A MATLAB implementation of a 1-D TEM inversion algorithm

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1   Introduction

Electromagnetic induction in the time domain is considered as a valuable tool for the recovery of the electrical conductivity of the sub-surface. Albeit highly sophisticated 3-D forward modelling algorithms exist, often a simple 1-D model parametrization seems appropriate to sufficiently well fit the observed data. In this paper we consider a 1-D inversion algorithm for TEM response data. The purpose of this study is two-fold: First, the desire for a simple but robust and user-friendly code for the interpretation of transient electromgnetic data has been addressed. Second, the interplay of numerous parameters associated with, e.g., data scaling, data transformation, convergence control etc. could be studied. The results of those studies provide good assumptions for the choice of specific inversion parameters also arising in 2-D and 3-D inversion codes that are based on the same numerical parameter estimation approach.

The model parameters to be recovered within the inversion are those of a horizontally layered half-space with isotropic electrical conductivities. Forward responses are obtained by evaluating Hankel integrals as typically arising in 1-D dipole induction problems. For the numerical computation of the integrals we apply a Fast Hankel Transform technique (FHT).

2   TEM forward response

We consider as model parameters the piecewise constant isotropic electrical conductivities and thicknesses of horizontal layers within the Earth. More precisely, in the case of an N-layered halfspace, we denote $\sigma_1, \ldots, \sigma_N$ as the conductivities within the conducting lower halfspace. In the insulating upper halfspace the electrical conductivity is assumed to be zero, i.e., $\sigma_0 \to 0$. Hence, jumps in the electrical conductivities occur at the boundary interfaces at $z_0, z_1, \ldots, z_{N-1}$. Without loss of generality we set $z_0 = 0$. The thickness of a layer may simply be calculated by $d_i = z_i - z_{i-1}$, $i = 1, \ldots, N-1$.

In the case of land-based TEM, the transmitter dipole is usually placed on the surface of the Earth, i.e., at $z = z_0$. Receivers may be located at the Earth's surface, i.e., at $z = z_0$, or within boreholes at arbitrary depths $z > z_0$.

A typical procedure for obtaining the layered halfspace time-domain response of a vertical magnetic dipole with dipole moment m located at $r = 0, z = 0$, i.e., the origin of a cylindrical coordinate system, can be outlined as follows: We consider as response the

voltage induced in an induction coil receiver located at $\mathbf{r} = (r, z), r > 0, z \geq 0$. The voltage per coil area is proportional to the time derivative of the magnetic flux, i.e., $\partial B_z(r, z, t)/\partial t$. First, we obtain an expression for $B_z(r, z, \omega)$ in the frequency domain

$$B_z(r, z, \omega) = \frac{\mu m}{2\pi} \int_0^\infty \frac{\lambda^3}{\lambda + a_1} e^{-a_1 z} J_0(\lambda r) \, \mathrm{d}\lambda, \quad z \geq 0, \tag{1}$$

where a_1 is the admittance at the surface of an N-layered halfspace. The admittance a_1 can be calculated recursively by the following algorithm:

$$\lambda_n = \sqrt{\lambda^2 + i\omega\mu\sigma_n}, \quad 1 \leq n \leq N \tag{2}$$

$$a_N = \lambda_N \tag{3}$$

$$a_n = \frac{a_{n+1} + \lambda_n \tanh(\lambda_n d_n)}{\lambda_n + a_{n+1} \tanh(\lambda_n d_n)}, \quad N - 1 \geq n \geq 1. \tag{4}$$

The Hankel integral (1) can be evaluated using, e.g., a Fast Hankel Transform algorithm. Moreover, for a fixed point $\mathbf{r} = (r, z)$, the integral (1) can be evaluated for arbitrary frequencies $\omega > 0$.

Next, we obtain the time domain expression for the vertical magnetic dipole with a current step waveform. The frequency and time domain responses are Fourier transform pairs. In the following we demonstrate, how the inverse Fourier transform or Fourier synthesis can be carried out. The current waveform can be described by the Heaviside step function

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0. \end{cases} \tag{5}$$

Since $B_z(r, z, \omega)$ is a complex-valued function, but $B_z(r, z, t)$ is real and causal, i.e., $B_z(r, z, t) = 0$ for $t < 0$, we can express the inverse Fourier transform equivalently as a Cosine or Sine Transform

$$B_z(t) = \frac{2}{\pi} \int_0^\infty \mathrm{Im}\frac{B_z(\omega)}{\omega} \cos(\omega t) \, \mathrm{d}\omega \tag{6a}$$

$$= \frac{2}{\pi} \int_0^\infty \mathrm{Re}\frac{B_z(\omega)}{\omega} \sin(\omega t) \, \mathrm{d}\omega. \tag{6b}$$

The sine and cosine functions can be exactly represented as Bessel functions of half order. More precisely,

$$J_{-1/2}(\omega t) = \sqrt{\frac{2}{\pi\omega t}} \cos(\omega t) \tag{7a}$$

$$J_{+1/2}(\omega t) = \sqrt{\frac{2}{\pi\omega t}} \sin(\omega t). \tag{7b}$$

Using (7), the calculation of $\partial B_z(r, z, t)/\partial t$ can finally be obtained by evaluating the integral

$$\frac{\partial B_z(r, z, t)}{\partial t} = \sqrt{\frac{2}{\pi t}} \int_0^\infty \mathrm{Re}(B_z(r, z, \omega))\sqrt{\omega} J_{-1/2}(\omega t) \, \mathrm{d}\omega. \tag{8}$$

Integrals of the form (8) can also be evaluated using a Fast Hankel Transform (Christensen, 1990).

A typical transient response of a layered halfspace is illustrated in Fig. 1.

In what follows, we denote as *forward response* the function $f(\mathbf{m}) := \partial B_z(\mathbf{m}, t)/\partial t$ for any fixed point $\mathbf{r} = (r, z)$.
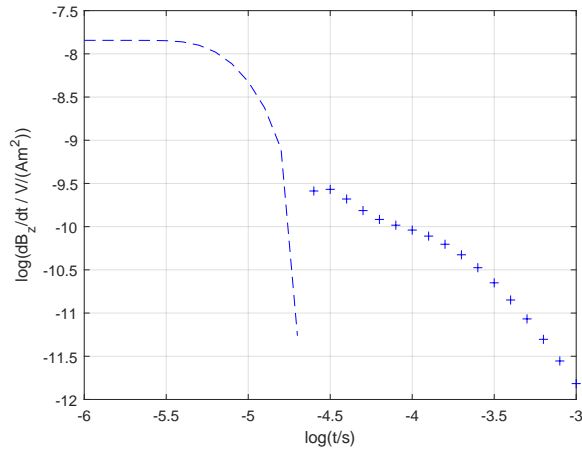
Figure 1: Time derivative of the vertical magnetic field 100 m from a vertical magnetic dipole. Transmitter and receiver on the surface of a $100 \, \Omega \cdot m$ homogeneous halfspace with a thin $10 \, \Omega \cdot m$ layer embedded between 60 and 80 m depth. Dashed line negative.

3   The TEM inverse problem

The TEM data are either observations of electric fields or voltages induced in an induction coil for arbitrary, generally non-equidistant times $t > 0$. In this paper we consider as data the induced voltage which is proportional to the time-derivative of the magnetic induction, i.e., $\partial B_z / \partial t$. We collect the data in a vector

$$\mathbf{d} \in \mathbb{R}^n$$

corresponding to the observation times $t_1 < t_2 < \cdots < t_n$ at which the data has been sampled.

The vector $\mathbf{m} \in \mathbb{R}^N$ stores the values of the model parametrization for a layered half-space. In the following, however, we keep the layer thicknesses fixed and use the transformation $m_i = \log \sigma_i, i = 1, \ldots, N$ to ensure the positivity of the reconstructed electrical conductivities during the inversion.

The *inverse problem* can then be stated as follows:

Find a parameter distribution $\mathbf{m}$ which minimizes the functional

$$\Phi(\mathbf{m}) = \frac{1}{2} \| d - f(\mathbf{m}) \|_2^2. \tag{9}$$

The necessary conditon for the existence of a minimum is

$$\nabla \Phi(\mathbf{m}) = 0. \tag{10}$$

This gradient has to be computed numerically. In our case, we approximate the partial derivatives by finite differences. We denote the $n \times N$ matrix of the partial derivatives

$$\mathbf{J}(\mathbf{m}) = \frac{\partial f_i(\mathbf{m})}{\partial m_j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, N \tag{11}$$

as *sensitivity matrix* or *Jacobian*.

The *damped Gauß-Newton* method yields a search direction $\mathbf{p} \in \mathbb{R}^M$ by solving the system of normal equations

$$[\mathbf{J}(\mathbf{m})^\top \mathbf{J}(\mathbf{m}) + \lambda \mathbf{I}]\mathbf{p} = \mathbf{J}(\mathbf{m})^\top (\mathbf{d} - f(\mathbf{m})). \tag{12}$$

The large dynamic range of the data $\mathbf{d}$ requires a proper mathematical transformation. As the data exhibit a logarithmic scaling and possibly sign changes, a transformation of the form

$$g(f(t)) = \mathrm{asinh}(f(t)/a) \tag{13}$$

is appropriate.

Further, it is advisable to check the validity of the derivatives of $f(\mathbf{m})$ appearing in the gradient $\nabla \Phi(\mathbf{m})$ by a simple test (Taylor test):

We choose $\mathbf{m}$ and a random vector $\Delta \mathbf{m}$ with $\|\Delta m\|_2 = 1$. For $h \to 0+$ there holds

$$e_0(h) := \|f(\mathbf{m} + h\Delta\mathbf{m}) - f(\mathbf{m})\|_2 \qquad\qquad = O(h) \tag{14}$$

$$e_1(h) := \|f(\mathbf{m} + h\Delta\mathbf{m}) - f(\mathbf{m}) - hf'(\mathbf{m})\Delta\mathbf{m}\|_2 \qquad = O(h^2). \tag{15}$$

The above algorithm generates numbers illustrated in Fig. 2. Note that for small values of the step size h (for $h < 10^{-6}$), the expected behaviour deteriorates due to the accumulation of round-off errors in floating-point arithmetics.
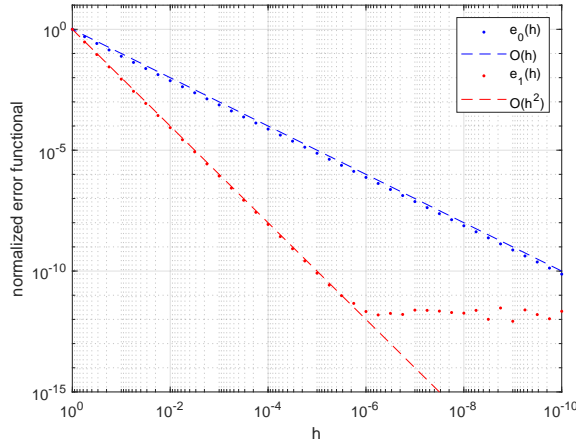
Figure 2: Result of the Taylor test applied to $f(\mathbf{m})$.

Once a descent search direction $\mathbf{p}$ is obtained, then a parameter α is selected to fulfil the so-called Armijo condition

$$\Phi(\mathbf{m}_k + \alpha\mathbf{p}) < \Phi(\mathbf{m}_k) - \gamma\alpha\mathbf{p}^\top \mathbf{J}^\top (\mathbf{d} - f(\mathbf{m}_k)) \tag{16}$$

where $\gamma \in (0,1)$ is a given constant (here, $\gamma = 10^{-4}$).

Finally, to obtain a stable solution of (12), the value of the regularization paramter λ has to be choosen carefully. We propose a scheme in which the decrease of λ is controlled by the convergence rate, i.e., the relative drop of the functional (9) for subsequent Gauß-Newton iterations. More precisely, the value of λ will be decreased by, e.g., a factor of 10 as soon as the relative drop falls below a pre-defined threshold ξ with $0 < \xi < 1$, i.e, $\xi = 0.2$. For small values of ξ we observe more Gauß-Newton iterations carried out at a fixed value of λ, while for $\xi \to 1$ the value of λ will be reduced at every Gauß-Newton step, until a pre-defined lower bound of λ is reached.

4   Numerical experiments

To provide an experimental data set, we consider a thin conductive layer with an electrical resistivity of 10 $\Omega \cdot m$ embedded in a homogeneous half-space of 100 $\Omega \cdot m$. The layer is located at a depth of $60 \leq z \leq 80$ m. Synthetic transient data **d** have been computed for 31 logarithmically equidistant times t with $10^{-6} \leq t \leq 10^{-3}$ s. The synthetic data have been contaminated with noise, i.e.,

$$d_i := d_i(1 + \epsilon_i \sqrt{t_i}), \quad i = 1, \ldots, n,$$

where the ϵ_i are random numbers from a normal distribution with zero mean and a standard deviation of 10^{-1}. The horizontal offset r between the vertical magnetic dipole transmitter source and the receiver is 100 m. Both transmitter and receiver are at the plane $z = 0$ m. Hence, a horizontal co-planar configuration is considered (cf. Fig. 1).

For our numerical experiments we have choosen a layered half-space with 10 layers, i.e., $N = 10$. The layer thicknesses have been kept fixed and choosen small enough such that the late time data still has sufficient sensitivity to recover the substratum.

The inversion results (Fig. 3) reveal that the convergence of the functional $\Phi(\mathbf{m})$ has a pronounced dependency on the starting model $\mathbf{m}_0$.

Independent of the actual choosen values for this experiment, the model reponses can be well fitted to the noisy synthetic data.

Further, the convergence and model roughness are controlled by the regularization parameter. As can be seen from Fig. 3 (right column), a small value of $\lambda = 10^{-6}$ has already established after a few Gauß-Newton iterations. The numerical solution of the system of normal equations (12) yields oscillating model parameter updates, which introduce, e.g., the highly resistive layer above the conductive layer.

5   Summary

An implementation of a TEM data inversion scheme based on a damped Gauß-Newton approach has been presented. The algorithm is able to recover model parameters of a layered half-space with fixed layer boundaries. The method works well for noisy data. The regularization parameter can be modified during the inversion to improve the rate of convergence.

The MATLAB implementation will be made available from the author at `www.github.com/ruboerner/tem1dinv`.

References

Christensen, N. B. (1990). Optimized Fast Hankel Transform Filters. *Geophysical Prospecting, 38*(5), 545–568.
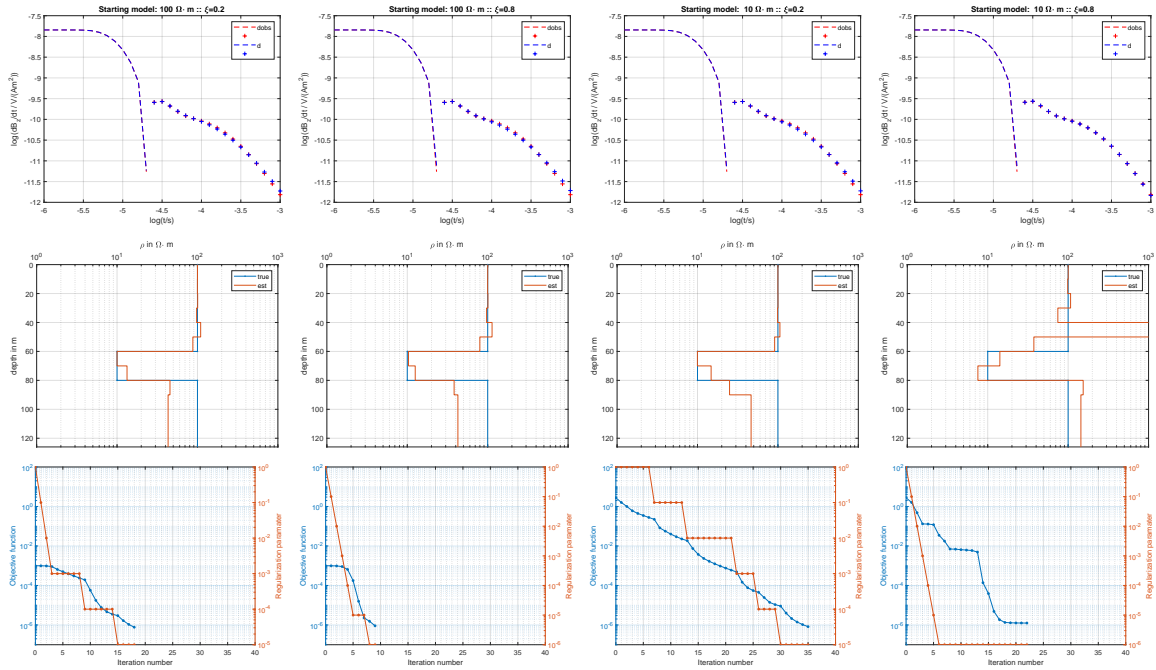
Figure 3: Inversion results for two different homogeneous starting models $\mathbf{m}_0$ and two different regularizaton control parameters ξ. Top row: Transient data (red) and model response (blue). Middle row: Estimated model (red) and true model (blue). Bottom row: Convergence (blue) and regularization parameter history (red). Columns correspond to the combinations ($100\ \Omega \cdot m$, $\xi = 0.2$), ($100\ \Omega \cdot m$, $\xi = 0.8$), ($10\ \Omega \cdot m$, $\xi = 0.2$), ($10\ \Omega \cdot m$, $\xi = 0.8$).