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On the differential properties of internal magnetic field models at the Earth's surface and at satellite altitudes

Wigor A. Webers*

GeoForschungsZentrum Potsdam, Telegrafenberg, D-14473 Potsdam, Germany

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Abstract

The inverse theory of potential fields shows that the correspondence between the internal magnetic field of the Earth and its field sources is unique when the potential field is known in all points of the three-dimensional space including all points of the source region (cp., e.g. Diesselhorst, H., 1939. *Magnetische Felder und Kräfte*. Johann Ambrosius Barth Verlag, Leipzig). Thus, to determine the sources of the field it is not sufficient to know the potential field in the space external to the sources. Moreover, field models derived from finite sets of potential field observations emphasize different source properties because of measurement errors. In this study, I argue that improved internal field models can be developed from multi-altitude magnetic observations by imposing more effective constraints on the poorly conditioned downward continuation problem. In particular, the convergence behaviour of spherical harmonic field models can be used to improve the downward continuation of the higher truncation index terms. A high quality approximation of the field continuation is essential when the field models are interpreted for relatively small field contributions such as from the lithospheric sources.

The relations between the potential field and its sources including the problems of potential field continuations – upward and downward – are governed by the theory of ill-posed inverse problems (cp., e.g. Anger, G., 1990. *Inverse Problems in Differential Equations*. Akademie/Plenum Press, Berlin/London; Anger, G., Gorenflo, R., Jochmann, H., Moritz, H., Webers, W. (Eds.), 1993. *Inverse Problems: Principles and Applications in Geophysics, Technology, and Medicine*. Akademie, Berlin; Huestis, S.P., Parker, R.L., 1979. Upward and downward continuations as inverse problems. *Geophys. J. R. Astr. Soc.* 57, 171–188; Rösler, R., 1981. Über die Fehlerfortpflanzung bei Potentialfeldtransformationen. *Gerlands Beitr. Geophys.* 90, 47–57).

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1. Introduction

The magnetic field recorded at irregularly distributed observatories and stations, contains internal and external field contributions. The internal includes components dominated by the Earth's main or core field, as well as relatively smaller contributions from the Earth's mantle and lithosphere.

These magnetic field constituents are differently represented in observations taken at the Earth's surface and satellite altitude due to the different measurement errors and mathematical properties. The external magnetic field effects, for

* Fax: +49 331 288 2321.

E-mail address: wigor@gfz-potsdam.de.

example, tend to contaminate the lithospheric components of the internal magnetic field much more severely at satellite altitude than at the Earth's surface. Consequently, field models of the two data sets will reflect fundamentally different source effects.

However, imposing effective constraints on the convergence behaviour in the downward continuation of satellite altitude model can improve the utility of the model for representing the internal field at the Earth's surface.

2. Mathematical formulation

From potential field theory, the internal magnetic field of the Earth may be represented as the gradient of the potential V given by

$$B_{\text{int}} = -\nabla V, \quad (1)$$

where the spherical harmonic expansion (SHA) of the potential is

$$V = \sum_{n=1}^{\infty} \sum_{m=0}^n a \left(\frac{a}{r} \right)^{n+1} (g_n^m \cos \lambda + h_n^m \sin m\lambda) P_n^m(\cos \vartheta) \quad (2)$$

and

(r, ϑ, λ) are the spherical polar coordinates

a is the radius of the Earth's surface (nominally 6371.2 km)

P_n^m is a Schmidt quasi-normalized associated Legendre function of degree n and order m , and

$g_n^m; h_n^m$ are the Gauss coefficients.

(3)

In the space external to the source region (i.e. in free space), the potential V satisfies the Laplace equation, so that

$$\Delta V = 0 \quad \text{for } r \geq a \quad (4)$$

In practice, the series expansion (2) is customarily referred to an Earth sphere of mean radius $a = 6371.3$ km, or an ellipsoidal or other appropriate reference surface of the Earth.

Furthermore, the infinite series expansion of Eq. (2) must be approximated by the partial sum with the truncation index N , given by

$$V = \sum_{n=0}^N \sum_{m=0}^n a \left(\frac{a}{r} \right)^{n+1} (g_n^m \cos m\lambda + h_n^m \sin m\lambda) P_n^m(\cos \vartheta). \quad (5)$$

Using the common index k instead of the indices n and m for the respective degree and order of the associated Legendre functions $P_n^m(\cos \vartheta)$ allows Eq. (5) to be expressed in terms of the orthogonal functional system $\{f_k\}$ and the coefficients $\{C_k\} = \{g_n^m; h_n^m\}$.

$$\begin{aligned} f_1 &= a \cdot P_1^0 \\ f_2 &= a \cdot \cos \lambda P_1^1 \\ f_3 &= a \cdot \sin \lambda P_1^1 \\ f_4 &= a \cdot P_2^0 \\ f_5 &= a \cdot \cos \lambda P_2^1 \\ f_6 &= a \cdot \sin \lambda P_2^1 \\ f_7 &= a \cdot \cos 2\lambda P_2^2 \\ f_8 &= a \cdot \sin 2\lambda P_2^2 \\ &\vdots \end{aligned}$$

for all $k = 1, 2, \dots, N \cdot (N+2)$.

(6)

The least squares method is applied to the derivatives of the potential V (Eq. (5)) for numerically calculating the Gauss coefficients $\{g_n^m; h_n^m\} = \{C_k\}$ of the SHA field model.

For field components taken at the Earth's surface the least squares determination references the functional system and Gauss coefficients to the sphere of radius $r=a$. However, the determination for observations taken at satellite altitude h are referenced to $r=a+h$. Obviously, the two data models are based on different functional systems and Gauss coefficients as can be seen by considering the first few terms of the potential (Eq. (5)) shown below.

	for $r = a$	for $r = a + h$
$n = 1$	$f_1 = a \cdot P_1^0$	$f_1 = a \left(1 + \frac{h}{a}\right)^{-2} P_1^0$
	$f_2 = a \cdot \cos\lambda P_1^1$	$f_2 = a \left(1 + \frac{h}{a}\right)^{-2} \cos\lambda P_1^1$
	$f_3 = a \cdot \sin\lambda P_1^1$	$f_3 = a \left(1 + \frac{h}{a}\right)^{-2} \sin\lambda P_1^1$
$n = 2$	$f_4 = a \cdot P_2^0$	$f_4 = a \left(1 + \frac{h}{a}\right)^{-3} P_2^0$
	$f_5 = a \cdot \cos\lambda P_2^1$	$f_5 = a \left(1 + \frac{h}{a}\right)^{-3} \cos\lambda P_2^1$
	$f_6 = a \cdot \sin\lambda P_2^1$	$f_6 = a \left(1 + \frac{h}{a}\right)^{-3} \sin\lambda P_2^1$
	$f_7 = a \cdot \cos 2\lambda P_2^2$	$f_7 = a \left(1 + \frac{h}{a}\right)^{-3} \cos 2\lambda P_2^2$
	$f_8 = a \cdot \sin 2\lambda P_2^2$	$f_8 = a \left(1 + \frac{h}{a}\right)^{-3} \sin 2\lambda P_2^2$
$n = 3$	$f_9 = a \cdot P_3^0$	$f_9 = a \left(1 + \frac{h}{a}\right)^{-4} P_3^0$
	$f_{10} = a \cdot \cos\lambda P_3^1$	$f_{10} = a \left(1 + \frac{h}{a}\right)^{-4} \cos\lambda P_3^1$

, etc. with the set of Gauss coefficients C_k and C'_k , respectively, being different due to the relevant functional systems.

Relating SHA field models to each other commonly involves upward or downward data continuation based on the ratios of the radii of the data reference surfaces. Downward continuation in particular is notoriously problematic and unstable, often yielding predictions with frequency components that are too high to have detected in the higher altitude data (e.g. Anger, 1990; Huestis and Parker, 1979; Rösler, 1981). Taking into account the different convergence properties of the power series at the reference surface (e.g. Knopp, 1922), however, can improve the accuracy of field continuation over the first order predictions from the conventional ratios of the reference surface radii. This approach is especially useful for lithospheric anomaly predictions that require high accuracy predictions near the Earth's surface of magnetic field models derived from satellite observations.

3. Relating SHA magnetic field models derived at the Earth's surface and satellite altitudes

The well-known spatial spectrum W_n (Mauersberger, 1956, 1961; Lucke, 1957; Lowes, 1974; Meyer, 1986) given by

$$W_n = (n + 1) \sum_{m=0}^n [(g_n^m)^2 + (h_n^m)^2] \quad (7)$$

expresses the mean energy density of the multipole field in the same degree n terms of the potential V (Eq. (2)). The linear approximation

$$\log W_n = F(n) = a_0 + a_1 n \quad (8)$$

has the lower slope at the ground (i.e. $r=a$) than at satellite altitude ($r=a+h$). These slope variations, respectively, reflect the slower convergence of the series expansion near the source region relative to its more rapid convergence in the far field.

To relate the different functional systems and their coefficients in Eq. (6), note by the orthogonality of the system $\{f_k\}$ for the series expansion of the potential V (Eq. (2)) that it follows

$$\varphi_k = C_k f_k \quad (9)$$

where

$$\langle \varphi_i, \varphi_k \rangle = \begin{cases} \frac{4\pi a^2}{2n+1} C_k^2 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases}$$

and $k = 1, 2, \dots, N(n+2)$.

The two functional systems $\{f_k\}$ (cp. Eq. (6)) can be treated by comparing the contribution to a volume in the functional space being spanned by these functions that is made analogously to the geometrical understanding of the vector analysis where the volume of the k dimensional vector space is calculated by the GRAM determinant (cp., e.g. Fichtenholz, 1972; Schröder, 1966). The GRAM determinant has elements formed by the scalar products of the k vectors spanning the functional space given by

$$G = \psi(k) = \begin{vmatrix} \langle \varphi_1, \varphi_1 \rangle & \langle \varphi_1, \varphi_2 \rangle & \langle \varphi_1, \varphi_3 \rangle & \dots & \langle \varphi_1, \varphi_k \rangle \\ \langle \varphi_2, \varphi_1 \rangle & \langle \varphi_2, \varphi_2 \rangle & \langle \varphi_2, \varphi_3 \rangle & \dots & \langle \varphi_2, \varphi_k \rangle \\ \langle \varphi_3, \varphi_1 \rangle & \langle \varphi_3, \varphi_2 \rangle & \langle \varphi_3, \varphi_3 \rangle & \dots & \langle \varphi_3, \varphi_k \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle \varphi_k, \varphi_1 \rangle & \langle \varphi_k, \varphi_2 \rangle & \langle \varphi_k, \varphi_3 \rangle & \dots & \langle \varphi_k, \varphi_k \rangle \end{vmatrix} \quad (10)$$

Because of the orthogonality of Eq. (9), the GRAM determinant (Eq. (10)) consists only of the non-zero-products $\langle \varphi_k, \varphi_k \rangle$ in the principal diagonal so that the product of the elements in the principal diagonal gives the volume. Therefore, it makes good sense to study the stepwise contribution of each $\langle \varphi_k, \varphi_k \rangle$, $k = 1, 2, \dots, N(N+2)$, to the volume G in functional dependence on the index k for both functional systems $\{f_k\}$ of Eq. (6).

The functional dependence of G on the index k is an inherent mathematical invariant of the SHA model made by its functional system $\{f_k\}$ and relevant Gauss coefficients C_k that also characterizes its convergence behaviour. Analogous to the discussion of the spatial spectrum W_n (Eq. (7)), the SHA models are usefully compared in terms of $\log \psi(k)$ that is expressed approximately as the polynomial in k given by:

$$\log \psi(k) = c_0 + c_1 k + c_2 k^2 + c_3 k^3 + \dots \quad (11)$$

Eq. (11) is a considerably more comprehensive extension of the linear approximation that Webers (2002) developed to transform the Gauss coefficients for the downward continuation of satellite altitude magnetic data. Extending the transformation of Gauss coefficients $\{C_k\} = \{g_n^m; h_n^m\}$ from the linear to the more general polynomial in Eq. (11) gives

$$\begin{aligned} & C_{\text{reg}}(n, k, C_k, r_1, r_2, \gamma, c_0^{(2)}) \\ &= \text{sign}(C_k) \left[(2n+1) 10^{(1-\cos \gamma) c_0^{(2)} - (k + (c_2/c_1)k^2 + (c_3/c_1)k^3 + \dots) \sin \gamma} \left(\frac{C_k^2}{2n+1} \left(\frac{r_1}{r_2} \right)^{2(n+1)} \right)^{\cos \gamma} \frac{(4\pi r_1^2)^{\cos \gamma}}{4\pi r_1^2} \right]^{1/2}, \\ & \text{for } r_2 < r_1 \text{ and } \gamma = \alpha_2 - \alpha_1 \end{aligned} \quad (12)$$

where r_1 is the radius of the satellite altitude reference surface, r_2 the radius of the ground reference surface, α_1 the slope of Eq. (12) for the satellite altitude data; $c_1^{(1)} = \tan \alpha_1$, α_2 the slope of Eq. (12) for the ground data; $c_1^{(2)} = \tan \alpha_2$, γ the difference of the slopes where $\gamma = \alpha_2 - \alpha_1$, C_k the Gauss coefficients for the satellite altitude data and $c_0^{(2)}$ is the first (constant) term of Eq. (12) for the ground data.

The reg index in the downward continued Gauss coefficients C_{kreg} (Eq. (12)) indicates the fact that the downward field continuation mathematically means an ill-posed inverse problem that is solved by a regularization process. Here, the regularization parameters γ , $c_0^{(2)}$, $c_j^{(1)}$ for $j = 0, 1, 2, \dots$ (cp. Eqs. (11) and (12)) have been derived from the inherent mathematical characteristics, i.e. the different convergence property at the ground in comparison to the satellite altitude. In this respect no further explanations are necessary, what the consequences of these regularizing parameters would mean.

In downward continuation the regularization of Eq. (12) increasingly dampens the effects of coefficients with increasing k indices. Consequently, the higher frequency terms of the SHA field model are numerically reduced so that the downward field continuation does not introduce shorter wavelengths of the internal magnetic field that can be observed at satellite altitude. Moreover, errors may not be increased by a transformation of potential fields (Rösler, 1981).

The transformation formula (Eq. (12)) reduces to the simple transformation by the ratio of the reference radii r_1 and r_2 , respectively, where the slope difference is zero, i.e. for $\gamma = 0$. However, where the two data sets involve different convergence properties, downward continuation using Eq. (12) provides improved estimates relative to the first order predictions from the simple ratio of reference surface radii.

4. Examples of the downward continuation procedure of satellite altitude magnetic fields

For demonstrating these calculations, let us compare the multi-altitude magnetic field predictions from the 1990 epoch Definitive International Geomagnetic Reference Field (*DGRF* 1990) of truncation index $N = 10$ (IAGA, 1996) and for the Ørsted *oe6a* model (epoch 2000) of truncation index $N = 20$ (www.dsri.dk/Oersted/Field-models). The *oe6a* Ørsted model coefficients were determined from satellite observation data at the mean altitude of $h = 750$ km assuming convergence critical consistent with downward field continuation. Both field models have been published as being valid for the Earth's surface, i.e. for $r = a$ despite of the fact that observation data from the Earth's surface and satellite altitudes had been considered. Discussing the presented examples here the convergence for the Earth's surface is supposed to be correctly used neglecting the mixture of differential convergence properties when the models were calculated.

In order to show the better approximation for a downward continuing procedure by Eq. (12) both field models, *DGRF* 1990 and *oe6a*, respectively, were calculated for the satellite altitudes of $h = 400$ km (comparable with mean satellite altitude of CHAMP satellite) and $h = 750$ km (comparable with the mean Ørsted satellite altitude), i.e. mathematically by the ratio of reference surface radii as simple upward continuations. This means a rough approximation for an upward potential field continuation that, in general, is supposed as being sufficient for the examples given here.

The field model *DGRF* 1990 ($r = a$) calculated for the satellite altitude $h = 400$ km is denoted by *DGRFs* 1990, the field model *oe6a* ($r = a$) calculated for the satellite altitude $h = 750$ km is denoted by *oe6as*.

The standard procedure of downward continuations from satellite altitudes down to the ground ($r = a$) by the ratio of reference surface radii exactly reproduces the original field models *DGRF* 1990 and *oe6a*, respectively. Mathematically, upward and downward continuations by the ratio of radii are completely inverse to each other and mutually reproduce the other fields. This is significantly in contradiction to the physical fact that the internal magnetic field is a potential field that decreases with increasing distance from the Earth as its source region. Therefore, field continuations by the ratio of reference surface radii are only very rough approaches.

The new downward continuation procedure according to Eq. (12) takes into account the decreasing of the potential field with increasing distances and calculates dampened Gauss coefficients based on the differential convergence properties of the SHA field models.

The difference chart of Fig. 1 (*DGRF* 1990 – *DGRFs*d (400) 1990, Z-component) shows approximately the decreasing of the internal magnetic field (Z-component) that is determined when *DGRFs* 1990 had been downward continued by Eq. (12). In this way *DGRF* 1990 cannot exactly be reproduced according to the physical conditions so that the difference chart of Fig. 1 is not identically zero. *DGRFs*d (400) 1990 denotes the downward continuation of *DGRFs*

Fig. 1. *DGRF* 1990: difference chart *DGRF* 1990 – *DGRFsd* (400) 1990, Z-component in nT: *DGRF* 1990 mathematically upward continued to the satellite altitude of $h = 400$ km as *DGRFs* (400) 1990, downward continued to the ground as *DGRFsd* (400) 1990.

1990 from the altitude $h = 400$ km down to the ground by Eq. (12). The standard continuations by the ratio of the reference surface radii gives a difference chart identically zero, i.e. Fig. 1 shows in which way the downward continuation by Eq. (12) is approximately improved.

In Webers (2002) for the *DGRF* 1990 model also the charts for *DGRF* ($r = a$), *DGRFs* ($h = 400$ km), *DGRFsd* (from $h = 400$ km down to $r = a$) are presented and discussed.

Fig. 2 gives the difference chart *DGRF* 1990 – *DGRFsd* (750) 1990, Z-component and analogously shows the decreasing of the internal magnetic field model (Z-component) for a satellite altitude of $h = 750$ km. *DGRFsd* (750) 1990 denotes the downward continuation of *DGRFs* 1990 from $h = 750$ km down to the ground by Eq. (12).

The difference chart of Fig. 2 shows much higher differences in comparison to Fig. 1 because of the higher satellite altitude. This reflects well the physical condition of the decreasing potential field. In opposite to that the standard downward field continuation by the ratio of reference surfaces shows no difference field despite of different altitudes as mentioned before. Because of the truncation index $N = 10$ for the *DGRF* model the linear approach for Eq. (11) proved to be sufficient.

Figs. 1 and 2 both demonstrate that there is an essential decrease of the internal magnetic field for these altitudes that must not be neglected. Moreover, this decrease shows a large variety of its spatial dependence and also illustrates how far essential smaller constituents of the internal field at the ground do not reach definite satellite altitudes.

Fig. 3 shows approximately the decreasing of the internal magnetic field (Z-component) for the *oea6a* Oersted field model (Z-component). The difference chart *oea6a20* – *oea6a20sd* had been calculated for the downward continuation of *oea6a20s* by Eq. (12) from the satellite altitude of $h = 750$ km down to the ground and with a linear approach for Eq. (11) denoted by *oea6a20sd*. Because of the higher truncation index, here $N = 20$, it proves as useful, to determine the downward field continuation by Eq. (12) when a polynomial of third degree had been used for Eq. (11). This better approach is denoted by *oev320sd*. Fig. 4 gives the difference chart *oea6a20* – *oev320sd*, Z-component and presents analogously a better approach for an internal magnetic field model of higher truncation index, here $N = 20$.

Fig. 5 shows the improvement by the better approach of Eq. (11), means for the Z-component of Fig. 3 in comparison to Fig. 4.

The examples claim to demonstrate the problems of downward field continuation as well as how the new procedure of Eq. (12) improves the downward continued field models in comparison to the standard procedure by the ratio of the

Fig. 2. *DGRF* 1990: difference chart *DGRF* 1990 – *DGRFs*(750) 1990, Z-component in nT: *DGRF* 1990 mathematically upward continued to the satellite altitude of $h = 750$ km as *DGRFs*(750) 1990, downward continued to the ground as *DGRFs*(750) 1990.

Fig. 3. *oe6a*: Difference chart *oe6a20* – *oe6a20sd*, Z-component in nT: *oe6a* mathematically upward continued to the satellite altitude of $h = 750$ km as *oe6as*, downward continued to the ground as *oe6asd*.

Fig. 4. *oe6a*: Difference chart *oe6a20* – *oev320sd*, Z-component in nT: *oe6a* mathematically upward continued to the satellite altitude of $h = 750$ km as *oe6as*, downward continued to the ground referred to a polynomial of third degree as *oev320sd*.

reference surface radii. This is not concerned by the fact that both models, *DGRF* 1990 and *oe6a*, respectively, refer to different epochs. A downward continuation is calculated separately for its related model.

Both field models had been calculated together with their statistical evaluations as rms, etc. as had been published. When both field models had been calculated for relevant satellite altitudes there is no additional statistical evaluation.

Fig. 5. *oe6a*: Difference chart *oev320sd* – *oe6a20sd*, Z-component according to Figs. 3 and 4 showing the improved downward continuation by applying Eq. (11) up to the third degree in comparison to the first degree.

The transformation by Eq. (12) gives a downward field continuation of the Gauss coefficients by an approach as an analytical mathematical formula. The field components of the downward continued model are calculated as derivatives of the potential V (cp. Eqs. (1)–(5)) as a form of field synthesis by a series expansion approach of analytical formulae using the new Gauss coefficients by Eq. (12).

An additional statistical evaluation more than the original rms, etc. can only be made by discussing consequences of any form of continuation of errors. This is planned as next step of studying the problems of downward field continuation.

The global maps of Figs. 1–5 are given in form of contour lines (isolines) where the spacing of the contour intervals are selected adequate to the spatial dependent difference values given in nT. The computer program allows to vary suitably the contour intervals for detailed studies.

5. Conclusions

The discussion of the examples demonstrates that the downward field continuation can in general only be calculated as reasonable approaches because of the mathematical theoretical background of the ill-posed inverse problems. Furthermore, the quality of the approaches depends on the steps with which quality also these approaches are used.

The theoretical background as well as the examples make obvious that the decreasing of the internal magnetic field in dependence on altitudes h is very essential for high quality internal magnetic field models and must not be neglected when field data of multi-altitude observations are used. Consequently, field continuations must be applied. The quality of their approaches must be considered and the parameters of all the approximations used are to be selected in a suitable way.

These problems are of special importance when the smaller contributions from the lithosphere to the internal magnetic field model are studied and regional field models are interpreted for their geophysical sources.

Moreover, it is very useful to study the differences of separate field models referred to different satellite altitudes but for the simultaneous epoch and to use them for geophysical interpretation.

In particular, simultaneous satellite missions that use different satellite orbits and altitudes as it is planned for the swarm mission give outstanding chances to use these data for a detailed research.

Here, it is also worth to study the different contributions of internal and external magnetic field contributions to the recorded data sets at different altitudes. A combination of upward and downward field continuations with the related difference fields is possible that enables also conclusions for the physical constituents of the fields.

If magnetic field data of different reference surfaces are used commonly for a field model without any reasonable transformation procedure then only a mean field model of first approximation quality is calculated.

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