, the induced internal mass redistribution $\varrho_0\|\vec{u}\|$ and the applied surface load σ^L are of the same order in magnitude:

$$\sigma^L = O(\varrho_0\|\vec{u}\|). \tag{15}$$

This allows us to approximate $\boldsymbol{c}^L(t)$ with the same accuracy as that applied in the linearization of $\boldsymbol{C}^R(t)$. Substituting eq. (12) into eq. (10) and taking into account eq. (15), the inertia tensor $\boldsymbol{c}^L(t)$, correct to the first order in $\|\vec{u}\|$, can be expressed as

$$\boldsymbol{c}^L(t) = \int_{\partial V} \sigma^L(\vec{x},t)[(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}\,]\,dS(\vec{x}). \tag{16}$$

In summary, we consider that the perturbations of the initial inertia tensor $\boldsymbol{C}_0$ are associated with the surface-mass load redistribution on the external boundary $\partial V(t)$ and with the internal mass redistribution in $V(t)$ induced by a time-varying surface load. The total increment $\boldsymbol{c}(t)$ of the inertial tensor $\boldsymbol{C}_0$ is then expressed as the sum of two constituents:

$$\boxed{\boldsymbol{c}(t) = \boldsymbol{c}^L(t) + \boldsymbol{c}^R(t).} \tag{17}$$

2.4 Rigid-body translation and rotation

The deformation field $\vec{u}(\vec{x}, t)$ resulting from solving boundary-value problems of gravito-viscoelastodynamics is determined uniquely up to a rigid-body translation and rotation. In general, there are six rigid-body degrees of freedom of the configuration $\kappa(t)$. We will now specify them.

The position of the centre of mass O of the initial configuration κ_0, as viewed in the rotating frame $O(x_1, x_2, x_3)$, may be written as the volume integral

$$\vec{p}_0 = \frac{1}{M}\int_V \varrho_0(\vec{x})\vec{x}\,dV(\vec{x}), \tag{18}$$

where M is the mass of the configuration κ_0. In the instantaneous configuration $\kappa(t)$, the centre of mass $O(t)$ is located at the position

$$\vec{p}(t) = \frac{1}{M(t)}\left[\int_{V(t)} \varrho(\vec{r}, t)\vec{r}\,dV(\vec{r}, t) + \int_{\partial V(t)} \sigma(\vec{r}, t)\vec{r}\,dS(\vec{r}, t)\right], \tag{19}$$

where $M(t)$ is the mass of the configuration $\kappa(t)$. Applying the principle of mass conservation (4) and introducing the surface-mass density $\sigma^L(\vec{x}, t)$ of the surface load by eq. (7), the Eulerian variables in $\vec{p}(t)$ can be changed to Lagrangian variables as

$$\vec{p}(t) = \frac{1}{M}\left[\int_V \varrho_0(\vec{x})\vec{r}(\vec{x}, t)\,dV(\vec{x}) + \int_{\partial V} \sigma^L(\vec{x}, t)\vec{r}(\vec{x}, t)\,dS(\vec{x})\right]. \tag{20}$$

Expressing the instantaneous position $\vec{r}(\vec{x}, t)$ of the material particle $\vec{x}$ in terms of the displacement $\vec{u}(\vec{x}, t)$ according to eq. (11), the vector $\vec{p}(t)$ can be written in the form

$$\vec{p}(t) = \vec{p}_0 + \vec{d}(t), \tag{21}$$

where $\vec{d}(t)$ displaces the origin O to the origin $O(t)$:

$$\vec{d}(t) = \frac{1}{M}\left[\int_V \varrho_0(\vec{x})\vec{u}(\vec{x}, t)\,dV(\vec{x}) + \int_{\partial V} \sigma^L(\vec{x}, t)\vec{x}\,dS(\vec{x})\right]. \tag{22}$$

In view of the estimate (15), the second integral on the right-hand side was approximated correct to the first order in $\|\vec{u}\|$. The requirement, raised in Section 2.1, that the position of the centre of mass $O(t)$ for all $t > 0$ must coincide with the position of the centre of mass O at the time $t = 0$ can now be expressed as $\vec{d}(t) = \vec{0}$. This will ensure that there is no rigid-body translation of the configuration $\kappa(t)$ with respect to the rotating frame $O(x_1, x_2, x_3)$. In addition, if we assume that $\vec{p}_0 = \vec{0}$, the rotating frame $O(x_1, x_2, x_3)$ is geocentric at for all t.

The instantaneous angular-momentum vector $\vec{H}(t)$ of the configuration $\kappa(t)$, rotating with the instantaneous angular velocity $\vec{\omega}(t)$ about the centre of mass O is given by (e.g. Munk & MacDonald 1960; Moritz & Mueller 1987)

$$\vec{H}(t) = \boldsymbol{C}(t)\cdot\vec{\omega}(t) + \vec{h}(t), \tag{23}$$

where the inertia tensor $\boldsymbol{C}(t)$ of the configuration $\kappa(t)$ is equal to the sum of the inertia tensor $\boldsymbol{C}_0$ of the configuration κ_0 and the inertia-tensor increment $\boldsymbol{c}(t)$:

$$\boldsymbol{C}(t) = \boldsymbol{C}_0 + \boldsymbol{c}(t). \tag{24}$$

The relative angular-momentum vector $\vec{h}(t)$ is

$$\vec{h}(t) = \int_{V(t)} \varrho(\vec{r}, t)[\vec{r}\times\vec{v}(\vec{r}, t)]\,dV(\vec{r}, t) + \int_{\partial V(t)} \sigma(\vec{r}, t)[\vec{r}\times\vec{v}(\vec{r}, t)]\,dS(\vec{r}, t) \tag{25}$$

or, changing from Eulerian to Lagrangian variables,

$$\vec{h}(t) = \int_V \varrho_0(\vec{x})\left[\vec{r}(\vec{x}, t)\times\frac{d\vec{r}(\vec{x}, t)}{dt}\right]dV(\vec{x}) + \int_{\partial V} \sigma^L(\vec{x}, t)\left[\vec{r}(\vec{x}, t)\times\frac{d\vec{r}(\vec{x}, t)}{dt}\right]dS(\vec{x}). \tag{26}$$

Here $\vec{v}(\vec{r}(\vec{x}, t), t) := d\vec{r}(\vec{x}, t)/dt$ is the Eulerian velocity in the volume $V(t)$ and on the surface $\partial V(t)$, respectively, as measured in the rotating frame $O(x_1, x_2, x_3)$. Because of the assumption of mechanical equilibrium of κ_0, the position of a material particle in κ_0 does not change with respect to the rotating frame $O(x_1, x_2, x_3)$ and $\vec{v}(\vec{r}(\vec{x}, t), t) = d\vec{u}(\vec{x}, t)/dt$ due to eq. (11). Taking into account eq. (15), the linearization of eq. (26) results in

$$\vec{h}(t) = \int_V \varrho_0(\vec{x})\left[\vec{x}\times\frac{d\vec{u}(\vec{x}, t)}{dt}\right]dV(\vec{x}), \tag{27}$$

which is correct to first order in $\|\vec{u}\|$.

We may use the freedom of choosing a rigid-body rotation and constrain the field $d\vec{u}(\vec{x}, t)/dt$ such that the relative angular-momentum vector $\vec{h}(t)$ is equal to a prescribed vector. If, in particular, we put $\vec{h}(t) = \vec{0}$, there is no rigid-body rotation of the configuration $\kappa(t)$ with respect to the rotating frame $O(x_1, x_2, x_3)$ and the geocentric coordinate axes x_1, x_2, x_3 are called the Tisserand axes (Munk & MacDonald 1960, p. 10).

2.5 Linearized Liouville equation

In this section, for the sake of completeness, we will give a brief overview of the Liouville equation, its linearization and a time-domain solution to the linearized Liouville equation. A more detailed treatment of this subject can be found in Munk & MacDonald (1960) or Moritz & Mueller (1987).

If no external torque is applied, the rotational motion of the deformed Earth is governed by the principle of angular-momentum conservation, which results in the well-known Liouville equation:

$$\frac{d}{dt}[\boldsymbol{C}(t)\cdot\vec{\omega}(t)+\vec{h}(t)]+\vec{\omega}(t)\times[\boldsymbol{C}(t)\cdot\vec{\omega}(t)+\vec{h}(t)]=\vec{0}, \tag{28}$$

which applies to the rotating frame $O(x_1, x_2, x_3)$. The instantaneous angular velocity $\vec{\omega}(t)$ can be decomposed into the uniform angular velocity $\vec{\Omega}_0$ and a small perturbation $\Omega_0\vec{m}(t)$:

$$\vec{\omega}(t)=\vec{\Omega}_0+\Omega_0\vec{m}(t). \tag{29}$$

The dimensionless quantities m_1 and m_2 express the deviations of the instantaneous rotation axis from the equilibrium rotation axis and the quantity m_3 the variations in the rotational speed. We consider $\boldsymbol{c}(t)$, $\vec{m}(t)$ and $\vec{h}(t)$ as small quantities whose products will be neglected. Substituting the decompositions (24) and (29) into eq. (28) and retaining linear terms only, we obtain the linearized Liouville equation:

$$\begin{aligned} m+\frac{i}{\sigma_e}\frac{dm}{dt} &= \chi-\frac{i}{\Omega_0}\frac{d\chi}{dt},\\ \frac{dm_3}{dt} &= -\frac{d\chi_3}{dt}, \end{aligned} \tag{30}$$

where the two coupled equations for m_1 and m_2 are expressed in complex notation using

$$\begin{aligned} m(t) &:= m_1(t)+im_2(t), \quad \chi(t):=\chi_1(t)+i\chi_2(t),\\ c(t) &:= c_{13}(t)+ic_{23}(t), \quad h(t):=h_1(t)+ih_2(t) \end{aligned} \tag{31}$$

and $\chi(t)$ and $\chi_3(t)$ are the angular excitation functions defined by

$$\chi(t):=\frac{1}{(C-A)\Omega_0}[\Omega_0 c(t)+h(t)], \tag{32}$$

$$\chi_3(t):=\frac{1}{C\Omega_0}[\Omega_0\, c_{33}(t)+h_3(t)]. \tag{33}$$

The linearized Liouville eq. (30) can be solved by direct time integration. Assuming $\vec{m}(0)=\vec{0}$, the solution at the time $t>0$ is (Moritz & Mueller 1987, Section 5.4.1)

$$m(t)=-\frac{\sigma_e}{\Omega_0}\chi(t)-i\sigma_e\left(1+\frac{\sigma_e}{\Omega_0}\right)\int_0^t\chi(\tau)e^{i\sigma_e(t-\tau)}d\tau, \tag{34}$$

$$m_3(t)=-\chi_3(t), \tag{35}$$

where the parameter $\sigma_e=\Omega_0(C-A)/A$ is the Euler wobble frequency of the rigid Earth. Inspection of eqs (31)–(35) shows that the time evolution of the unknowns $m_i(t)$ is completely determined by the Cartesian components $c_{13}(t)$, $c_{23}(t)$ and $c_{33}(t)$ of the inertia-tensor increment $\boldsymbol{c}(t)$ and by the relative angular-momentum vector $\vec{h}(t)$.

2.6 Removal of the Chandler wobble

The wander of the rotation vector induced by a non-oscillatory long-time redistribution of the surface-mass load consists of periodic oscillations with the Chandler-wobble frequency superimposed on long-time variations. If we require the removal of the Chandler wobbling of the rotation vector from the rotation-response time-series $m(t)$, we can achieve this by moving average filtering of $m(t)$ over the Chandler-wobble period T_e according to

$$\overline{m}(t)=\frac{1}{T_e}\int_{t_1=t-T_e/2}^{t+T_e/2}m(t_1)\,dt_1, \tag{36}$$

where $T_e:=2\pi/\sigma_e$. Substitution for $m(t)$ from eq. (34) yields

$$\overline{m}(t)=-\frac{\sigma_e}{\Omega_0}\overline{\chi}(t)-i\sigma_e\left(1+\frac{\sigma_e}{\Omega_0}\right)\frac{1}{T_e}\int_{t_1=t-T_e/2}^{t+T_e/2}e^{i\sigma_e t_1}\int_{\tau=0}^{t_1}\chi(\tau)e^{-i\sigma_e\tau}d\tau\,dt_1. \tag{37}$$

If only the long-time redistribution of surface masses, for example during GIA, is considered, the angular excitation function $\chi(t)$ may be represented, with a high degree of accuracy, by a linear change over the Chandler-wobble period,

$$\chi(t_1)=\alpha t_1+\beta \quad \text{for } t-T_e/2\le t_1\le t+T_e/2, \tag{38}$$

where α and β are constants. Multiplying $\chi(t)$ by $e^{-i\sigma_e t}$ and integrating over time results in

$$\int_{\tau=0}^{t_1}\chi(\tau)e^{-i\sigma_e\tau}d\tau=\frac{1}{\sigma_e}\left[i\,\chi(t_1)+\frac{\alpha}{\sigma_e}\right]e^{-i\sigma_e t_1}-\frac{1}{\sigma_e}\left(i\beta+\frac{\alpha}{\sigma_e}\right). \tag{39}$$

Multiplying this again by $e^{i\sigma_e t_1}$, integrating over the Chandler-wobble interval and noting that

$$\int_{t_1=t-T_e/2}^{t+T_e/2} e^{i\sigma_e t_1} dt_1 = 0 \tag{40}$$

for $\sigma_e \neq 0$, yields

$$\frac{1}{T_e}\int_{t_1=t-T_e/2}^{t+T_e/2} e^{i\sigma_e t_1}\int_{\tau=0}^{t_1}\chi(\tau)e^{-i\sigma_e\tau}d\tau\, dt_1 = \frac{1}{\sigma_e}\left[i\,\overline{\chi}(t) + \frac{\alpha}{\sigma_e}\right]. \tag{41}$$

Finally, eq. (37) can be recast into

$$\overline{m}(t) = \overline{\chi}(t) - \frac{i\alpha}{\sigma_e}\left(1 + \frac{\sigma_e}{\Omega_0}\right) \tag{42}$$

or, upon substituting for α from eq. (38),

$$\overline{m}(t) = \overline{\chi}(t) - \frac{i}{\sigma_e}\frac{C}{A}\frac{d\chi(t)}{dt}. \tag{43}$$

Let us estimate the size of the second term on the right-hand side for GIA-induced rotational response. For this long-term geophysical process, the magnitude of the function $\chi(t)$ and its time derivative can be estimated by the following values (see the numerical example in Section 6):

$$|\chi(t)| < 3\times 10^{-4}, \qquad \frac{1}{\sigma_e}\left|\frac{d\chi(t)}{dt}\right| < 1\times 10^{-8}. \tag{44}$$

Consequently, the second term on the right-hand side of eq. (43) can be safely neglected, causing an error that is smaller than the error of the spherical approximation discussed in Section 4.2. Finally,

$$\overline{m}(t) = \chi(t), \tag{45}$$

where we have used the fact that the time averaging according to eq. (36) applied to a linear function yields the function itself. Inspecting the first linearized Liouville eq. (30), eq. (45) shows that, for periods much longer than the period of the Chandler wobble, the second term on the left-hand side and the second term on the right-hand side of the linearized Liouville eq. (30), that is, the time-derivative terms, can be dropped. The same has recently been concluded by Vermeersen & Sabadini (1996) and Mitrovica & Milne (1998), who clarified a misunderstanding over the removal of this term from the Liouville equation used by Wu & Peltier (1984).

3 INERTIA-TENSOR PERTURBATIONS FROM EXTERNAL GRAVITATIONAL-POTENTIAL CHANGES

3.1 The Eulerian density increment

Time changes of the volume–mass density in the instantaneous configuration $\kappa(t)$ can be described by the Eulerian increment ϱ^E of the initial mass density (e.g. Wolf 1991; Dahlen & Tromp 1998, Section 3.2.1):

$$\varrho^E := \varrho(\vec{r}, t) - \varrho_0(\vec{x}(\vec{r}, t)). \tag{46}$$

This increment can be expressed in terms of the displacement $\vec{u}(\vec{x}, t)$ by linearizing the mass-conservation law (4):

$$\varrho^E = -\mathrm{div}\,[\varrho_0(\vec{x})\vec{u}(\vec{x}, t)], \tag{47}$$

which is correct to first order in $\|\vec{u}\|$. Since, in linearized theory, it is irrelevant whether the increment ϱ^E is regarded as a function of $\vec{r}$ or $\vec{x}$, we consider, in the following, that ϱ^E depends on the position vector $\vec{x}$, that is $\varrho^E(\vec{x}, t)$.

Let us compute the inertia tensor $\boldsymbol{c}^{\varrho}(t)$ associated with the Eulerian density increment in the initial volume V:

$$\boldsymbol{c}^{\varrho}(t) := \int_V \varrho^E(\vec{x}, t)[(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}\,]\,dV(\vec{x}). \tag{48}$$

To derive an alternative form of $\boldsymbol{c}^{\varrho}(t)$, we consider Green's theorem for a differentiable vector $\vec{v}$ and a differentiable tensor $\boldsymbol{T}$ in the form

$$\int_V(\mathrm{div}\,\vec{v})\boldsymbol{T}\,dV = \int_{\partial V}(\vec{n}\cdot\vec{v}^{\,-})\boldsymbol{T}\,dS - \int_{\Sigma}[(\vec{n}\cdot\vec{v})\boldsymbol{T}]_-^+\,d\Sigma - \int_V(\vec{v}\cdot\mathrm{grad}\,\boldsymbol{T})\,dV, \tag{49}$$

where $\vec{n}$ is the outward unit vector normal to ∂V or to an internal discontinuity Σ, $\vec{v}^{\,-}$ denotes $\vec{v}$ on the interior side of ∂V and the symbol $[f]_-^+$ indicates the jump of quantity f at the discontinuity Σ. Considering ϱ^E in the form (47) and using Green's theorem (49) for $\vec{v} = -\varrho_0(\vec{x})\vec{u}(\vec{x}, t)$ and $\boldsymbol{T} = (\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}$ yields

$$\boldsymbol{c}^{\varrho}(t) = \int_{\Sigma}\sigma^{\Sigma}(\vec{x}, t)[(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}\,]\,d\Sigma(\vec{x}) - \int_{\partial V}\sigma^{\partial V}(\vec{x}, t)[(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}\,]\,dS(\vec{x})$$
$$+ \int_V \varrho_0(\vec{x})\vec{u}(\vec{x}, t)\cdot\mathrm{grad}\,[(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}\,]\,dV(\vec{x}), \tag{50}$$

where the surface-mass densities $\sigma^{\Sigma}(\vec{x},t)$ and $\sigma^{\partial V}(\vec{x},t)$ are defined by

$$\begin{aligned}\sigma^{\Sigma}(\vec{x},t) &:= [\varrho_0(\vec{x})(\vec{n}(\vec{x})\cdot\vec{u}(\vec{x},t))]_-^+ \quad \text{for } \vec{x}\in\Sigma,\\ \sigma^{\partial V}(\vec{x},t) &:= \varrho_0(\vec{x}^-)\big(\vec{n}(\vec{x})\cdot\vec{u}(\vec{x}^-,t)\big) \quad \text{for } \vec{x}\in\partial V\end{aligned} \tag{51}$$

and where $\varrho_0(\vec{x}^-)$ and $\vec{u}(\vec{x}^-,t)$ denote the volume–mass density and the displacement, respectively, on the interior side of ∂V. Using tensor differential identities

$$\begin{aligned}\vec{u}\cdot\text{grad}\,[(\vec{x}\cdot\vec{x})\boldsymbol{I}] &= 2(\vec{x}\cdot\vec{u})\boldsymbol{I},\\ \vec{u}\cdot\text{grad}\,(\vec{x}\otimes\vec{x}) &= \vec{x}\otimes\vec{u}+\vec{u}\otimes\vec{x}\end{aligned} \tag{52}$$

and inspecting eq. (14), the last term on the right-hand side of eq. (50) is found to be equal to $\boldsymbol{c}^R(t)$, and eq. (50) can be rewritten in the form

$$\boxed{\boldsymbol{c}^R(t) = \boldsymbol{c}^{\varrho}(t) - \boldsymbol{c}^{\Sigma}(t) + \boldsymbol{c}^{\partial V}(t).} \tag{53}$$

The inertia tensors $\boldsymbol{c}^{\Sigma}(t)$ and $\boldsymbol{c}^{\partial V}(t)$, accounting for the vertical displacement of the internal discontinuity Σ and the external boundary ∂V, respectively, are given by

$$\boldsymbol{c}^{\Sigma}(t) := \int_{\Sigma}\sigma^{\Sigma}(\vec{x},t)[(\vec{x}\cdot\vec{x})\boldsymbol{I}-\vec{x}\otimes\vec{x}\,]\,d\Sigma(\vec{x}), \tag{54}$$

$$\boldsymbol{c}^{\partial V}(t) := \int_{\partial V}\sigma^{\partial V}(\vec{x},t)[(\vec{x}\cdot\vec{x})\boldsymbol{I}-\vec{x}\otimes\vec{x}\,]\,dS(\vec{x}). \tag{55}$$

In Section 3.4, we will use the decomposition (53) to express $\boldsymbol{c}^R(t)$ by the MacCullagh's formula.

3.2 Alternative forms of inertia tensors

To determine polar motion and changes in the length of day, the Cartesian components $c_{13}(t)$, $c_{23}(t)$ and $c_{33}(t)$ of the inertia-tensor increment $\boldsymbol{c}(t)$ must be specified explicitly. Inspecting eqs (16), (48), (54) and (55), we can see that the inertia tensors $\boldsymbol{c}^L(t)$, $\boldsymbol{c}^{\varrho}(t)$, $\boldsymbol{c}^{\Sigma}(t)$ and $\boldsymbol{c}^{\partial V}(t)$ have the same tensor integration kernel of the form $(\vec{x}\cdot\vec{x})\boldsymbol{I}-\vec{x}\otimes\vec{x}$. The Cartesian components (1, 3), (2, 3) and (3, 3) of this tensor are

$$[(\vec{x}\cdot\vec{x})\boldsymbol{I}-\vec{x}\otimes\vec{x}]_{i3} = \begin{cases}-x_1x_3 & i=1,\\ -x_2x_3 & i=2,\\ x_1^2+x_2^2 & i=3.\end{cases} \tag{56}$$

Introducing spherical coordinates (r,Ω), $\Omega := (\vartheta,\varphi)$, and considering scalar spherical harmonics of degree 2 and orders 1 and 0, $Y_{21}(\Omega) = -\sqrt{15/8\pi}\,\sin\vartheta\,\cos\vartheta\,e^{i\varphi}$ and $Y_{20}(\Omega) = \sqrt{5/16\pi}(3\cos^2\vartheta-1)$, respectively, the products on the right-hand side of eq. (56) can be expressed as

$$\begin{aligned}x_1x_3+ix_2x_3 &= r^2\sin\vartheta\cos\vartheta\,e^{i\varphi} = -2\sqrt{\frac{2\pi}{15}}r^2Y_{21}(\Omega),\\ x_1^2+x_2^2 &= r^2(1-\cos^2\vartheta) = \frac{2}{3}r^2\left[1-2\sqrt{\frac{\pi}{5}}Y_{20}(\Omega)\right].\end{aligned} \tag{57}$$

Substitution of eq. (57) into eq. (16) results in

$$c^L(t) = 2\sqrt{\frac{2\pi}{15}}\int_{\partial V}\sigma^L(\vec{x},t)Y_{21}(\Omega)r^2dS(\vec{x}), \tag{58}$$

$$c_{33}^L(t) = \frac{2}{3}\int_{\partial V}\sigma^L(\vec{x},t)\left[1-2\sqrt{\frac{\pi}{5}}Y_{20}(\Omega)\right]r^2dS(\vec{x}), \tag{59}$$

where $c^L(t) := c_{13}^L(t) + i\,c_{23}^L(t)$. The inertia tensors $\boldsymbol{c}^{\Sigma}(t)$, $\boldsymbol{c}^{\partial V}(t)$ and $\boldsymbol{c}^{\varrho}(t)$ can be expressed in the same forms as eqs. (58) and (59) by replacing the surface-mass density $\sigma^L(\vec{x},t)$ by $\sigma^{\Sigma}(\vec{x},t)$, $\sigma^{\partial V}(\vec{x},t)$ and $\varrho^E(\vec{x},t)$, respectively, and the surface integral over ∂V by the surface integral over Σ for evaluating $\boldsymbol{c}^{\Sigma}(t)$ or by the volume integral over V for evaluating $\boldsymbol{c}^{\varrho}(t)$. In view of the decomposition (53), the following linear combinations are particularly helpful

$$\begin{aligned}c^{\varrho}(t)-c^{\Sigma}(t)+c^{\partial V}(t) = 2\sqrt{\frac{2\pi}{15}}\bigg[&\int_V\varrho^E(\vec{x},t)Y_{21}(\Omega)r^2dV(\vec{x}) - \int_{\Sigma}\sigma^{\Sigma}(\vec{x},t)Y_{21}(\Omega)r^2d\Sigma(\vec{x})\\ &+\int_{\partial V}\sigma^{\partial V}(\vec{x},t)Y_{21}(\Omega)r^2dS(\vec{x})\bigg],\end{aligned} \tag{60}$$

$$\begin{aligned}c_{33}^{\varrho}(t)-c_{33}^{\Sigma}(t)+c_{33}^{\partial V}(t) = \frac{2}{3}\bigg\{&\int_V\varrho^E(\vec{x},t)\left[1-2\sqrt{\frac{\pi}{5}}Y_{20}(\Omega)\right]r^2dV(\vec{x})\\ &-\int_{\Sigma}\sigma^{\Sigma}(\vec{x},t)\left[1-2\sqrt{\frac{\pi}{5}}Y_{20}(\Omega)\right]r^2d\Sigma(\vec{x}) + \int_{\partial V}\sigma^{\partial V}(\vec{x},t)\left[1-2\sqrt{\frac{\pi}{5}}Y_{20}(\Omega)\right]r^2dS(\vec{x})\bigg\}.\end{aligned} \tag{61}$$

It is important to emphasize that a similar rearrangement cannot be carried out for the inertia-tensor increment $\boldsymbol{c}^R(t)$, since, as eq. (14) shows, the integration kernel of $\boldsymbol{c}^R(t)$ differs from $(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}$.

3.3 The Eulerian gravitational-potential increment

For a surface-mass induced deformation, the Eulerian increment ϕ^E of the initial gravitational potential satisfies the following boundary-value problem (e.g. Wu & Peltier 1982):

$$\nabla^2\phi^E = 4\pi G\,\varrho^E \quad \text{in } V - \Sigma, \tag{62}$$

subject to the interface conditions on an internal discontinuity Σ:

$$\left.\begin{aligned} [\phi^E]_-^+ &= 0 \\ [\vec{n}\cdot\operatorname{grad}\phi^E]_-^+ &= -4\pi G\sigma^\Sigma \end{aligned}\right\} \quad \text{on } \Sigma, \tag{63}$$

and the boundary conditions on the external boundary ∂V:

$$\left.\begin{aligned} [\phi^E]_-^+ &= 0 \\ [\vec{n}\cdot\operatorname{grad}\phi^E]_-^+ &= 4\pi G\sigma^{\partial V} + 4\pi G\sigma^L \end{aligned}\right\} \quad \text{on } \partial V. \tag{64}$$

Outside the volume V, the Eulerian density increment ϱ^E vanishes and the incremental gravitational potential is harmonic, $\nabla^2\phi^E = 0$. The solution to the boundary-value problem defined by eqs (62)–(64) is expressed as the sum of the gravitational potential $\phi^{E,L}(\vec{x},t)$ of the surface-mass load and the gravitational potential $\phi^{E,R}(\vec{x},t)$ of internal mass redistribution induced by the surface load:

$$\phi^E(\vec{x},t) = \phi^{E,L}(\vec{x},t) + \phi^{E,R}(\vec{x},t), \tag{65}$$

where the particular terms are expressed as Newton integrals:

$$\phi^{E,L}(\vec{x},t) = -G\int_{\partial V}\frac{\sigma^L(\vec{x}',t)}{L}\,dS(\vec{x}'), \tag{66}$$

$$\phi^{E,R}(\vec{x},t) = -G\int_V\frac{\varrho^E(\vec{x}',t)}{L}\,dV(\vec{x}') + G\int_\Sigma\frac{\sigma^\Sigma(\vec{x}',t)}{L}\,d\Sigma(\vec{x}') - G\int_{\partial V}\frac{\sigma^{\partial V}(\vec{x}',t)}{L}\,dS(\vec{x}'), \tag{67}$$

and L is the distance between the computation point $\vec{x}$ and an integration point $\vec{x}'$,

$$L := \|\vec{x} - \vec{x}'\|. \tag{68}$$

The first term in eq. (67) accounts for the volume–mass density ϱ^E in V, the second term accounts for the surface-mass density σ^Σ due to the normal displacement of the internal discontinuity Σ and the third term accounts for the surface-mass density $\sigma^{\partial V}$ due to the normal displacement of the external boundary ∂V.

3.4 MacCullagh's formulae

We now express the Eulerian gravitational-potential increment ϕ^E outside the volume V as a series of solid spherical harmonics. The expansion of the reciprocal distance in terms of solid spherical harmonics can be found in Kellogg (1929, Section 5.2). For $r > r'$, it holds

$$\frac{1}{L} = \frac{4\pi}{r}\sum_{j=0}^{\infty}\frac{1}{2j+1}\left(\frac{r'}{r}\right)^j\sum_{m=-j}^{j}Y_{jm}(\Omega)Y_{jm}^*(\Omega'), \tag{69}$$

where $Y_{jm}(\Omega)$ is the scalar spherical harmonic of degree j and order m and the asterisk denotes complex conjugation. Substituting this expansion into the Newton integrals (66) and (67), interchanging the order of summation over j and m with integration over r' and Ω', the gravitational-potential increment ϕ^E at a point outside the volume V can be expressed as a series of solid spherical harmonics:

$$\phi^E(\vec{x},t) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\left(\frac{a}{r}\right)^{j+1}\phi_{jm}^E(t)Y_{jm}(\Omega), \tag{70}$$

where $\phi_{jm}^E(t)$ consists of the 'load' and 'response' gravitational-potential coefficients,

$$\phi_{jm}^E(t) = \phi_{jm}^{E,L}(t) + \phi_{jm}^{E,R}(t), \tag{71}$$

with

$$\phi_{jm}^{E,L}(t) = -\frac{4\pi G}{(2j+1)a^{j+1}}\int_{\partial V}\sigma^L(\vec{x},t)Y_{jm}^*(\Omega)r^j\,dS(\vec{x}), \tag{72}$$

$$\begin{aligned}\phi_{jm}^{E,R}(t) = -\frac{4\pi G}{(2j+1)a^{j+1}}\Bigg[&\int_V\varrho^E(\vec{x},t)Y_{jm}^*(\Omega)r^j\,dV(\vec{x}) - \int_\Sigma\sigma^\Sigma(\vec{x},t)Y_{jm}^*(\Omega)r^j\,d\Sigma(\vec{x}) \\ &+ \int_{\partial V}\sigma^{\partial V}(\vec{x},t)Y_{jm}^*(\Omega)r^j\,dS(\vec{x})\Bigg].\end{aligned} \tag{73}$$

Note that we have introduced the mean radius of the Earth a to normalize the potential coefficients. In particular, the second-degree coefficients are

$$\phi_{2m}^{E,L}(t) = -\frac{4\pi G}{5a^3}\int_{\partial V}\sigma^L(\vec{x},t)Y_{2m}^*(\Omega)r^2 dS(\vec{x}), \tag{74}$$

$$\phi_{2m}^{E,R}(t) = -\frac{4\pi G}{5a^3}\left[\int_V \varrho^E(\vec{x},t)Y_{2m}^*(\Omega)r^2 dV(\vec{x}) - \int_\Sigma \sigma^\Sigma(\vec{x},t)Y_{2m}^*(\Omega)r^2 d\Sigma(\vec{x}) + \int_{\partial V}\sigma^{\partial V}(\vec{x},t)Y_{2m}^*(\Omega)r^2 dS(\vec{x})\right]. \tag{75}$$

Combining eqs (58) and (60) with eqs (74) and (75) for order $m = 1$, we obtain

$$c^L(t) = -\sqrt{\frac{5}{6\pi}}\frac{a^3}{G}\left[\phi_{21}^{E,L}(t)\right]^*, \tag{76}$$

$$c^\varrho(t) - c^\Sigma(t) + c^{\partial V}(t) = -\sqrt{\frac{5}{6\pi}}\frac{a^3}{G}\left[\phi_{21}^{E,R}(t)\right]^*. \tag{77}$$

In view of eq. (53), the left-hand side of eq. (77) is equal to $c^R(t)$. Hence, the 'load' inertia tensor and the 'response' inertia tensor can be expressed in the same form:

$$c^{L,R}(t) = -\sqrt{\frac{5}{6\pi}}\frac{a^3}{G}\left[\phi_{21}^{E,\{L,R\}}(t)\right]^*. \tag{78}$$

By summing them, we finally have

$$\boxed{c(t) = -\sqrt{\frac{5}{6\pi}}\frac{a^3}{G}\left[\phi_{21}^{E}(t)\right]^*.} \tag{79}$$

This is the first MacCullagh's formula for the Eulerian gravitational-potential increment ϕ^E, which relates the Cartesian components $c_{13}(t)$ and $c_{23}(t)$ of the inertia-tensor increment $\boldsymbol{c}(t)$ with the gravitational potential-increment coefficient $\phi_{21}^E(t)$.

A similar procedure for degree $j = 2$ and order $m = 0$ results in

$$c_{33}^L(t) = \frac{1}{3}\sqrt{\frac{5}{\pi}}\frac{a^3}{G}\phi_{20}^{E,L}(t) + \frac{2}{3}\int_{\partial V}\sigma^L(\vec{x},t)r^2\, dS(\vec{x}), \tag{80}$$

$$c_{33}^R(t) = \frac{1}{3}\sqrt{\frac{5}{\pi}}\frac{a^3}{G}\phi_{20}^{E}(t) + \frac{2}{3}\left[\int_V \varrho^E(\vec{x},t)r^2 dV(\vec{x}) - \int_\Sigma \sigma^\Sigma(\vec{x},t)r^2 d\Sigma(\vec{x}) + \int_{\partial V}\sigma^{\partial V}(\vec{x},t)r^2 dS(\vec{x})\right]. \tag{81}$$

The integrals on the right-hand side of the last equation can be further rearranged by the following Green's theorem:

$$\int_V (\operatorname{div}\vec{v})r^2 dV = \int_{\partial V}(\vec{n}\cdot\vec{v}^-)r^2 dS - \int_\Sigma [(\vec{n}\cdot\vec{v})]_-^+ r^2 d\Sigma - 2\int_V (\vec{x}\cdot\vec{v})\, dV, \tag{82}$$

where $\vec{v}$ is a differentiable vector function. Applying this theorem to $\vec{v} = -\varrho_0(\vec{x})\vec{u}(\vec{x},t)$ and considering the definitions (47) and (51) of the volume–mass and surface-mass densities, we find

$$\int_V \varrho^E(\vec{x},t)r^2\, dV(\vec{x}) = \int_\Sigma \sigma^\Sigma(\vec{x},t)r^2\, d\Sigma(\vec{x}) - \int_{\partial V}\sigma^{\partial V}(\vec{x},t)r^2\, dS(\vec{x}) + 2\int_V \varrho_0(\vec{x})(\vec{x}\cdot\vec{u}(\vec{x},t))\, dV(\vec{x}). \tag{83}$$

In view of this, eq. (81) has the form

$$c_{33}^R(t) = \frac{1}{3}\sqrt{\frac{5}{\pi}}\frac{a^3}{G}\phi_{20}^{E}(t) + \frac{4}{3}\int_V \varrho_0(\vec{x})(\vec{x}\cdot\vec{u}(\vec{x},t))dV(\vec{x}). \tag{84}$$

The integrals on the right-hand sides of eqs (80) and (84) can be expressed in terms of the trace of the inertia tensors. Applying the trace operator to eqs (16) and (14) and the identities

$$\begin{aligned}\operatorname{Tr}[(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}] &= 2(\vec{x}\cdot\vec{x}),\\ \operatorname{Tr}[2(\vec{x}\cdot\vec{u})\boldsymbol{I} - \vec{x}\otimes\vec{u} - \vec{u}\otimes\vec{x}] &= 4(\vec{x}\cdot\vec{u}),\end{aligned} \tag{85}$$

we obtain

$$\operatorname{Tr}\boldsymbol{c}^L(t) = 2\int_{\partial V}\sigma^L(\vec{x},t)r^2\, dS(\vec{x}), \tag{86}$$

$$\operatorname{Tr}\boldsymbol{c}^R(t) = 4\int_V \varrho_0(\vec{x})(\vec{x}\cdot\vec{u}(\vec{x},t))\, dV(\vec{x}). \tag{87}$$

In view of this, eqs (80) and (84) for the 'load' inertia tensor and the 'response' inertia tensor can be expressed in the same form:

$$c_{33}^{L,R}(t) = \frac{1}{3}\sqrt{\frac{5}{\pi}}\frac{a^3}{G}\phi_{20}^{E,\{L,R\}}(t) + \frac{1}{3}\operatorname{Tr}\boldsymbol{c}^{L,R}(t). \tag{88}$$

By summing them, we finally have

$$\boxed{c_{33}(t) = \frac{1}{3}\sqrt{\frac{5}{\pi}}\frac{a^3}{G}\phi_{20}^E(t) + \frac{1}{3}\mathrm{Tr}\,\boldsymbol{c}(t).} \tag{89}$$

This is the second MacCullagh's formula for the Eulerian gravitational-potential increment ϕ^E, which relates the Cartesian component $c_{33}(t)$ of the inertia-tensor increment $\boldsymbol{c}(t)$ with the gravitational potential-increment coefficient $\phi_{20}^E(t)$ and the trace of $\boldsymbol{c}(t)$.

The MacCullagh's formulae are very convenient for computing the inertia-tensor increment $\boldsymbol{c}(t)$ in GIA modelling. This is because they make use of the Eulerian gravitational-potential increment, which is a field variable computed in GIA studies.

3.5 The Eulerian centrifugal-potential increment

The sea-level response of the Earth induced by GIA loading is governed by the so-called sea-level equation (e.g. Milne & Mitrovica 1998), which contains, among other field variables, the Eulerian gravity-potential increment Φ^E evaluated at the Earth's surface. The increment Φ^E is equal to the sum of the Eulerian gravitational-potential increment Φ^E plus the Eulerian centrifugal-potential increment ψ^E:

$$\Phi^E(\vec{x}, t) := \phi^E(\vec{x}, t) + \psi^E(\vec{x}, t), \tag{90}$$

where ψ^E defines the change of the centrifugal potential due to deformation,

$$\psi^E := \psi(\vec{r}, t) - \psi_0(\vec{x}(\vec{r}, t)). \tag{91}$$

Substituting for the centrifugal potentials for the initial equilibrium configuration κ_0 and the instantaneous configuration $\kappa(t)$, respectively, according to

$$\begin{aligned}\psi_0(\vec{x}) &= -\frac{1}{2}\left[\Omega_0^2(\vec{x}\cdot\vec{x}) - (\vec{\Omega}_0\cdot\vec{x})^2\right],\\ \psi(\vec{r}, t) &= -\frac{1}{2}[\omega^2(\vec{r}\cdot\vec{r}) - (\vec{\omega}\cdot\vec{r})^2]\end{aligned} \tag{92}$$

and making use of eq. (11), the increment ψ^E can be expressed as a function of m_i (e.g. Dahlen 1976; Moritz & Mueller 1987, Section 3.2),

$$\psi^E = \Omega_0^2\left[m_1x_1x_3 + m_2x_2x_3 - m_3\left(x_1^2 + x_2^2\right)\right], \tag{93}$$

which is correct to first order in $\|\vec{m}\|$. As for the Eulerian density increment ϱ^E, the increment ψ^E can be regarded as a function of $\vec{x}$ and t, that is $\psi^E(\vec{x}, t)$. In terms of the zeroth- and second-degree spherical harmonics, eq. (93) can be rewritten as

$$\psi^E(\vec{x}, t) = \left(\frac{r}{a}\right)^2\psi_{00}^E(t)Y_{00}(\Omega) + \left(\frac{r}{a}\right)^2\sum_{m=-1}^{1}\psi_{2m}^E(t)Y_{2m}(\Omega), \tag{94}$$

where

$$\begin{aligned}\psi_{00}^E(t) &= -\frac{2}{3}\sqrt{4\pi}\,\Omega_0^2a^2m_3(t),\\ \psi_{20}^E(t) &= \frac{4}{3}\sqrt{\frac{\pi}{5}}\Omega_0^2a^2m_3(t),\\ \psi_{21}^E(t) &= -\sqrt{\frac{2\pi}{15}}\Omega_0^2a^2[(m_1(t) - im_2(t)],\\ \psi_{2,-1}^E(t) &= -\left[\psi_{21}^E(t)\right]^*.\end{aligned} \tag{95}$$

4 APPROXIMATIONS USED IN GIA

4.1 Static-deformation approximation for a fluid core

So far, we have used the linearized Lagrangian description for the fluid core and the solid mantle. Since the GIA is a long-term process, the effect of the fluid core on the viscoelastic response of the Earth to surface glacial loading can alternatively be viewed in terms of a static deformation. This description results from two assumptions:

(i) the fluid core is in hydrostatic equilibrium in both the initial and the instantaneous configurations and

(ii) core fluid is inviscid. Based on these assumptions, Dahlen (1974) and Crossley & Gubbins (1975) showed that the Eulerian density increment in the fluid core is

$$\varrho^E = \frac{\vec{\nu}\cdot\mathrm{grad}\varrho_0}{\vec{\nu}\cdot\mathrm{grad}\Phi_0}\,\Phi^E, \tag{96}$$

where $\vec{\nu}$ is the unit normal to a level surface of density ϱ_0 and gravity potential Φ_0 in the initial configuration and Φ^E is the Eulerian gravity potential increment defined by eq. (90).

The instantaneous inertia tensor of the fluid core in the configuration $\kappa(t)$, as viewed in the rotating reference frame $O(x_1, x_2, x_3)$, can be written as

$$\boldsymbol{C}^R_{\text{core}}(t) = \int_{V_{\text{core}}(t)} \varrho(\vec{r}, t)[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dV(\vec{r}, t), \tag{97}$$

where $V_{\text{core}}(t)$ is the instantaneous volume of the core. Decomposing $\varrho(\vec{r}, t)$ into initial and incremental parts, that is, $\varrho(\vec{r}, t) = \varrho_0(\vec{r}) + \varrho^E(\vec{r}, t)$, and splitting the integral over the volume $V_{\text{core}}(t)$ for $\varrho_0(\vec{r})$ into the integral over the volume of the core in the initial configuration, V_{core}, and the incremental volume $V_{\text{core}}(t) - V_{\text{core}}$, we have

$$\boldsymbol{C}^R_{\text{core}}(t) = \boldsymbol{C}^R_{0,\text{core}} + \boldsymbol{c}^R_{\text{core}}(t), \tag{98}$$

where $\boldsymbol{C}^R_{0,\text{core}}$ is the inertia tensor of the core in the initial configuration,

$$\boldsymbol{C}^R_{0,\text{core}} = \int_{V_{\text{core}}} \varrho_0(\vec{r})[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]dV(\vec{r}), \tag{99}$$

and the inertia-tensor increment $\boldsymbol{c}^R_{\text{core}}(t)$ is

$$\boldsymbol{c}^R_{\text{core}}(t) = \int_{V_{\text{core}}(t)} \varrho^E(\vec{r}, t)[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dV(\vec{r}, t) + \int_{V_{\text{core}}(t)-V_{\text{core}}} \varrho_0(\vec{r})[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dV(\vec{r}, t). \tag{100}$$

Let ∂V_{core} and $\partial V_{\text{core}}(t)$ denote the core–mantle boundary in the initial and instantaneous configurations, respectively, and let $\vec{n}(\vec{r})\cdot\vec{u}(\vec{r}, t), \vec{r}\in\partial V_{\text{core}}$, be the displacement component along the unit normal $\vec{n}(\vec{r})$ with respect to ∂V_{core} which takes ∂V_{core} to $\partial V_{\text{core}}(t)$. Correct to the first order in $\|\vec{n}\cdot\vec{u}\|$, the integral over the volume $V_{\text{core}}(t) - V_{\text{core}}$ in eq. (100) can be expressed as an integral over the surface ∂V_{core}:

$$\int_{V_{\text{core}}(t)-V_{\text{core}}} \varrho_0(\vec{r})[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dV(\vec{r}, t) = \int_{\partial V_{\text{core}}} \varrho_0(\vec{r}^{\,-})(\vec{n}(\vec{r})\cdot\vec{u}(\vec{r}^{\,-}, t))[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dS(\vec{r}). \tag{101}$$

Moreover, since the Eulerian density increment is considered as a first-order field variable, the integral over the instantaneous volume $V_{\text{core}}(t)$ in eq. (100) can be approximated by the integral over the initial volume: V_{core}

$$\int_{V_{\text{core}}(t)} \varrho^E(\vec{r}, t)[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dV(\vec{r}, t) = \int_{V_{\text{core}}} \varrho^E(\vec{r}, t)[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dV(\vec{r}), \tag{102}$$

which is correct to the first order in $\|\vec{n}\cdot\vec{u}\|$. Finally, not permitting cavitation or overlap between the core and the mantle, that is, $\vec{n}(\vec{r})\cdot\vec{u}(\vec{r}^{\,-}, t) = \vec{n}(\vec{r})\cdot\vec{u}(\vec{r}^{\,+}, t)$ for $\vec{r}\in\partial V_{\text{core}}$, the inertia-tensor increment due to the static deformation of the fluid core is

$$\boldsymbol{c}^R_{\text{core}}(t) = \int_{V_{\text{core}}} \varrho^E(\vec{r}, t)[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dV(\vec{r}) + \int_{\partial V_{\text{core}}} \varrho_0(\vec{r}^{\,-})(\vec{n}(\vec{r})\cdot\vec{u}(\vec{r}^{\,+}, t))[(\vec{r}\cdot\vec{r})\boldsymbol{I} - \vec{r}\otimes\vec{r}\,]\, dS(\vec{r}). \tag{103}$$

4.2 Spherical approximation

Though the MacCullagh's formulae should exclusively be used in computing the inertia-tensor increment $\boldsymbol{c}(t)$, it may be necessary to compute the two constituents of $\boldsymbol{c}(t)$, that is, the inertia tensors $\boldsymbol{c}^L(t)$ and $\boldsymbol{c}^R(t)$, separately. Moreover, to compute $c_{33}(t)$ and to solve the linearized Liouville equation, we also need to express the trace of $\boldsymbol{c}(t)$ and the relative angular-momentum vector $\vec{h}(t)$ in terms of the displacement $\vec{u}(\vec{x}, t)$. All these expressions are simplified if the spherical approximation is applied to $\boldsymbol{c}(t)$ and $\vec{h}(t)$.

Since the Earth's topography deviates from the mean sphere by the order of the Earth's flattening, we can approximate the external boundary ∂V in $\boldsymbol{c}^R(t)$ and $\boldsymbol{c}^L(t)$ by this sphere. The relative error introduced by this spherical approximation is of the order of 3×10^{-3}, which results in an absolute error of the order of 10^{29} kg m^2 in terms of the inertia tensors $\boldsymbol{c}^R(t)$ and $\boldsymbol{c}^L(t)$.

Moreover, a laterally heterogeneous density distribution $\varrho_0(\vec{x})$ in the initial configuration κ_0 requires a non-zero initial deviatoric stress for its support. Because the deviatoric stresses within the Earth are not well known yet, they are commonly omitted in GIA modelling. In addition, the inversion of the long-wavelength geoid (e.g. Čadek & Fleitout 2003) and the results of seismic tomography (e.g. Su *et al.* 1994) combined with laboratory experiments (e.g. Karato & Wu 1993) indicate that lateral heterogeneities in the mass density do not exceed 2 per cent of the spherically symmetric density distribution. Both these facts suggest that the laterally heterogeneous density $\varrho_0(\vec{x})$ in the inertia tensor $\boldsymbol{c}^R(t)$ can be replaced by a spherically symmetric density. The absolute error of this approximation is of the order of 10^{30} kg m^2 viewed in terms of the inertia-tensor increments.

Taken together, the two aspects of the spherical approximation considered result in the following approximate formulae:

$$\boldsymbol{c}^L(t) = \int_{\partial V} \sigma^L(\vec{x}, t)[(\vec{x}\cdot\vec{x})\boldsymbol{I} - \vec{x}\otimes\vec{x}\,]\, dS(\vec{x}), \tag{104}$$

$$\boldsymbol{c}^R(t) = \int_V \varrho_0(r)[2(\vec{x}\cdot\vec{u}(\vec{x}, t))\boldsymbol{I} - \vec{x}\otimes\vec{u}(\vec{x}, t) - \vec{u}(\vec{x}, t)\otimes\vec{x}\ \,]\, dV(\vec{x}), \tag{105}$$

$$\vec{d}(t) = \int_V \varrho_0(r)\vec{u}(\vec{x}, t)dV(\vec{x}) + \int_{\partial V} \sigma^L(\vec{x}, t)\vec{x}\; dS(\vec{x}), \tag{106}$$

$$\vec{h}(t) = \int_V \varrho_0(r)\left[\vec{x} \times \frac{d\vec{u}(\vec{x},t)}{dt}\right] dV(\vec{x}), \tag{107}$$

where V is now the mean Earth's sphere with the spherical external boundary ∂V, $\varrho_0(r)$ is the spherically symmetric density with a jump at the spherical internal discontinuity Σ and r is the radial distance from the origin O. In spherical coordinates (r, Ω), the volume and surface elements are $dV(\vec{x}) = r^2 dr d\Omega$, $dS(\vec{x}) = a^2 d\Omega$, $d\Sigma(\vec{x}) = a_\Sigma^2 d\Omega$, $d\Omega = \sin\vartheta\, d\vartheta\, d\varphi$, and a and a_Σ are the radii of ∂V and Σ, respectively. The spherical approximation of the inertia tensors $\boldsymbol{c}^\Sigma(t)$ and $\boldsymbol{c}^{\partial V}(t)$ can be expressed in the same form as $\boldsymbol{c}^L(t)$ by replacing the surface-mass density $\sigma^L(\vec{x}, t)$ by $\sigma^\Sigma(\vec{x}, t)$ and $\sigma^{\partial V}(\vec{x}, t)$, respectively, and the surface integral over ∂V by the surface integral over Σ for evaluating the tensor $\boldsymbol{c}^\Sigma(t)$. The spherical approximation of the surface-mass densities $\sigma^\Sigma(\vec{x}, t)$ and $\sigma^{\partial V}(\vec{x}, t)$ is

$$\sigma^\Sigma(\vec{x}, t) = [\varrho_0(r)(\vec{e}_r \cdot \vec{u}(\vec{x}, t))]_-^+, \quad \sigma^{\partial V}(\vec{x}, t) = \varrho_0(r^-)(\vec{e}_r \cdot \vec{u}(\vec{x}^-, t)), \tag{108}$$

where $\vec{e}_r$ is the unit vector in the radial direction.

5 SPHERICAL HARMONIC FORMULATION

5.1 Spherical harmonic representation of inertia tensors

Having introduced the spherical approximation, the formalism of vector and tensor spherical harmonics may be applied to the inertia tensors. We represent the Lagrangian displacement vector $\vec{u}(\vec{x}, t)$ in terms of vector spherical harmonic as

$$\vec{u}(\vec{x}, t) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\left[U_{jm}(r,t)\vec{S}_{jm}^{(-1)}(\Omega) + V_{jm}(r,t)\vec{S}_{jm}^{(1)}(\Omega) + W_{jm}(r,t)\vec{S}_{jm}^{(0)}(\Omega)\right], \tag{109}$$

where $\vec{S}_{jm}^{(-1)}(\Omega)$, $\vec{S}_{jm}^{(1)}(\Omega)$ and $\vec{S}_{jm}^{(0)}(\Omega)$ are spheroidal and toroidal vector spherical harmonics, respectively (their definition and basic properties are given in e.g. Martinec 2000, Appendix B). Likewise, the surface-mass distributions are represented in terms of scalar spherical harmonics $Y_{jm}(\Omega)$ as

$$\begin{bmatrix} \sigma^L(\vec{x}, t) \\ \sigma^\Sigma(\vec{x}, t) \\ \sigma^{\partial V}(\vec{x}, t) \end{bmatrix} = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\begin{bmatrix} \sigma_{jm}^L(t) \\ \sigma_{jm}^\Sigma(t) \\ \sigma_{jm}^{\partial V}(t) \end{bmatrix} Y_{jm}(\Omega). \tag{110}$$

Making use of the orthonormality property of scalar spherical harmonics, the expansion coefficients are expressed in the form

$$\begin{bmatrix} \sigma_{jm}^L(t) \\ \sigma_{jm}^\Sigma(t) \\ \sigma_{jm}^{\partial V}(t) \end{bmatrix} = \int_{\Omega_s}\begin{bmatrix} \sigma^L(\vec{x}, t) \\ \sigma^\Sigma(\vec{x}, t) \\ \sigma^{\partial V}(\vec{x}, t) \end{bmatrix} Y_{jm}^*(\Omega)\, d\Omega, \tag{111}$$

where Ω_s is the full solid angle. Substitution of eq. (108) into eq. (111) yields the spherical harmonic coefficients $\sigma_{jm}^\Sigma(t)$ and $\sigma_{jm}^{\partial V}(t)$ of the surface-mass densities $\sigma^\Sigma(\vec{x}, t)$ and $\sigma^{\partial V}(\vec{x}, t)$, respectively, as

$$\sigma_{jm}^\Sigma(t) := [\varrho_0(r)U_{jm}(r,t)]_{a_\Sigma^-}^{a_\Sigma^+}, \quad \sigma_{jm}^{\partial V}(t) := \varrho_0(a^-)U_{jm}(a^-, t). \tag{112}$$

Writing the position vector as $\vec{x} = r\vec{e}_r$ and the identity tensor as $\boldsymbol{I} = \vec{e}_r \otimes \vec{e}_r + \vec{e}_\vartheta \otimes \vec{e}_\vartheta + \vec{e}_\varphi \otimes \vec{e}_\varphi$, where $\vec{e}_r$, $\vec{e}_\vartheta$ and $\vec{e}_\varphi$ are the spherical unit base vectors, the integration kernels of $\boldsymbol{c}^L(t)$ and $\boldsymbol{c}^R(t)$ can be expressed in the forms

$$\sigma^L(\vec{x}, t)[(\vec{x} \cdot \vec{x})\boldsymbol{I} - \vec{x} \otimes \vec{x}\,] = -a^2 \sum_{j=0}^{\infty}\sum_{m=-j}^{j} \sigma_{jm}^L(t)\boldsymbol{Z}_{jm}^{(5)}(\Omega), \tag{113}$$

$$2(\vec{x} \cdot \vec{u}(\vec{x}, t))\boldsymbol{I} - \vec{x} \otimes \vec{u}(\vec{x}, t) - \vec{u}(\vec{x}, t) \otimes \vec{x} = - 2r\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\left[U_{jm}(r,t)\boldsymbol{Z}_{jm}^{(5)}(\Omega) + V_{jm}(r,t)\boldsymbol{Z}_{jm}^{(2)}(\Omega) + W_{jm}(r,t)\boldsymbol{Z}_{jm}^{(3)}(\Omega)\right], \tag{114}$$

where $\boldsymbol{Z}_{jm}^{(\lambda)}(\Omega)$, $\lambda = 2, 3, 5$, are the tensor spherical harmonics defined by

$$\boldsymbol{Z}_{jm}^{(2)}(\Omega) := \frac{\partial Y_{jm}(\Omega)}{\partial\vartheta}\boldsymbol{e}_{r\vartheta} + \frac{1}{\sin\vartheta}\frac{\partial Y_{jm}(\Omega)}{\partial\varphi}\boldsymbol{e}_{r\varphi},$$

$$\boldsymbol{Z}_{jm}^{(3)}(\Omega) := -\frac{1}{\sin\vartheta}\frac{\partial Y_{jm}(\Omega)}{\partial\varphi}\boldsymbol{e}_{r\vartheta} + \frac{\partial Y_{jm}(\Omega)}{\partial\vartheta}\boldsymbol{e}_{r\varphi},$$

$$\boldsymbol{Z}_{jm}^{(5)}(\Omega) := -Y_{jm}(\Omega)\left(\boldsymbol{e}_{\vartheta\vartheta} + \boldsymbol{e}_{\varphi\varphi}\right) \tag{115}$$

and $\boldsymbol{e}_{ij}$, $i, j \in \{r, \vartheta, \varphi\}$, are the symmetric parts of the dyadic products of the spherical unit base vectors. More details on the tensor spherical harmonics $\boldsymbol{Z}_{jm}^{(\lambda)}(\Omega)$ can be found in Martinec (2000, Appendix B). Note that the definition of $\boldsymbol{Z}_{jm}^{(5)}(\Omega)$ used here differs from that used by

Martinec (2000) by factor $j(j+1)$. Straightforward, but extensive mathematical manipulations result in the analytical expressions for the angular integrals over tensor spherical harmonics:

$$\int_{\Omega_s} \boldsymbol{Z}^{(2)}_{jm}(\Omega)\,d\Omega = 6\sqrt{\frac{2\pi}{15}}\delta_{j2}\left[\frac{1}{2}\left(\delta_{m,-2} - \sqrt{\frac{2}{3}}\delta_{m0} + \delta_{m2}\right)\boldsymbol{e}_{11} - \frac{1}{2}\left(\delta_{m,-2} + \sqrt{\frac{2}{3}}\delta_{m0} + \delta_{m2}\right)\boldsymbol{e}_{22} + \sqrt{\frac{2}{3}}\delta_{m0}\boldsymbol{e}_{33} - i(\delta_{m,-2} - \delta_{m2})\boldsymbol{e}_{12} + (\delta_{m,-1} - \delta_{m1})\boldsymbol{e}_{13} - i(\delta_{m,-1} + \delta_{m1})\boldsymbol{e}_{23}\right],$$

$$\int_{\Omega_s} \boldsymbol{Z}^{(3)}_{jm}(\Omega)\,d\Omega = 0,$$

$$\int_{\Omega_s} \boldsymbol{Z}^{(5)}_{jm}(\Omega)\,d\Omega = -\frac{4}{3}\sqrt{\pi}\,\delta_{j0}(\boldsymbol{e}_{11} + \boldsymbol{e}_{22} + \boldsymbol{e}_{33}) + \frac{1}{3}\int_{\Omega_0} \boldsymbol{Z}^{(2)}_{jm}(\Omega)\,d\Omega, \tag{116}$$

where δ_{ij} is the Kronecker delta and $\boldsymbol{e}_{ij}$, $i, j \in \{1, 2, 3\}$, are the symmetric parts of the dyadic products of the Cartesian unit base vectors. Substituting the representations (109) and (110) into eqs (104) and (105) and using the integral relation (116), we find that

$$\boldsymbol{c}^L(t) = \frac{4}{3}\sqrt{\pi}\,a^4(\boldsymbol{e}_{11} + \boldsymbol{e}_{22} + \boldsymbol{e}_{33})\sigma^L_{00}(t) - 2\sqrt{\frac{2\pi}{15}}\,a^4 \sum_{m=-2}^{2}\left[\frac{1}{2}\left(\delta_{m,-2} - \sqrt{\frac{2}{3}}\delta_{m0} + \delta_{m2}\right)\boldsymbol{e}_{11} - \frac{1}{2}\left(\delta_{m,-2} + \sqrt{\frac{2}{3}}\delta_{m0} + \delta_{m2}\right)\boldsymbol{e}_{22} + \sqrt{\frac{2}{3}}\delta_{m0}\boldsymbol{e}_{33} - i(\delta_{m,-2} - \delta_{m2})\boldsymbol{e}_{12} + (\delta_{m,-1} - \delta_{m1})\boldsymbol{e}_{13} - i(\delta_{m,-1} + \delta_{m1})\boldsymbol{e}_{23}\right]\sigma^L_{2m}(t), \tag{117}$$

$$\boldsymbol{c}^R(t) = \frac{8}{3}\sqrt{\pi}(\boldsymbol{e}_{11} + \boldsymbol{e}_{22} + \boldsymbol{e}_{33})\int_{r=0}^{a} \varrho_0(r)U_{00}(r,t)r^3\,dr - 4\sqrt{\frac{2\pi}{15}} \sum_{m=-2}^{2}\left[\frac{1}{2}\left(\delta_{m,-2} - \sqrt{\frac{2}{3}}\delta_{m0} + \delta_{m2}\right)\boldsymbol{e}_{11} - \frac{1}{2}\left(\delta_{m,-2} + \sqrt{\frac{2}{3}}\delta_{m0} + \delta_{m2}\right)\boldsymbol{e}_{22} + \sqrt{\frac{2}{3}}\delta_{m0}\boldsymbol{e}_{33} - i(\delta_{m,-2} - \delta_{m2})\boldsymbol{e}_{12} + (\delta_{m,-1} - \delta_{m1})\boldsymbol{e}_{13} - i(\delta_{m,-1} + \delta_{m1})\boldsymbol{e}_{23}\right] \times \int_{r=0}^{a} \varrho_0(r)[U_{2m}(r,t) + 3V_{2m}(r,t)]r^3\,dr. \tag{118}$$

The inertia tensors $\boldsymbol{c}^\Sigma(t)$ and $\boldsymbol{c}^{\partial V}(t)$ can be expressed in the same form as $\boldsymbol{c}^L(t)$ by replacing the surface-mass density $\sigma^L_{jm}(t)$ by $\sigma^\Sigma_{jm}(t)$ and $\sigma^{\partial V}_{jm}(t)$, respectively, and the radius *a* by a_Σ in the case of $\boldsymbol{c}^\Sigma(t)$.

To model polar motion, the $c_{13}(t)$ and $c_{23}(t)$ components of the inertia tensors must be expressed explicitly. For the complex variables $c^{L,R}(t) = c^{L,R}_{13}(t) + i\,c^{L,R}_{23}(t)$, eqs (117) and (118) yield

$$c^L(t) = 2\sqrt{\frac{2\pi}{15}}a^4\left[\sigma^L_{21}(t)\right], \tag{119}$$

$$c^R(t) = 4\sqrt{\frac{2\pi}{15}}\int_{r=0}^{a} \varrho_0(r)[U^*_{21}(r,t) + 3V^*_{21}(r,t)]r^3 dr. \tag{120}$$

To model a change in the length of day, the Cartesian component $c_{33}(t)$ of the inertia tensors must be specified. Eqs (117) and (118) provide

$$c^L_{33}(t) = \frac{4}{3}\sqrt{\frac{\pi}{5}}a^4\left[\sqrt{5}\,\sigma^L_{00}(t) - \sigma^L_{20}(t)\right], \tag{121}$$

$$c^R_{33}(t) = \frac{8}{3}\sqrt{\frac{\pi}{5}}\int_{r=0}^{a} \varrho_0(r)[\sqrt{5}\,U_{00}(r,t) - U_{20}(r,t) - 3V_{20}(r,t)]r^3 dr. \tag{122}$$

These equations show that, for a spherically symmetric density distribution, only the spheroidal part of displacement vector $\vec{u}(\vec{x}, t)$ contributes to $\boldsymbol{c}^R(t)$, while the toroidal part of $\vec{u}(\vec{x}, t)$ does not affect it.

The trace of the inertia tensors $\boldsymbol{c}^R(t)$ and $\boldsymbol{c}^L(t)$ can also be deduced from eqs (117) and (118):

$$\mathrm{Tr}\,\boldsymbol{c}^L(t) = 4\sqrt{\pi}a^4\sigma^L_{00}(t), \tag{123}$$

$$\mathrm{Tr}\,\boldsymbol{c}^R(t) = 8\sqrt{\pi}\int_{r=0}^{a} \varrho_0(r)U_{00}(r,t)r^3 dr. \tag{124}$$

We can see that the trace of $\boldsymbol{c}^R(t)$ vanishes if the zeroth-degree coefficient $U_{00}(r,t)$ of the vertical displacement is equal to zero. This applies when the Earth is considered to be incompressible. Since $\sigma^L_{00}(0) = 0$ at the initial time, the trace of $\boldsymbol{c}^L(t)$ vanishes if the total mass of the surface load is conserved during glaciation cycles.

5.2 Spherical-harmonic representation of rigid-body translation and rotation

The final step in our theoretical considerations is to express the rigid-body translation vector $\vec{d}(t)$ and the relative angular-momentum vector $\vec{h}(t)$ in terms of vector spherical harmonics. Both vectors will be considered in the spherical approximations (106) and (107). Representing the displacement vector $\vec{u}(\vec{x},t)$ and the surface-mass density $\sigma^L(\vec{x},t)$ in the forms (109) and (110), respectively, considering the definition of the vector spherical harmonics $\vec{S}^{(-1)}_{jm}(\Omega) := Y_{jm}(\Omega)\vec{e}_r$ and making use of the angular integral over vector spherical harmonics,

$$\int_{\Omega_s} \vec{S}^{(\lambda)}_{jm}(\Omega)\, d\Omega = \sqrt{\frac{4\pi}{3}}\delta_{j1}(\delta_{\lambda,-1} + 2\delta_{\lambda,1})\vec{e}_m, \tag{125}$$

where $\vec{e}_m$, $m = \pm 1, 0$, are the cyclic covariant base vectors (Varshalovich *et al.* 1989, Section 1.1.), we find that the rigid-body translation vector $\vec{d}(t)$ is given by

$$\vec{d}(t) = \sqrt{\frac{4\pi}{3}}\sum_{m=-1}^{1}\vec{e}_m\left\{\int_{r=0}^{a}\varrho_0(r)[U_{1m}(r,t) + 2V_{1m}(r,t)]r^2 dr + a^3\sigma^L_{1m}(t)\right\}. \tag{126}$$

Likewise, using the formulae for the cross products of $\vec{e}_r$ with vector spherical harmonics,

$$\begin{aligned}
\vec{e}_r \times \vec{S}^{(-1)}_{jm}(\Omega) &= 0,\\
\vec{e}_r \times \vec{S}^{(1)}_{jm}(\Omega) &= \vec{S}^{(0)}_{jm}(\Omega),\\
\vec{e}_r \times \vec{S}^{(0)}_{jm}(\Omega) &= -\vec{S}^{(1)}_{jm}(\Omega),
\end{aligned} \tag{127}$$

the cross product occurring in $\vec{h}(t)$ is

$$\vec{x} \times \frac{d\vec{u}(\vec{x},t)}{dt} = r\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\left[\frac{dV_{jm}(r,t)}{dt}\vec{S}^{(0)}_{jm}(\Omega) - \frac{dW_{jm}(r,t)}{dt}\vec{S}^{(1)}_{jm}(\Omega)\right]. \tag{128}$$

Multiplying both sides by $\varrho_0(r)$, integrating the result over the full solid angle and making use of the angular integral (125), we find that the relative angular-momentum vector is

$$\vec{h}(t) = -2\sqrt{\frac{4\pi}{3}}\sum_{m=-1}^{1}\vec{e}_m\int_{r=0}^{a}\varrho_0(r)\frac{dW_{1m}(r,t)}{dt}r^3\, dr. \tag{129}$$

To obtain the Cartesian components of $\vec{h}(t)$, we use the relationship between the cyclic covariant base vectors $\vec{e}_m$, $m = \pm 1, 0$, and the Cartesian unit base vectors $\vec{e}_i$, $i = 1, 2, 3$:

$$\begin{aligned}
\vec{e}_{+1} &= -\frac{1}{\sqrt{2}}(\vec{e}_1 + i\,\vec{e}_2),\\
\vec{e}_{-1} &= \frac{1}{\sqrt{2}}(\vec{e}_1 - i\,\vec{e}_2),\\
\vec{e}_0 &= \vec{e}_3.
\end{aligned} \tag{130}$$

By this, the spherical harmonic representation of the Cartesian components of $\vec{h}(t)$ is found to be

$$h_1(t) + i\,h_2(t) = 2\sqrt{\frac{8\pi}{3}}\int_{r=0}^{a}\varrho_0(r)\frac{dW_{11}(r,t)}{dt}r^3 dr, \tag{131}$$

$$h_3(t) = -2\sqrt{\frac{4\pi}{3}}\int_{r=0}^{a}\varrho_0(r)\frac{dW_{10}(r,t)}{dt}r^3 dr. \tag{132}$$

We can see that, for a spherically symmetric density distribution only, the toroidal part of the displacement vector $\vec{u}(\vec{x},t)$ contributes to $\vec{h}(t)$, while the spheroidal part of $\vec{u}(\vec{x},t)$ affects the centre-of-mass translation vector $\vec{d}(t)$.

6 NUMERICAL VALIDATION OF THE THEORY

6.1 1-D case

It is useful to consider briefly the special case when not only the initial density $\varrho_0(r)$ but also the viscoelastic parameters vary only radially. Then, the gravitational potential $\phi^{E,L}(\vec{x},t)$ of the surface-mass load and the gravitational potential $\phi^{E,R}(\vec{x},t)$ of the internal mass redistribution induced by the surface load can be related by the load Love numbers. This relation, expressed in terms of the spherical-harmonic gravitational-potential coefficients of angular degree j and azimuthal order m is (e.g. Wu & Peltier 1982)

$$\phi^{E,R}_{jm}(t) = k^L_j(t) * \phi^{E,L}_{jm}(t), \tag{133}$$

where * denotes the time convolution and $k^L_j(t)$ is the surface-load Love number of angular degree j for the gravitational potential. As far as the rotational deformation is concerned, the gravitational-potential increment $\phi^{E,CF}$ induced by centrifugal-potential perturbations, expressed in terms of the second-degree spherical-harmonic coefficients, is given by (Munk & MacDonald 1960, Section 5.2)

$$\phi^{E,CF}_{2m}(t) = k^T_2(t) * \psi^E_{2m}(t), \tag{134}$$

where $k^T_2(t)$ is the second-degree tidal Love number for the gravitational potential and $\psi^E_{2m}(t)$ are the second-degree spherical harmonics of the centrifugal-potential increment ψ^E. Eqs (133) and (134) reflect the fact that spherical symmetry of an earth model causes that surface-mass loading and/or tidal forcing with a prescribed spherical-harmonic component of angular degree j induces a viscoelastic response characterized by a spherical harmonic of the same angular degree j only. This effect is usually called the *j-degeneracy*. In addition, for a spherically symmetric earth, both the surface-load and tidal-load Love numbers are independent of the azimuthal order m (the so-called *m-degeneracy*), which means that the viscoelastic Green's response functions for a given j are identical for all $m = -j, \ldots, j$. A laterally heterogeneous earth model removes both degeneracies.

The gravitational-potential coefficients for degree $j = 2$ and order $m = 1$ can be expressed in terms of the (1, 3) and (2, 3) Cartesian components of the load inertia tensor $\boldsymbol{c}^L(t)$ and the response inertia-tensor increment $\boldsymbol{c}^R(t)$ by the first MacCullagh's formula (78). Combining this formula with eq. (133) for $j = 2$ and $m = 1$ results in

$$c^R(t) = k^L_2(t) * c^L(t). \tag{135}$$

Likewise, substituting eq. (134) for order $m = 1$ into eq. (79) and using eq. (95) for $\psi^E_{21}(t)$ yields the (1, 3) and (2, 3) Cartesian components of the inertia-tensor increment due to centrifugal-potential perturbations:

$$c^{CF}(t) = (C - A)\frac{k^T_2(t)}{k_s} * m(t), \tag{136}$$

where the secular Love number is defined by $k_s := 3G(C - A)/\Omega_0^2 a^5$ (Munk & MacDonald 1960, Section 5.3; Moritz & Mueller 1987, Section 3.2).

Finally, the second MacCullagh's formula (88) combined with eq. (134) for degree $j = 2$ and order $m = 0$ yields

$$c^R_{33}(t) - \frac{1}{3}\mathrm{Tr}\,[\boldsymbol{c}^R(t)] = k^L_2(t) * \left[c^L_{33}(t) - \frac{1}{3}\mathrm{Tr}\,[\boldsymbol{c}^L(t)]\right]. \tag{137}$$

A similar procedure results in the (3, 3) Cartesian component of the inertia-tensor increment due to centrifugal-potential perturbations:

$$c^{CF}_{33}(t) - \frac{1}{3}\mathrm{Tr}\,[\boldsymbol{c}^{CF}(t)] = \frac{4\pi}{15}(C - A)\frac{k^T_2}{k_s}(t) * m_3(t). \tag{138}$$

Eqs (135)–(137) with $\mathrm{Tr}\,[\boldsymbol{c}^R(t)] = \mathrm{Tr}\,[\boldsymbol{c}^L(t)] = 0$ have been widely used in modelling the rotational response of a spherically symmetric earth model to surface-mass loading and centrifugal-potential perturbations (see Section 1 for references). The contribution of $c^{CF}_{33}(t)$ to $m_3(t)$ is often neglected, because it is about two orders of magnitude smaller than the principal moment of inertia C.

6.2 Direct numerical integration

We now validate the first and second MacCullagh's formulae (79) and (89) against forward computations of the inertia tensors $\boldsymbol{c}^L(t)$ and $\boldsymbol{c}^R(t)$ according to formulae (119)–(122). Since these formulae assume spherical symmetry of the earth model, the calculations are carried out using a spherically symmetric, self-gravitating, incompressible, Maxwell-viscoelastic earth model. To find the relaxation times and associated amplitudes of all normal modes and, therefore, to carry out the numerical convolution in the case of the Laplace-domain method, applied in the next section, we use a three-layer earth model of mass density and elastic rigidity. The mass densities of the core and mantle are 10.9869×10^3 kg m^{-3} and 4.4494×10^3 kg m^{-3}, respectively. The elastic shear moduli of the mantle and lithosphere are 1.4519×10^{11} Pa and 0.67×10^{11} Pa, respectively. This model is a simplified version of the original PREM model, yet it provides inertia-tensor perturbations that are sufficiently realistic for the Earth. Moreover, the reader may use the simpler earth model to validate their numerical programming.

The upper-mantle viscosity is 5×10^{20} Pa s, the lower-mantle viscosity is 1×10^{22} Pa s, and the thickness of the elastic lithosphere is 100 km. The fluid core is incorporated as a boundary condition at the core–mantle boundary. The ice model used is based on the global ICE-3G deglaciation history proposed by Tushingham & Peltier (1991). We construct a 100-kyr glaciation phase by scaling the maximum extend of the ICE-3G with relative sea level from far-field sites (e.g. Lambeck & Chappell 2001) followed by the ICE-3G deglaciation history.

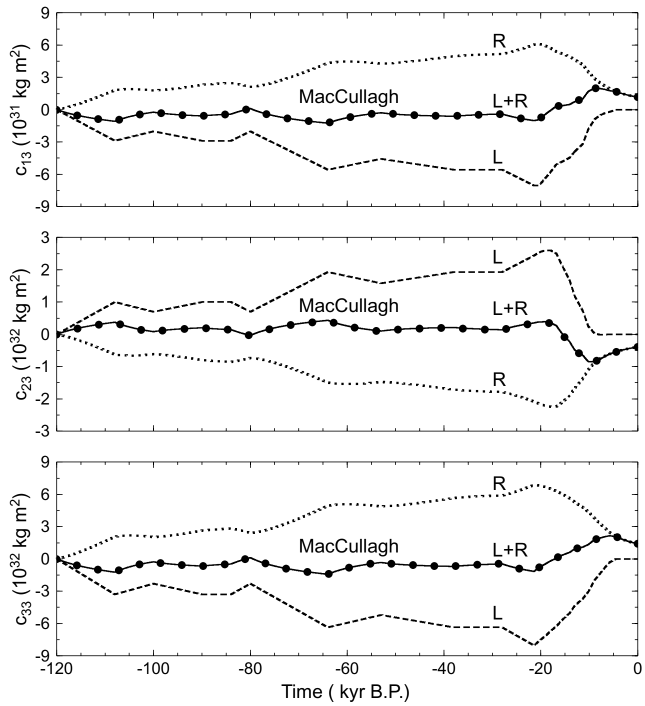

Figure 2. Comparison of the $c_{j,3}$, $j = 1, 2, 3$, components of the inertia-tensor increment $\boldsymbol{c}(t)$ computed by the MacCullagh's formulae (79) and (89) (circles) and by summing two contributions $c^L_{j,3}$ and $c^R_{j,3}$ (solid lines). The components $c^L_{j,3}$ (dashed lines) are computed from the second-degree spherical-harmonic coefficients of surface-mass load according to eqs (119) and (121), while the components $c^R_{j,3}$ (dotted lines) are computed by the direct numerical integration of the second-degree spherical-harmonic coefficients of the displacement field according to eqs (120) and (122). The results apply to an earth model with a 100-km-thick elastic lithosphere, and the upper-mantle and lower-mantle viscosities of 5×10^{20} Pa s and 1×10^{22} Pa s, respectively. The inertia-tensor increment due to the static deformation of the fluid core is computed by eq. (103).

Fig. 2 shows the Cartesian components (1, 3), (2, 3) and (3, 3) of the inertia tensors $\boldsymbol{c}^L(t)$, $\boldsymbol{c}^R(t)$ and $\boldsymbol{c}\,(t)$, respectively, as functions of time. The components $c^L_{j,3}$ and $c^R_{j,3}$, $j = 1, 2, 3$, are first computed separately according to eqs (119)–(122), and then the sums $c^L_{j,3} + c^R_{j,3}$ are compared with the components $c_{j,3}$ computed according to the MacCullagh's formulae (79) and (89). We can see that there is excellent agreement between the results $c_{j,3}$ obtained by the two different approaches.

6.3 Laplace-domain method

The GIA-induced rotational response of the Earth is conventionally computed in the Laplace-transform domain using the method proposed by Wu & Peltier (1984). For the purpose of comparison, the time-domain method of computing the temporal perturbation of the inertia tensor, presented in Section 3, is now compared with the conventional Laplace-domain method.

For an earth model consisting of uniform incompressible linear viscoelastic layers or shells, the surface-load Love number for a delta-function load history can be written in the form

$$k^L_j(t) = k^{L,E}_j \delta(t) + \sum_{k=1}^{M} r^L_{j,k} e^{s_{j,k} t}. \tag{139}$$

This equation expresses a time-dependent viscoelastic response characterized by an immediate elastic effect, described by the elastic load Love number $k^{L,E}_j$, followed by a multiplicity of M exponentially decaying normal modes. These modes of viscous-gravitational relaxation have negative inverse-relaxation times $s_{j,k}$ and amplitudes $r^L_{j,k}$.

The time convolution (135) can be expressed by the definite integral

$$c^R(t) = \int_{t_0}^{t} k^L_2(t - \tau) c^L(\tau)\, d\tau, \tag{140}$$

where t_0 is the time when an ice–water mass is applied on the Earth's surface. Substituting from eq. (139) for $j = 2$, we find

$$c^R(t) = k^{L,E}_2 c^L(t) + \sum_{k=1}^{M} r^L_{2,k} e^{s_{2,k} t} \int_{t_0}^{t} e^{-s_{2,k}\tau} c^L(\tau)\, d\tau. \tag{141}$$

Note that a similar procedure can be applied to eq. (137) for the (3, 3) Cartesian component of the response inertia-tensor increment.

For the viscoelastic earth model introduced in the previous section, the inverse relaxation times and amplitudes have been computed by the matrix-propagator method (e.g. Martinec & Wolf 1998). The model is characterized by five relaxation modes associated with the jumps in the

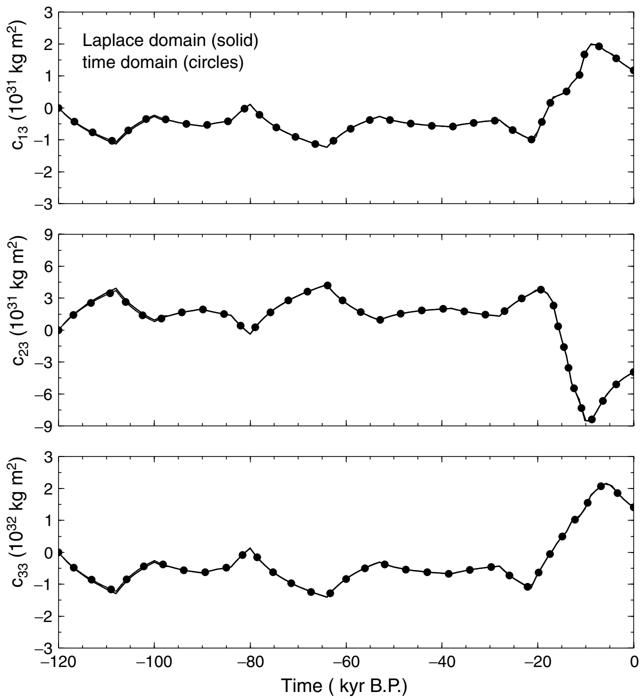

Figure 3. Comparison of the $c_{j,3}$, $j = 1, 2, 3$, components of the inertia-tensor increment $\boldsymbol{c}(t)$ computed by the MacCullagh's formulae (79) and (89) (circles) and by summing two contributions $c^L_{j,3}$ and $c^R_{j,3}$ (solid lines). The components $c^L_{j,3}$ are computed from the second-degree spherical harmonic of the global ICE-3G deglaciation history according to eqs (119) and (121), while $c^R_{j,3}$, $j = 1, 2, 3$, are computed by the numerical convolution (141). The results apply to model considered in Fig. 2.

density and the viscoelastic parameters at the outer surface, the lithosphere-upper mantle discontinuity, the upper-lower mantle discontinuity and the core–mantle boundary. We have found numerically that the negative inverse relaxation times of the five modes for angular degree $j = 2$ are $s_1 = -2.71538427 \times 10^{-10}$ s^{-1}, $s_2 = -2.06179711 \times 10^{-10}$ s^{-1}, $s_3 = -7.564780 \times 10^{-12}$ s^{-1}, $s_4 = -5.023441 \times 10^{-12}$ s^{-1} and $s_5 = -1.103326 \times 10^{-12}$ s^{-1}. Moreover, we have evaluated the load inertia tensor $c^L(t)$ at equally spaced time steps according to eq. (119) applied to the ICE-3G ice model. Finally, the convolution integrals (141) for various inverse relaxation times have been computed numerically by the trapezoidal-rule quadrature (e.g. Press *et al.* 1992).

Fig. 3 shows the Cartesian components (1, 3), (2, 3) and (3, 3) of the inertia tensor $\boldsymbol{c}(t)$ as a function of time. The components $c^R_{j,3}$, $j = 1, 2, 3$, are first computed separately according to eq. (141), and then the sums $c^L_{j,3} + c^R_{j,3}$ are compared with the components $c_{j,3}$ computed according to the MacCullagh's formulae (79) and (89). We can see that there is excellent agreement between $c_{j,3}$ obtained by the two different approaches.

7 CONCLUSIONS

This paper has been motivated by two problems. First, the existing theory of the GIA-induced rotational response of the Earth is based on modelling in the Laplace-transform domain. Since a new generation of software treats GIA in the time domain, our task has been to establish the theory for the GIA-induced rotational response in the time domain, thus avoiding the necessary application of the Laplace-domain method. Second, the existing theory for the GIA-induced rotational response also makes use of the m-degeneracy of the surface-load and tidal-load Love numbers, that is, the earth model must be spherically symmetric. Recent efforts to model GIA for laterally heterogeneous earth models have motivated us to develop the theory for the GIA-induced rotational response without applying the formalism of load Love numbers.

The theory presented in this paper satisfies both tasks. The GIA-induced rotational response of the Earth is computed in the time domain, making use of the analytical solution of the linearized Liouville equation. The time-domain solution of both the GIA (e.g. Martinec 2000) and the induced rotational response is easily combined with a time-domain solution of the sea level equation with time-varying shoreline geometry. Moreover, the theory is valid for any type of Earth model, because it does not require any assumption as to how the gravitational-potential increment, employed to generate the inertia-tensor increments, is computed. That is, $\phi^E(t)$ may be computed by the Laplace-domain method for the case of spherically symmetric earth models, or by numerical software recently being developed to treat laterally heterogeneous earth models.

The MacCullagh's formulae derived in this paper provide a convenient, practical tool to transform gravitational-potential perturbations into inertia-tensor variations. A subsequent application of the Liouville equation yields time variations in the Earth's rotation parameters. Conversely, a time-varying rotation induces perturbations of the centrifugal force and, subsequently, variations in sea level. However, the coupling between the Earth's rotational variations and GIA-induced deformations through ice–water redistribution is only one part of the connection between these two geophysical phenomena. In the follow-up paper, we will develop the theory and present numerical examples for the second part of the coupling between variations in the Earth's rotation and GIA, namely, the case when perturbations in the centrifugal force additionally deform a 3-D viscoelastic earth and, as a result affects the gravitational-potential perturbations.

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