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Geodetic Boundary Value Problems

III

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Summary

The author extends and completes his investigations about the solution of the boundary value problem of Molodenskij found by means of the identity of Green during the last 30 years. These derivations are developed here in a clear, comprehensive and systematic order. It is the inversion of the fundamental equation of physical geodesy which is treated here. The mapping between the telluroid and the Earth's surface happens by vertical point shifts. The final result allows the calculation of the height anomalies exact to 1 cm; thus, it is usefull for the determination of the decimeter- and centimeter- geoid.

The solution has the shape of a closed expression. It does not imply series developments which have a dubions convergence or which do not allow to evaluate the amount of the residual term of it. All the here introduced series developments have a quick, clear and guaranteed convergence. Iteration procedures are avoided. The final result expresses the height anomalies or the perturbation potential at the Earth's surface in terms of the free-air anomalies of the gravity at the Earth's surface.

The main term of the solution is the Stokes integral which has the Faye - anomalies in the integrand. These anomalies consist in the sum of the free-air anomalies and the plane terrain reduction of the gravity. Further, these Faye - anomalies are supplemented by a small and smoothed term which can be disregarded in most cases, which has positive and negative amounts, and which implies the vertical gradient of the refined Bouguer - anomalies. Further on, this main term has to be supplemented by the addition of 3 or 4 relative small terms, only one of them can reach about 50 cm in extreme cases.

The final solution of the boundary value problem has a shape which is distinguished by the special property of the additives that a clear separation between the terms linear and quadratic in the heights takes place. The terms quadratic in the heights can be neglected for test points situated in plane countries or in low mountain ranges.

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Only for test points situated in high mountains, the terms quadratic in the heights can be of interest. Even in this case, these terms have only relative small amounts, and the integration area can be restricted on the near surroundings of the test point, to a distance of not more than some tens of kilometers.

The final solution of the boundary value problem is convenient for routine applications, and it meets all theoretical requirements. The physical boundary values are not subsided downwards from the surface to the sphere, but the geometrical terms come upwards from the sphere to the surface.

Zusammenfassung

Die in den letzten 30 Jahren, seit 1959, ausgeführten Untersuchungen des Autors zum Problem der Darstellung der Lösung des Randwertproblems von Molodenskij mittels der Identität von Green werden erweitert und in eine endgültige Form gebracht. Alle diese Untersuchungen werden hier in systematischer Weise vollständig zusammengefaßt. Es handelt sich also um die Inversion der Fundamentalgleichung der Physikalischen Geodäsie. Die Punktverschiebungen zwischen dem Telluroid und der Erdoberfläche erfolgen nur in vertikaler Richtung.

Im Mittelpunkt der Untersuchungen steht die Erfassung aller Glieder, die den Betrag von etwa 1 cm bei den Höhenanomalien erreichen. Die Lösung wird also soweit entwickelt, daß sie für die Bestimmung des Dezimeter - und des Zentimetergeoids geeignet ist. Die Lösung hat die Form eines geschlossenen Ausdrucks. Es werden keine Reihenentwicklungen eingeführt, deren Konvergenz fraglich ist, und bei denen sich die Größe des Restgliedes nicht abschätzen läßt. Soweit Reihenentwicklungen tatsächlich eingeführt werden, haben sie eine sehr schnelle und gesicherte Konvergenz. Iterationsprozesse werden vermieden.

Die erhaltene Lösung drückt die Höhenanomalien oder das Störpotential an der Erdoberfläche als Funktion von den Schwereanomalien an der Erdoberfläche aus. Das Hauptglied der Lösung wird durch das Stokes-sche Integral gebildet, das über die Faye-Anomalien zu erstrecken ist. Bei diesen Anomalien ist zu den Freiluftanomalien die ebene Geländereduktion der Schwere hinzuaddiert worden.

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Zu diesen Faye-Anomalien tritt noch ein kleiner glatter Ausdruck, der positiv und negativ sein kann, und der sich aus dem Vertikalgradienten der Bougueranomalien ableitet. Ferner treten zum Hauptglied noch 3 oder 4 kleine Nebenglieder hinzu, eins von ihnen kann den Betrag von etwa 50 cm erreichen. Bei diesen Entwicklungen wird eine klare Trennung vorgenommen zwischen den Gliedern die linear und denen die quadratisch in der Höhe sind; die quadratischen sind nur für Aufpunkte im Hochgebirge von Interesse.

Die gefundene Lösung ist für Routineanwendungen geeignet, und sie befriedigt auch die theoretischen Erfordernisse.

Die physikalischen Randwerte an der Erdoberfläche werden nicht herabgesenkt auf die Bezugskugel oder auf das Bezugsellipsoid, sondern die geometrischen Ausdrücke unterliegen Prozeduren, bei denen sie von der Bezugskugel (-ellipsoid) zur Erdoberfläche kommen.

Es wird vorausgesetzt, daß die Geländeneigung endliche und stetige Werte hat, - so wie man sie aus den topographischen Karten entnehmen kann. Jeder Punkt an der Erdoberfläche hat eine eindeutig definierte Tangentialebene.

Резюме

Проводимые за последние 30 лет, начиная с 1989 года, исследования автора по проблеме изложения решения краевой задачи Молоденского посредством идентичности Грина даются в расширенной и доведенной до окончательного вида форме. Дается полное и систематическое объединение всех исследований. Речь, таким образом, идет об инверсии фундаментального уравнения в области физической геодезии. Смещение точек между теллуроидом и поверхностью земли реализуется только в вертикальном направлении.

В центре исследований находится учет всех составляющих, которые при высотных аномалиях достигают приблизительной величины в 1 см. Решение, следовательно, разрабатывается до такой степени, чтобы оно стало пригодным для определения дециметрового и сантиметрового геоида. Решение это имеет форму законченного выражения. Не вводится ни одно разложение в ряд, конвергенция которого стоит под вопросом и величину остаточного члена которого не удается определить заранее. Поскольку же разложения в ряд действительно вводятся, у них имеется очень быстрая и надежная конвергенция. Обходятся без процессов ите-

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рации. Полученное решение выражает высотные аномалии или потенциал помех на поверхности земли в качестве функции гравитационных аномалий на поверхности земли. Главное составляющее решения образуется при помощи интеграла Стокса, который следует распространить и на аномалию Фая. Что касается данных аномалий, то тут к аномалиям Фая прибавлена плоская топографическая поправка на гравитацию. К этой аномалии Фая прибавляется еще небольшое гладкое выражение, которое может быть как положительным, так и отрицательным, и которое является производным от вертикальных градиентов гравиметрической аномалии Буге.

Далее к главному члену добавляются еще 3 или 4 небольших побочных составляющих, одно из которых может достичь приблизительной величины в 50 см. При данном развитии производится недвусмысленное деление между линейными членами и членами - квадратными на высоте; квадратные составляющие представляют интерес только для начальных точек в высокогорьи.

Найденное решение пригодно для рядового использования. Оно также отвечает теоретическим требованиям.

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1. Introduction

The boundary value problem of the physical geodesy deals with the inversion of the fundamental differential equation of the physical geodesy,

$$\Delta g_T = -\frac{\partial T}{\partial r} - \frac{2}{r} T \qquad (1)$$

T is the perturbation potential,
r is the geocentric radius, and
Δg_T is the free-air anomaly.

(1) expresses the free-air anomaly in terms of the perturbation potential. Vice versa, the solution of the boundary value problem gives the perturbation potential in terms of the free-air anomalies. Approaching the problem by the consideration of a spherical model Earth, the solution of the boundary value problem is reduced to the Stokes formula,

$$T = \frac{R}{4\pi} \iint_{1} \Delta g_T \; S(p) \; dl \; . \qquad (2)$$

R is the radius of the globe,
S(p) is the Stokes function,
p is the spherical distance
between the test point P and the point Q running over the sphere in the course of the integration by (2). 1 denotes the unit sphere.

Recognizing the great improvements in the precision of the geodetic measurements, it is no more allowed to introduce a spherical Earth as a substitute for the real Earth as boundary surface. It is necessary to consider the boundary values as continuous functions along the Earth's surface shaped by the topography. This type of a boundary value problem is discussed in the following lines. Thus, the matter to be treated now consists in the problem to find the inversion of the equation (1). The empirically obtained boundary values Δg_T are given along the real surface of the Earth u.
The T values along u are to be represented in terms of these Δg_T values. At the end of this publication, the following solution of this problem is obtained, (267), (268),

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$$T = \frac{R}{4\pi} \iint\limits_{l} \left[\Delta g_T + C + C_1(M) \right] S(p) \quad dl + \left\{ \Omega(M) \right\} . \qquad (3)$$

In case, the test point is situated in low mountain ranges or in the lowlands, the supplementary term $\{\Omega (M)\}$ can be replaced by the term $\{\Omega^*(M)\}$ which can be computed more easily, (272) (273).

In (3), C is the plane terrain reduction of the gravity, C_1 (M) results from the vertical gradient of the refined Bouguer - anomalies, (291) (292),

$$C_1(M) = -\left(H_Q - H_P \right) \frac{\partial}{\partial H} \Delta g_{Bouguer} \cong$$

$$\cong -\left(H_Q - H_P\right) \frac{R^2}{2\pi} \iint\limits_{l} \frac{1}{e_{oo}^3} \left[\left(\Delta g_{Bouguer}\right)_Y - \left(\Delta g_{Bouguer}\right)_Q \right] dl . \qquad (4)$$

H is the height above the globe v, Fig. 2. e_{oo} is the straight distance between the two points Q^* and Y^* on the globe v,

$$e_{oo} = \overline{Q^* , Y^*} = 2 R \sin p/2; \qquad (5)$$

p is here the spherical distance between the two points Q and Y .
In the integration of (4), the point Q is fixed and the point Y is moving.

In the mountains, C can reach an amount of 10 mgal or 50 mgal; in extreme cases, C can be greater than 50 mgal. C is always positive.

But, C_1 (M) has positive and negative amounts.
Only in extreme cases, C_1 (M) can reach an amount of 1 or 2 mgal; [4] pg 12,33,43. C_1 (M) has to be computed in terms of the smoothed potential M or in terms of the smoothed Bouguer - anomalies. M is the perturbation potential T minus the potential of the mountain masses B, having the standard density; the potential of these mountain masses condensed at the globe is in most cases an adequate substitute for B. Thus, the very small and in most cases negligibly small amount of C_1 (M) has the great advantage in our applications that the calculation of it can be handled easily. In the computation of C_1 (M), only the long

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wave-length constituents in the potential M or in the Bouguer-anomalies have to be included, having a wave-length much more great than the differences of the topographical heights. Indeed, these short - wave constituents have a very small impact on the final result for the T or the ζ value, the perturbation potential or the height anomaly of the test point. The impact this short - wave effect exerts on the final ζ value indirectly by way of $C_1(M)$ can be neglected, since it is always smaller than about 0.1 cm, see [6].

As to $\Omega(M)$, this term can be computed by the expressions (268) (224). Probably, the absolute amount of $\Omega(M)$ will never be greater than 0,5 m or 1 m. The right hand side of (3) is, in any case, dominated by the first term of it, being the Stokes integral.

The parentheses $\{\}$ in (3) stand for the regulation that the share of the spherical harmonics of the oth and 1st degree is split off.

As to the philosophy of the equations (1) (2) (3), they base on a mapping between the telluroid t and the surface of the Earth u by means of a vertical point shift, Fig. 1.
The length of the point shift vector is equal to the height anomaly.

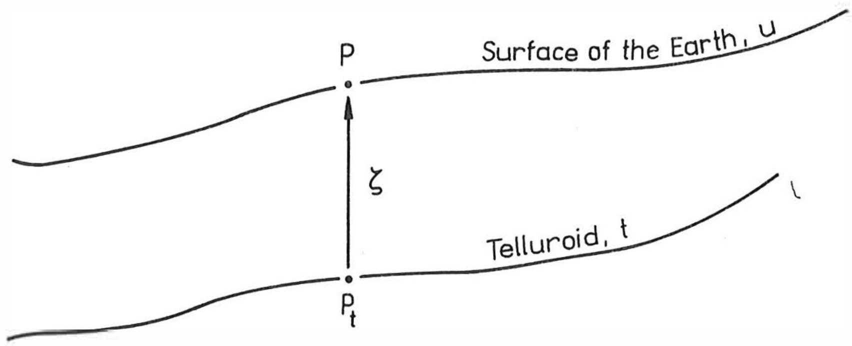

Fig. 1.

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The empirically obtained gravity g refers to the surface point P, the corresponding normal gravity g' is computed for the telluroid point P_t , Fig. 1. Thus,

$$\Delta g_T = g - g' = (g)_P - (g')_{P_t} . \quad (6)$$

After the above lines which give a short description of the Molodenskij type boundary value problem, some other types of boundary value problems are to be sketched. For instance, the scalar gravity potential W and the gradient of W can be introduced as boundary values along the Earth's surface u,

$$W \quad \text{and} \quad \nabla W = \underline{\underline{g}} . \quad (7)$$

The gravity potential along the surface u,

$$W = (W)_u , \quad (8)$$

and the 3 components of the vector

$$\underline{\underline{g}} = (\nabla W)_u = (\underline{\underline{g}})_u \quad (9)$$

represent 4 two - parametric surface functions. If the boundary values (8) and (9) are given, it is possible to replace the vertical point shift vector by an oblique point shift vector. This procedure leads to the determination of the horizontal position of the point P. However along the continents and especially along the oceans, the full gravity vector (9) is given by measurements at rare places, only. Consequently, the boundary value problem having boundary values according to (8) and (9) is not of great importance in our applications.

A boundary value problem of another type (being in the vicinity of the Molodenskij problem) has two surface functions as boundary values. Here, along the surface u, the scalar gravity potential and the length of the gravity vector,

$$(W)_u \quad \text{and} \quad (g)_u , \quad (9a)$$

establish the boundary values. As to the boundary values of the type (9a), it is interesting to discuss the version at which $(g)_u$ is substituted by data derived by satellite observations. The methods of satellite geodesy allow the precise determination of the

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geocentric radius r of the test point P at the Earth's surface, exact to some centimeters. On the other hand, precise levellings lead to precise values of $(W)_u$,(More precise:The difference $(W)_u - (W)_{Geoid}$). From them, by well - known procedures, precise values of the normal heights h' can be computed exact to some centimeters. The following relation is self - explanatory,

$$r = r\,(r_\varepsilon\ ,\ h'\ ,\ \zeta\,)\ . \tag{10}$$

This is the well - known relation which connects the geocentric radius r of the surface u , the geocentric radius of the mean Earth ellipsoid r_ε , the normal height h' , and the height anomaly ζ . A rearrangement of (10) gives the explicit representation of ζ in terms of r, r_ε , h';

$$\zeta = \zeta\,(r,\ r_\varepsilon\ ,\ h')\ . \tag{11}$$

The ζ value of (11) leads to the perturbation potential T by

$$T = g'\zeta = g' \cdot \zeta(r,\ r_\varepsilon\ ,\ h')\ . \tag{12}$$

Thus, the approach considering the couple

$$(W)_u,\ (r)_u \qquad \text{or} \qquad h'\ ,\ (r)_u \tag{13}$$

gives directly the local value of T and ζ by local considerations, (12). Hence, the couple (13) seems to have certain advantages in comparsion with (9a). But this fact is valid, then and then only, if the special occasion is given in which both the values $r = (\,r\,)_u$ and h' are determined within some centimeters.

The solution of the geodetic boundary value problem by the equation (3) is of use also for the solution of the mixed boundary value problems [4][5] .

In the subsequent investigations, the mean ellipsoid of the Earth is replaced by the globe v (with the radius R) being the mean sphere of the Earth. By a supplementary procedure, it is possible to add the transition from the sphere to the ellipsoid. Here, the formulas of Sagrebin and Bjerhammar, for instance, can be of use.

The equation (3) for the solution of our boundary value problem is free of any series development of dubious convergence. It is also free of any series development the residuum of which cannot be evaluated with sufficient precision. (See [4] , page (20)...(24)). It is also free of any series development which does not allow a clear insight into the upper bound of its residuum. A popular suggestion about this upper bound does not suffice in our applications.

Generally, power series developments for T, ξ , η , Δg_T imply certain difficulties; thus, they have a limited efficiency and a limited field of application, only.

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As to the here introduced heights H, they consist of the sum of the normal heights h' and of the height anomalies ζ ,

$$H = h' + \zeta \quad . \tag{14}$$

Since here the mean ellipsoid of the Earth is substituted by the globe v, the H values appear here as the height of the Earth's surface above the globe. In a more precise ellipsoidal consideration, H is the length of the exterior ellipsoidal normal describing the surface u. Beforehand, ζ is an unknown value, indeed. It is the value to be determined even by our here discussed procedure. For the execution of the first step, h' or h' + $\bar{\zeta}$ are convenient approximative substitutes for H, where $\bar{\zeta}$ is an approximative value for ζ .
For a second iteration step, the ζ value obtained by the first step can be introduced into the precise relation (14). But, these considerations are of theoretical value, only. Such an iteration procedure will change the ζ values computed by (3) by not more than about 0,1 cm. It is the effect the transition from h' to $h'+\zeta = H$ takes on C, $C_1(M)$, $\{\Omega(M)\}$, further, by it, on the T value, (3).

In case of a spherical Earth, (10) takes the form

$$r = R + h' + \zeta \quad . \tag{15}$$

ζ is equal to T/g'. The formula for C can be found in [4] , page 24, equation (17);there is valid : $Z = H_Y - H_Q \cong (h')_Y - (h')_Q$,(see Fig. 2,page 15).

2. The identity of Green

In the following developments, the second identity of Green is the basing mathematical relation[1] [3] [4] [5] . For a point $\bar{P}$ in the mass-free exterior space of the Earth, this identity has the subsequent shape,

$$T(\bar{P}) = \frac{1}{4\pi} \iint_u \frac{1}{e(\bar{P},Q)} \frac{\partial T}{\partial n} \, du - \frac{1}{4\pi} \iint_u T \frac{\partial}{\partial n} \left[1/e(\bar{P},Q) \right] du \; . \tag{16}$$

The meaning of the different symbols appearing in the equation (16) can be taken from Fig. 2.

In the subsequent investigations, the slopes of the terrain are considered to have finite and continuous amounts; these amounts of this kind can be taken from the topographical maps, of course. In each point, the surface of the Earth u has a clearly defined tangential plane.

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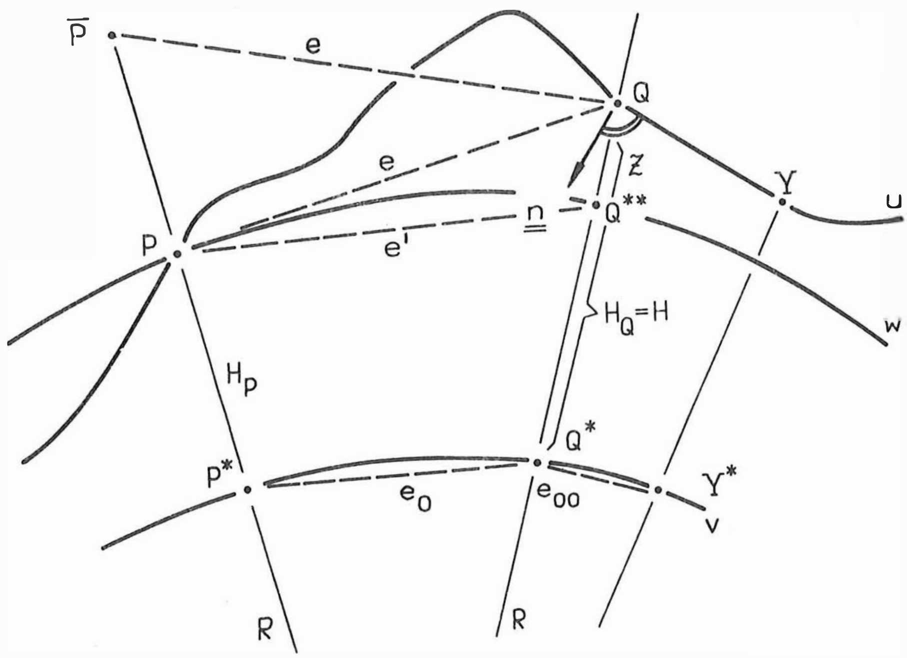

Fig. 2.

u:	: Surface of the Earth,
v	: Mean (geocentric) globe in sea level, R is the radius,
w	: Geocentric sphere, R + H_P is the radius,
P	: Fixed test point at the surface of the Earth u,
Q	: A point on u, moving during the integrations with P as fixed test point,
Y	: A point on u, moving during the integrations with Q as fixed test point,
P^*, Q^*, Y^*	: The vertical projections of the points P, Q, Y on v,
Q^{**}	: The perpendicular projection of the point Q on w,
$\bar{P}$	: A point perpendicular above the test point P,
e	: Straight distance between P and Q, ($\bar{P}$ and Q),
e', e_o, e_{oo}	: Straight distance between P and Q^{**}, resp. P^* and Q^*, resp. Q^* and Y^*,
H_P, H_Q	: Height of P, Q above the globe v,
Z	: The difference of H_Q minus H_P.

In (16), we have the 3. identity of Green. This identity contains the oblique derivatives with regard to the normal n of the oblique surface of the Earth u. Thus, in the course of our deductions, these oblique derivatives give rise to the fact that the slope of the terrain turns up in the formulas. This slope would be difficultly to handle in numerical computations. But, by the method of integrations by parts,(A 270), this slope can be avoided, and, instead of it, the deflections of the vertical appear. The smoothing procedure governed by the M potential turns these deflections into very smoothed values easily to compute. Thus, the "oblique" method makes no principle trouble, finally.

If the relation (16) is assigned to the class of the "oblique methods" , this is a more qualitative and mathematical depiction. It is not quantitative, but natural science is more quantitative than qualitative.

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$\underline{\underline{n}}$ is the unit vector normal to the Earth's surface u, it is heading into the interior of the Earth.

The test point $\bar{P}$ is subsided down to the surface of the Earth. Thereby, (16) turns to (17),

$$T(P) = \frac{1}{2\pi} \iint_u \frac{1}{e(P,Q)} \frac{\partial T}{\partial n} du - \frac{1}{2\pi} \iint_u T\left[\frac{\partial}{\partial n} \frac{1}{e(P,Q)}\right] du \; . \quad (17)$$

3. The spherical solution

The spherical solution of the relation (17) is obtained if the height values H are set equal to zero. If H does go to zero, the straight distance e does go to e_o, Fig. 2,3,

$$e_o = 2 R \sin p/2 \quad , \qquad (17a)$$

and the derivation with regard to n turns to the derivation with regard to r, but with the inverse sign.
Further, Fig. 3,

$$\frac{\partial}{\partial n} (1/e) \longrightarrow - \frac{\partial}{\partial r} (1/e_o) = \frac{1}{e_o^2} \frac{\partial e_o}{\partial r} \quad , \qquad (18)$$

and

$$\frac{de_o}{dr} = \sin p/2 = \frac{e_o}{2R} \quad . \qquad (19)$$

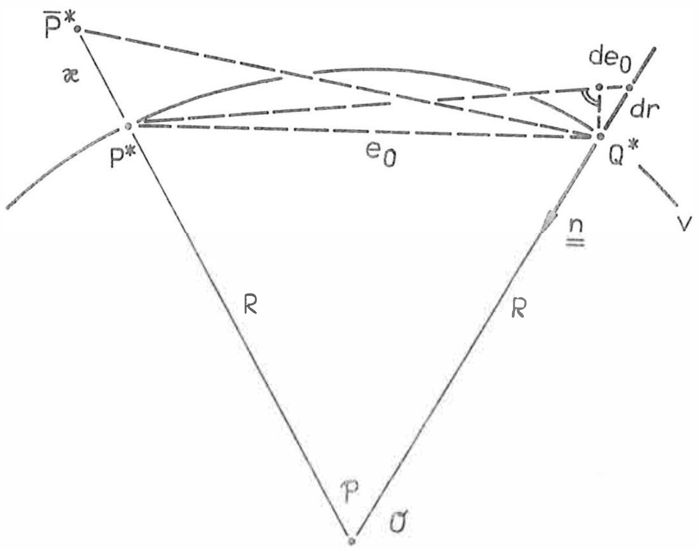

Fig. 3.

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Consequently, if the heights H diminish down to zero, the following transition behavior is valid,

$$\left[\frac{\partial}{\partial n}(1/e)\right] \rightarrow \frac{1}{2\, e_o R} \quad . \tag{20}$$

With (20), the spherical variant of the relation (17) gets the subsequent shape,

$$T = -\frac{1}{2\pi}\iint_v \frac{1}{e_o}\cdot\frac{\partial T}{\partial r}\cdot dv - \frac{1}{2\pi}\iint_v T\cdot\frac{1}{2\, e_o R}\cdot dv \ . \tag{21}$$

The spherical variant of (1) is,

(r = R),

$$\Delta g_T = -\frac{\partial T}{\partial r} - \frac{2}{R}\cdot T \quad . \tag{22}$$

(21) and (22) are combined to

$$T = \frac{1}{2\pi}\iint_v \frac{1}{e_o}\cdot\Delta g_T\cdot dv + \frac{3}{4\pi R}\iint_v \frac{1}{e_o}\cdot T\cdot dv \quad . \tag{23}$$

T and Δg_T are continuous functions along the surface of the Earth. They have the following spherical harmonics developments,

$$T = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[T_{1.n.m}\cdot R_{n.m}(\varphi,\lambda) + T_{2.n.m}\cdot S_{n.m}(\varphi,\lambda)\right] , \tag{24}$$

$$\Delta g_T = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[G_{1.n.m}\cdot R_{n.m}(\varphi,\lambda) + G_{2.n.m}\cdot S_{n.m}(\varphi,\lambda)\right] . \tag{25}$$

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The corresponding development for the inverse of the distance between the two points $\bar{P}^*$ and Q^*, Fig. 3, is,

$$\frac{1}{e\,(\bar{P}^*,\,Q^*)} = \frac{1}{\overline{\bar{P}^*,\,Q^*}} = \sum_{n=0}^{\infty} \frac{R^n}{(R+\varkappa)^{n+1}} P_n(\cos p),\ \varkappa > 0. \qquad (26)$$

P_n are the Legendre functions.

The decomposition formula of the spherical harmonics is introduced in (26). (27) follows,

$$\frac{1}{e\,(\bar{P}^*,\,Q^*)} = \sum_{n=0}^{\infty} \frac{R^n}{(R+\varkappa)^{n+1}} \cdot \frac{1}{2n+1} \sum_{m=0}^{n} \Xi_{n.m}\ , \qquad (27)$$

$$\Xi_{n.m} = R_{n.m}(\varphi,\lambda)\cdot R_{n.m}(\varphi',\lambda') + S_{n.m}(\varphi,\lambda)\cdot S_{n.m}(\varphi',\lambda')\ . \qquad (27a)$$

$T_{1.n.m}$, $T_{2.n.m}$ and $G_{1.n.m}$, $G_{2.n.m}$

are the Stokes constants of the developments (24) and (25). φ and λ is the geocentric latitude and longitude of the test point $\bar{P}^*$; φ' and λ' are the corresponding parameters for the point Q^*, moving over the globe in the course of the integration of (23), Fig. 3. $R_{n.m}(\varphi,\lambda)$ and $S_{n.m}(\varphi,\lambda)$ are the well-known normalized spherical harmonics of the degree n and of the order m,

$$\iint_v R_{n.m}(\varphi,\lambda)\cdot R_{i.k}(\varphi,\lambda)\,dv = \begin{cases} 0 & ;\ n \neq i \text{ or } m \neq k \text{ or both} \\ 4\pi R^2 & ;\ n = i,\ m = k \end{cases}, \qquad (28)$$

$$\iint_v S_{n.m}(\varphi,\lambda)\cdot S_{i.k}(\varphi,\lambda)\,dv = \begin{cases} 0 & ;\ n \neq i \text{ or } m \neq k \text{ or both} \\ 4\pi R^2 & ;\ n = i,\ m = k \end{cases}. \qquad (29)$$

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The relations from (24) to (29) are introduced into the integral equation (23).
The following equation for the Stokes constants is obtained, for $\varkappa \to 0$,

$$T_{1.n.m} = \frac{1}{2\pi R} \cdot \frac{1}{2n+1} \cdot G_{1.n.m} \cdot 4\pi R^2 + \tag{29a}$$
$$+ \frac{3}{4\pi R^2} \cdot \frac{1}{2n+1} \cdot T_{1.n.m} \cdot 4\pi R^2 ,$$

or

$$T_{1.n.m} = R \cdot \frac{1}{n-1} \cdot G_{1.n.m} \quad . \tag{29b}$$

And, in an analogous way,

$$T_{2.n.m} = R \cdot \frac{1}{n-1} \cdot G_{2.n.m} \quad . \tag{30}$$

By way of trial, it is supposed that the Stokes integral of the form (31) is a solution of the integral equation (23),

$$T = \frac{1}{4\pi R} \iint_{v} \Delta g_T \cdot S(p) \cdot dv \quad . \tag{31}$$

The correctness of (31) is easily verified in the following.
Indeed, the Stokes function S(p) has the relation

$$S(p) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \cdot P_n(\cos p) \quad . \tag{32}$$

$P_n(\cos p)$ are the Legendre functions.
The relations (24) (25) (26) (27) and (32) are put into (31).
The following equation yields

$$T_{1.n.m} = \frac{1}{4\pi R} \cdot G_{1.n.m} \cdot \frac{2n+1}{n-1} \cdot \frac{1}{2n+1} \cdot 4\pi R^2 \quad , \tag{33}$$

or

$$T_{1.n.m} = R \cdot \frac{1}{n-1} \cdot G_{1.n.m} \tag{34}$$

(34) corroborates (29b) and (30).

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Consequently, it is verified that the Stokes integral (31) is the solution of the identity of Green for a spherical model Earth, (23).

4. The decomposition of the identity of Green into the spherical and the topographical constituents

The identity of Green of the shape of (17) refers to the real surface of the Earth u. The oblique straight line e, the unit normal vector $\underline{\underline{n}}$ of the surface u, and the surface element du refer to the oblique surface of the Earth u shaped by the topography. All the two integrands on the right hand side of (17) come now to be multiplied with and divided through the term cos (g', n). $\sphericalangle$ (g', n) is the angle defined by the positive directions of the two vectors $\underline{\underline{g}}'$ and $\underline{\underline{n}}$, taken for points on the surface of the Earth u. $\underline{\underline{g}}'$ is the vector of the standard gravity heading into the interior of the Earth. In case of the here chosen spherical standard Earth, $\underline{\underline{g}}'$ points always to the centre O of this sphere. $\underline{\underline{n}}$ is also heading into the interior of the Earth, Fig. 4.

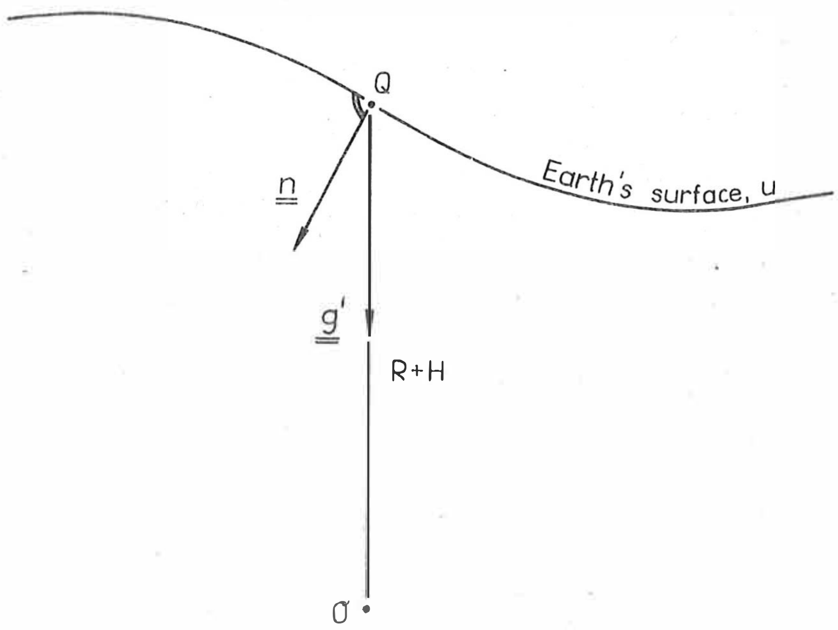

Fig. 4.

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Along these lines, (17) turns to

$$T(P) = \frac{1}{2\pi} \iint\limits_{u} \frac{1}{e(P,Q)} \cdot \frac{\partial T}{\partial n} \cdot \frac{1}{\cos(g', n)} \cdot du \cdot \cos(g', n) -$$

$$- \frac{1}{2\pi} \iint\limits_{u} T \cdot \frac{\partial \left(\frac{1}{e(P,Q)}\right)}{\partial n} \cdot \frac{1}{\cos(g', n)} \cdot du \cdot \cos(g', n) \quad . \qquad (35)$$

Now, the terms in the two integrands of (35) are decomposed into their spherical parts and into the residual non - spherical parts of them. The latter parts vanish if the heights H tend to zero, Fig. 2.
The following equations (36) to (39) governe the decomposition procedure,

$$\frac{\partial T}{\partial n} \cdot \frac{1}{\cos(g', n)} = - \frac{\partial T}{\partial r} + D(1.1) = K_1 + K_1' \quad , \qquad (36)$$

$$\frac{1}{e(P,Q)} = \frac{1}{e} = \frac{1}{e'} + D(1.2) = K_2 + K_2' \quad , \qquad (37)$$

$$\frac{\partial \frac{1}{e}}{\partial n} \cdot \frac{1}{\cos(g', n)} = - \frac{\partial \frac{1}{e'}}{\partial r} + D(1.3) = K_3 + K_3' \quad , \qquad (38)$$

$$du \cdot \cos(g', n) = dw + D(1.4) = K_4 + K_4' \quad ; \qquad (39)$$

$$dw = (R + H_P)^2 \cdot \cos\varphi \cdot d\varphi \cdot d\lambda \quad , \qquad (40)$$

$$e' = 2 \cdot (R + H_P) \cdot \sin p/2 \quad . \qquad (40a)$$

The meaning of the symbols K_1, K_1', K_2, K_2', K_3, K_3', K_4, K_4' follows even from the relations (36) to (39).
These relations, (36) to (39), are now introduced into (35).
(41) follows,

$$2\pi T = \iint\limits_{u} (K_2 + K_2') \cdot (K_1 + K_1') \cdot (K_4 + K_4') -$$

$$- \iint\limits_{u} T \cdot (K_3 + K_3') \cdot (K_4 + K_4') \quad . \qquad (41)$$

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Generally, the primed terms K_1', K_2', K_3', K_4' are much more small than the terms K_1, K_2, K_3, K_4. Hence, the multiplications in the integrands of (41) should consider only such products of three factors which have not more than one primed term K_1' or K_2' or K_3' or K_4'. There is only one exception, it is the product $K_2' \cdot K_1' \cdot K_4$.
Along these lines, the integrand of the first integral on the right hand side of (41) gets the form

$$K_2 \; K_1 \; K_4 + K_2 \; K_1 \; K_4' + K_2 \; K_1' \; K_4 + K_2 \; K_1' \; K_4' +$$

$$+ K_2' \; K_1 \; K_4 + K_2' \; K_1 \; K_4' + K_2' \; K_1' \; K_4 + K_2' \; K_1' \; K_4' \cong$$

$$\cong K_2 \; K_1 \; K_4 + K_2 \; K_1 \; K_4' + K_2 \; K_1' \; K_4 + K_2' \; K_1 \; K_4 + K_2' \; K_1' \; K_4 \; . \qquad (42)$$

Analogously, the two braces in the integrand of the second integral of (41) yield

$$K_3 \; K_4 + K_3 \; K_4' + K_3' \; K_4 + K_3' \; K_4' \cong$$

$$\cong K_3 \; K_4 + K_3 \; K_4' + K_3' \; K_4 \; . \qquad (43)$$

The introduction of (42) (43) and (36) (37) (38) (39) into (41) gives

$$2\pi T = -\iint\limits_W \frac{\partial T}{\partial r} \cdot \frac{1}{e'} \cdot dw - \iint\limits_W \frac{\partial T}{\partial r} \cdot \frac{1}{e'} \cdot D(1.4) -$$

$$- \iint\limits_W \frac{\partial T}{\partial r} D(1.2) \cdot dw + \iint\limits_W \frac{1}{e'} \cdot D(1.1) \cdot dw +$$

$$+ \iint\limits_W D(1.1) \cdot D(1.2) \cdot dw + \iint\limits_W T \frac{\partial \frac{1}{e'}}{\partial r} dw +$$

$$+ \iint\limits_W T \cdot \frac{\partial \frac{1}{e'}}{\partial r} \cdot D(1.4) - \iint\limits_W T \cdot D(1.3) \cdot dw. \qquad (44)$$

The 2nd, 3rd, 5th, 7 th, and the 8th term on the right hand side of (44) are put together under the denomination D(2.1),

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$$D(2.1) = - \iint_W \frac{\partial T}{\partial r} D(1.2) \quad dw - \iint_W \frac{\partial T}{\partial r} \frac{1}{e'} D(1.4) +$$

$$+ \iint_W T \frac{\partial \left(\frac{1}{e'}\right)}{\partial r} D(1.4) - \iint_W T \cdot D(1.3) \cdot dw +$$

$$+ \iint_W D(1.1) \cdot D(1.2) \cdot dw \;. \qquad (45)$$

The 5 expressions on the right hand side of (45) get individual denominations,

$$E\,(1) = - \iint_W \frac{\partial T}{\partial r} D(1.2) \cdot dw, \qquad (45a)$$

$$E\,(2) = - \iint_W T \cdot D(1.3) \cdot dw\,, \qquad (45b)$$

$$E\,(3) = - \iint_W \frac{\partial T}{\partial r} \frac{1}{e'} \cdot D(1.4)\,, \qquad (45c)$$

$$E\,(4) = \iint_W T \frac{\partial \frac{1}{e'}}{\partial r} D(1.4)\,, \qquad (45d)$$

$$E\,(5) = \iint_W D(1.1) \cdot D(1.2) \cdot dw \;. \qquad (45e)$$

Consequently, (45),

$$D(2.1) = E(1) + E(2) + E(3) + E(4) + E(5) \qquad (45f)$$

(44) and (45) are combined to

$$2 \pi T = \iint_W \left[- \frac{\partial T}{\partial r} + D(1.1) \right] \frac{1}{e'} \cdot dw +$$

$$+ \iint_W T \frac{\partial \frac{1}{e'}}{\partial r} \cdot dw + D(2.1) \qquad (46)$$

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The terms on the right hand side of (46) are now rearranged, in order to bring them into a shape which is convenient for numerical routine calculations.

In this context, the following relations are of use, Fig. 2 and 3, equation (1),

$$- \frac{\partial T}{\partial r} = \Delta g_T + \frac{2}{r} T , \qquad (47)$$

$$e' = 2(R + H_P) \sin p/2 = 2 R' \sin p/2, \qquad (48)$$

$$\frac{1}{e'} = \frac{1}{2 R' \sin p/2} , \qquad (49)$$

$$\frac{\partial e'}{\partial r} = \sin p/2 = \frac{e'}{2R'} , \qquad (50)$$

$$\frac{\partial\left(\frac{1}{e'}\right)}{\partial r} = - \left(\frac{1}{e'}\right)^2 \frac{\partial e'}{\partial r} , \qquad (51)$$

$$\frac{\partial\left(\frac{1}{e'}\right)}{\partial r} = - \frac{1}{2 e' R'} = - \frac{1}{4 R'^2 \sin p/2} , \qquad (52)$$

$$R' = R + H_P . \qquad (53)$$

p is the spherical distance, for instance between the points P and Q, or between Q and Y, Fig. 2. H_P is the height of the test point P above the sphere v, having the radius R, Fig. 2. The sphere w has the radius R', (48) (53), and the surface element dw is

$$dw = (R')^2 \, dl = (R')^2 \cos \varphi \, d\varphi \, d\lambda , \qquad (54)$$

dl is the surface element of the unit-sphere. φ and λ is the geocentric latitude and longitude. With (47) to (53), the equation (46) turns to (55),

$$2 \pi T = \iint_w \left[\Delta g_T + \frac{2}{r} T + D(1.1) \right] \frac{1}{2 R' \sin p/2} \, dw -$$

$$- \iint_w T \frac{1}{4(R')^2 \sin p/2} \, dw + D(2.1) . \qquad (55)$$

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Further,

$$4\cdot\pi\cdot R'\cdot T = \iint_w \left[\Delta g_T + D(1.1)\right]\cdot\frac{1}{\sin p/2}\cdot dw \; +$$

$$+ \iint_w \left[\frac{2}{r} - \frac{1}{2\cdot R'}\right]\cdot T\cdot\frac{1}{\sin p/2}\cdot dw + 2\cdot R'\cdot D(2.1) \quad . \tag{56}$$

The term in the brackets of the second integral on the right hand side of (56) is transformed and expressed by the heights. The radius r of the point Q is, Fig. 2,

$$r = R' + Z = R + H_P + Z \quad , \tag{57}$$

$$Z = H_Q - H_P \quad . \tag{57a}$$

Hence, (57),

$$r = R'\cdot(1 + \frac{Z}{R'}) \quad , \tag{58}$$

$$\frac{1}{r} = \frac{1}{R'}\cdot(1 + \frac{Z}{R'})^{-1} \quad , \tag{59}$$

$$\frac{1}{r} = \frac{1}{R'}\cdot\left[1 - \frac{Z}{R'} + \left(\frac{Z}{R'}\right)^2 - + \ldots\right], \quad \left|\frac{Z}{R'}\right| < 1 \quad . \tag{60}$$

Consequently, for the expression in the second brackets of (56),

$$\frac{2}{r} - \frac{1}{2\,R'} = \frac{3}{2\,R'} - \frac{2\,Z}{(R')^2} + \frac{2}{R'}\cdot\left(\frac{Z}{R'}\right)^2 \quad . \tag{61}$$

(56) and (61) yield

$$4\cdot\pi\cdot R'\cdot T = \iint_w \left[\Delta g_T + D(1.1)\right]\cdot\frac{1}{\sin p/2}\cdot dw \; +$$

$$+ \frac{3}{2\cdot R'}\iint_w T\cdot\frac{1}{\sin p/2}\cdot dw + 2\cdot R'\cdot D(2.1) \; +$$

$$+ \iint_w \frac{T}{R'}\cdot\left[-\frac{2\cdot Z}{R'} + 2\cdot\left(\frac{Z}{R'}\right)^2\right]\frac{1}{\sin p/2}\,dw \quad . \tag{61a}$$

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(61a) and (50) give rise to the following equation (61b) ,

$$4 \cdot \pi \cdot R' \cdot T = \iint_w \left[\Delta g_T + D(1.1) \right] \cdot \frac{1}{\sin p/2} \cdot dw +$$

$$+ \frac{3}{2R'} \cdot \iint_w T \cdot \frac{1}{\sin p/2} \cdot dw + 2 \cdot R' \cdot D(2.1) +$$

$$+ 2 \cdot R' \cdot \iint_w \left(\frac{T}{R'} \right) \cdot \left[- \frac{2Z}{R'} + 2 \cdot \left(\frac{Z}{R'} \right)^2 \right] \cdot \frac{1}{e'} \cdot dw. \quad (61b)$$

A lot of rearrangements, given in the appendix, leads to an expression for D(2.1) which is convenient for numerical routine calculations in our applications. D(2.1) has the subsequent development,

$$D(2.1) = F_1 + F_2 \quad . \qquad (62)$$

The explicit expression for F_1 is represented by the relations (A 484)(A 485)(A 471) (A 472)(A 473) in the appendix. These expressions of the appendix are convenient for routine calculations. The amount of F_1 is relative small. Obviously, F_1 will have an amount of not more than about a relative change in the height anomalies ζ by Z/R . With ζ = 100 m and Z = 3 km, a F_1 value of about 5 cm follows, only. By (A 485), F_1 can be computed by a global 10^o x 10^o compartment division of T, Δg_T , and H, - or by any equivalent procedure, for instance. The computation of F_1 by means of (A 485), introducing the T -, Δg_T -, and the H - values, can be handled easily by a computer. The formula (A 485) demands an extension of these calculations over the whole globe.

The F_2 value of (62) is described by the formula (A 486); the terms on the right hand side of (A 486) are represented by (A 474) ... (A 477). Also, the F_2 values are small.

For test point P situated in the lowlands or in low mountain ranges, the F_2 values will have negligible amounts, always. This fact is very probable.

The development for F_1 is of general importance. It is of importance for both cases, for high mountain test points and for lowland test points P . But, the development for F_2 is of practical importance for high mountain test points, only.

For test points P situated in high mountains, only in this case, the value of F_2 will reach such amounts which are of interest in our applications, possibly. But, to be sure and to avoid misunderstandings, also for high mountain test points P , the F_2 values will never take a dominating share on the finally computed T values

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values of the test points. Also in case of a topography with extreme cliffs, the computation of the F_2 term can be handled without any complication and without any singularity.

Also for test points P situated in high mountain ranges, the F_2 values will exert an impact on the height anomaly of the test point which is not greater than some centimeters, hardly surmounting the standard deviation of these ζ values in the high mountains.

The calculation procedure of (A 486) giving F_2 has to cover only the near surroundings of the test point up to a distance of 100 km or 1 000 km about.
These calculations can be handled easily by a computer which is fed with approximative amounts of T, Δg_T and Z.

Exterior of the high mountains, the simplified expression $F_{1.1}$ of (A 487) (A 495) (A 497) is always adequate in our applications, instead of $F_1 + F_2$.
Thus, (62) turns to

$$D(2.1) \cong F_{1.1} \quad , \text{ for : } x^2 = (Z/e')^2 \ll 1 , \tag{63}$$

for test points exterior of the high mountains. The computation of $F_{1.1}$ can be handled easily.

The 3rd and 4th term on the right hand side of (61b) gets the abbreviating denomination F, after a division through 2 R',

$$F = D(2.1) + \iint_w \left(\frac{T}{R'} \right) \left[- \frac{2Z}{R'} + 2 \left(\frac{Z}{R'} \right)^2 \right] \left(\frac{1}{e'} \right) dw \quad . \tag{64}$$

This expression for F, (64), implies only topographical terms, i. e. terms depending on the height differences Z. If Z does tend to zero all over the globe, in this case, F does tend to zero simultaneously.
The relations (62) and (64) can be combined,

$$F = F_1 + F_2 + \iint_w \left(\frac{T}{R'} \right) \left[- \frac{2Z}{R'} + 2 \left(\frac{Z}{R'} \right)^2 \right] \left(\frac{1}{e'} \right) dw \quad . \tag{65}$$

The reliefs, which follow by the transition from (62) to (63), are now put into the fore. This transition is governed by the condition that the test point P has to lie in the lowlands and not in the high mountains; thus, (A 487),

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$$x^2 = \left(\frac{Z}{e'}\right)^2 \ll 1 \quad . \qquad (66)$$

Further, this transition implies the neglection of relative errors of the order of Z/R. The details of this transition are described in the appendix by the equations from (A 485) to (A 497). Even these reliefs, ((66) and toleration of relative errors of Z/R), transform the equation (65) into the equation (67), by the transition

$$F \rightarrow F^{*} \quad ; \quad x^2 \ll 1 \quad . \qquad (66a)$$

Thus, (A 499),

$$F^{*} = F_{1.1} + \iint_{w} \left(\frac{T}{R'}\right)\cdot\left(-\frac{2Z}{R'}\right)\cdot\left(\frac{1}{e'}\right)\cdot dw \quad , \; x^2 \ll 1 \; , \qquad (67)$$

or, equating R' with R in sufficient approximation,

$$F^{*} = F_{1.1} + \iint_{w} \left(\frac{T}{R}\right)\cdot\left(-\frac{2Z}{R}\right)\cdot\left(\frac{1}{e'}\right)\cdot dw \quad , \; x^2 \ll 1 \; . \qquad (68)$$

After this consideration of the functions F, F_1, F_2, $F_{1.1}$, F^{*}, now the identity of Green is in the fore again. (61b) and (64) yield, dividing (61b) through 2 R',

$$2\cdot\pi\cdot T = \iint_{w}\left[\Delta g_T + D(1.1)\right]\left(\frac{1}{e'}\right)\cdot dw \; +$$

$$+ \iint_{w} \frac{3}{2}\cdot\frac{T}{R'}\cdot\frac{1}{e'}\cdot dw \qquad + F \; . \qquad (69)$$

In (69), only the two terms D(1.1) and F depend on the topographical heights H. All the other terms of the equation (69) do not depend on the heights H, they are described by pure spherical expressions.

A short discussion about the topographical terms D(1.1) and F, of (69), seems to be convenient to be added.

The term D(1.1) in (69) refers to the potential T. This speciality is denoted by the suffix T in the following lines, D_T (1.1), since later on, D(1.1) is also understood to refer to another potential. From the appendix , by the equation (A 21), $D_T(1.1)$ is

$$D_T(1.1) = e\cdot g\cdot \tan(g', n)\cdot \cos(A'' - A') \quad . \qquad (70)$$

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In the above equation (70), the symbol Θ is introduced, it is here the absolute amount of the plumb-line deflection, for the potential T,

$$\Theta^2 = \xi^2 + \eta^2 \quad , \qquad (71)$$

ξ and η : the north-south and the east-west component of the plumb-line deflection at the Earth's surface u,

$$\xi = -\frac{1}{R' + Z} \cdot \frac{1}{g} \cdot \frac{\partial T}{\partial \varphi} \quad , \qquad (72)$$

$$\eta = -\frac{1}{R' + Z} \cdot \frac{1}{\cos \varphi} \cdot \frac{1}{g} \cdot \frac{\partial T}{\partial \lambda} \quad . \qquad (73)$$

φ and λ are the geocentric latitude and longitude, in (72) and (73). Here, the globe v was taken as reference figure, φ and λ refer to this globe v, also. R' + Z is the geocentric radius r of the moving point Q at the Earth's surface u, Fig. 2. g is the real gravity at this point Q. A' is the azimuth of the slope of the terrain, counted clockwise from the north, (see Fig. A 1). A" is the azimuth of the plumb-line deflection Θ , counted clockwise from the north.

The north-south and the east-west derivatives of the perturbation potential T are understood that they are taken in horizontal direction ; thus, r is constant during these derivations of T.

As to the expression for F(T) , being equal to the function F of the equation (64), the detailed, complete, and comprehensive development for it, valid also in the high mountains, is given by (64), (A 461), (A 462) to (A 468), and from (A 471) to (A 477). Along these lines, the following universally valid formula for F(T) is found , neglecting the powers of $(Z/R')^2$ in 2. term on the right hand side of (64),

$$F(T) = D(2.1) + \iint_w \frac{T}{R'} \cdot \left[- \frac{2 \cdot Z}{R'} \right] \frac{1}{e'} \cdot dw \quad ,$$

$$F(T) = \sum_{i=1}^{8} f_i(T) \quad ; \qquad (74)$$

$$f_1(T) = \iint_w \Delta g_T \cdot \frac{Z}{R} \cdot \left[2 - \frac{1}{y + y^2} \right] \cdot \frac{1}{e'} \cdot dw \quad , \qquad (74a)$$

$$f_2(T) = \iint_w \frac{T}{R} \cdot \frac{Z}{R} \left[1 - \frac{2}{y + y^2} \right] \frac{1}{e'} \cdot dw \quad , \qquad (74b)$$

$$f_3(T) = \iint_w \frac{T}{R} \cdot \frac{v_1}{R} \cdot dw \quad , \qquad (74c)$$

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$$f_4(T) = - \iint_w \frac{\partial T}{R\,\partial p} \cdot \frac{1}{R} \cdot \frac{(\cos p/2)^2}{\sin p} \cdot b_7 \cdot dw \quad , \qquad (74d)$$

$$f_5(T) = - \iint \Delta g_T \cdot \frac{x^2}{y + y^2} \cdot de' \cdot dA \quad , \qquad (74e)$$

$$f_6(T) = \iint \frac{T}{R} \cdot \left[- \frac{2 \cdot x^2}{y + y^2} + v_3 \right] \cdot de' \cdot dA \quad , \qquad (74f)$$

$$f_7(T) = \iint \frac{\partial T}{\partial e'} \cdot (v_2 - b_{11}) \cdot de' \cdot dA \quad , \qquad (74g)$$

$$f_8(T) = - \iint g \cdot Z \cdot \Phi (x^* \cdot \xi , x^* \cdot \eta) \cdot de' \cdot dA \quad . \qquad (74h)$$

A is the azimuth, variating during the integration from the north, from zero to 2π , counted clockwise.

In the expressions for f_1, f_2, f_3, f_4, the integration covers whole the globe. But in the integrals for f_5, f_6, f_7, f_8, the integration has to be extended over the surroundings of the test point P, only, up to a distance of not more than about 100 km.

The equations from (74) up to (74h) contain the following abbreviations, (A 39) (A 40) (A 393) (A 395) (A 375),

$$x^* = \left[x^2 + \frac{e' \cdot x}{R'} \right] \cdot \frac{1}{x' + (x')^{1/2}} \quad , \qquad (75)$$

$$x = \frac{Z}{e'} \quad , \qquad (76)$$

$$x' = 1 + x^2 + \frac{Z}{R'} \quad , \qquad (77)$$

$$y^2 = 1 + x^2 \quad . \qquad (78)$$

The considerations connected with the transition procedure described by (66a), and also the deliberations about the validity of the equation (67), have demonstrated that the expression for F, (64), can be replaced by the more simple expression for F^* , (68), — at least in the lowlands and in not too rugged mountains. Only in high mountains, the universally valid formula (64) will be better than the simple form (68) of F^*.

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Thus, (66a), F^* has the following detailed expression which is convenient for routine calculations, (A 497) (66),

$$F^* = F^*(T) = \sum_{i=1}^{3} f_i^*(T) \; ; \qquad (79)$$

$$f_1^*(T) = \iint_w \Delta g_T \frac{Z}{R} \frac{3}{2} \frac{1}{e'} \, dw \; , \qquad (79a)$$

$$f_2^*(T) = \iint_w \frac{T}{R} \frac{Z}{R} \frac{1}{e'} \, dw \; , \qquad (79b)$$

$$f_3^*(T) = -\iint_w \frac{\partial T}{R \partial p} \frac{Z}{4R^2} \frac{\cos p/2}{(\sin p/2)^2} \, dw \; . \qquad (79c)$$

In the above lines, by the relations from (35) to (69), it was discussed how the pure spherical constituents (being free of the heights H) in the identity of Green can be separated from the topographical constituents D(1,1) and F (which tend to zero if the heights H tend to zero).

The functions v_1, v_2, v_3, b_7, b_{11}, which appear in the relations from (74) to (74h), should be given in detail, here. From (A 307) to(A 346) follows:

$$v_1 = (1/2)\cdot(x + \operatorname{arsinh} x) \; ; \qquad (80)$$

$$v_2 = -x\cdot(1/y) + \operatorname{arsinh} x + (\sin p/2)\cdot\left\{1 - (3/y) + 2\cdot y\right\} \; ,$$

$$, \; -\infty < x < +\infty \; , \quad e' < 1000 \text{ km} \; ; \qquad (81)$$

$$v_3 = 1 + (1/2)\cdot y - (3/2)\cdot(1/y) + x^2\cdot(1/2)\cdot\left\{-(1/y) + (1/y)^3\right\} +$$

$$+ x^3\cdot(1/y)^3\cdot(\sin p/2) + x^4\cdot(1/2)\cdot(1/y)^3 \; ,$$

$$, \; -\infty < x < +\infty \; , \quad e' < 1000 \text{ km} \; ; \qquad (82)$$

$$b_7 = \operatorname{arsinh} x \; ; \qquad (83)$$

$$b_{11} = x\cdot x^*(P,Q) = \left\{x^3 + \frac{(e')\cdot x^2}{R'}\right\}\cdot\frac{1}{x' + \sqrt{x'}} \; . \qquad (84)$$

b_{11} comes from (75) and (A 439).

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5. The representation of the perturbation potential T by the Stokes integral and the topographical supplements

It is generally acknowledged that the Stokes integral (31) is a good approximation to the precise shape of the solution of the integral equation (69).
Therefore, it is intended here to bring the precise solution of (69) in such a form which has the Stokes integral as the dominating main term, and which has to be completed by the addition of some supplementary topographical terms. The latter go to zero if the heights go to zero. Following up this problem, it is convenient to bring the relation (69) into the subsequent form,

$$T = \frac{1}{4\pi R'} \iint_w \frac{\alpha}{\sin p/2} \cdot dw + \frac{3}{8\pi (R')^2} \iint_w \frac{T}{\sin p/2} \cdot dw + \beta \quad . \qquad (85)$$

Here is, Fig. 2,

$$\alpha = \Delta g_T + D_T\,(1.1) \quad , \qquad (86)$$

$$\beta = \frac{1}{2\pi} \cdot F \quad , \qquad (87)$$

$$R' = R + H_P \quad , \qquad (88)$$

$$dw = R'^2 \cdot \cos\varphi \cdot d\varphi \cdot d\lambda \quad , \qquad (89)$$

$$dw = R'^2 \cdot \sin p \cdot dp \cdot dA \quad . \qquad (90)$$

R' is the geocentric radius of the test point P at the Earth's surface, Fig. 2. (85) can be brought into the following shape,

$$\frac{T}{R'} = \frac{1}{4\pi} \iint_1 \frac{\alpha}{\sin p/2} \cdot dl + \frac{3}{8\pi} \iint_1 \frac{T}{R'} \cdot \frac{1}{\sin p/2} \cdot dl + \frac{\beta}{R'} \quad , \qquad (91)$$

with

$$dl = \left(\frac{1}{R'}\right)^2 \cdot dw = \cos\varphi \cdot d\varphi \cdot d\lambda \quad . \qquad (92)$$

The functions α , $\frac{\beta}{R'}$, $\frac{T}{R'}$ and sin p/2, appearing in (91), can be represented in terms of the geocentric latitude and longitude, φ and λ , of the running point, Fig. 2. These functions can be given by series developments in spherical harmonics, because α and (β/R'), and (T/R') are continuous functions of φ and λ , (94) (95) (96) (97).

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For the sake of briefness and clarity in the further deductions, the following harmonics developments are not written down up to the last detail. Considering the harmonics of the degree n, not all the concerned zonal, tesseral and sectorial harmonics of the degree n are written down in the following lines. As usual, to have expressions easily to handle and to survey, only the zonal harmonics are written down; the tesseral and the sectorial harmonics of the same degree fulfill analogous relations, in this context.

With the substitution given by (93) , (see also (24) and (27)) ,

$$R_{n.o}(\varphi,\lambda) \longrightarrow Y_n(\varphi,\lambda) \quad , \tag{93}$$

the following developments for α, β , γ yield,

$$\alpha = \sum_{n=o}^{\infty} a_n \cdot Y_n(\varphi,\lambda) \quad , \tag{94}$$

$$\frac{\beta}{R'} = \frac{F}{2\pi R'} = \sum_{n=o}^{\infty} c_n \cdot Y_n(\varphi,\lambda) \quad , \tag{95}$$

$$\gamma = \frac{T}{R'} = \sum_{n=o}^{\infty} d_n \cdot Y_n(\varphi,\lambda) \quad . \tag{96}$$

In analogy to (27), the subsequent relation (97) is here introduced. This relation is of use for the representation of the inverse of sin p/2 which appears in (91);Fig.2, 3. The functions of (94)(95)(96) can be considered to be distributed along the unit sphere. The point P has the same latitude and longitude as the point P^* ; the same is valid for the points Q and Q^* . Thus, with Fig. 3,

$$\frac{2R}{e(\overline{P}^*,Q^*)} = \sum_{n=o}^{\infty} \left[\frac{R}{R+\varkappa}\right]^{n+1} \cdot \frac{2}{2n+1} \cdot Y_n(\varphi,\lambda)_{P^*} \cdot Y_n(\varphi,\lambda)_{Q^*} \,. \tag{97}$$

For $\varkappa \rightarrow o$, the point $\overline{P}^*$ subsides down to the point P^*, and the left hand side of (97) turns to the inverse of sin p/2, Fig. 3.

Furthermore, the relation (96) is inserted in (91), the equation (98) yields,

$$\gamma = \frac{1}{4\pi} \iint\limits_{l} \frac{\alpha}{\sin p/2} \cdot dl + \frac{3}{8\pi} \iint\limits_{l} \frac{\gamma}{\sin p/2} \cdot dl + \frac{\beta}{R'} \quad . \tag{98}$$

Further on, with (97), and accounting for (98a),

$$\lim_{\varkappa \rightarrow o} \left[\frac{2R}{e(\overline{P}^*,Q^*)}\right] = \frac{1}{\sin p/2} \quad , \tag{98a}$$

the expression (99), for the spherical harmonics developments, follows

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$$\sum_{n=0}^{\infty} d_n \cdot Y_n(\varphi,\lambda)_{P^*} = \frac{1}{4\pi} \sum_{n=0}^{\infty} a_n \frac{2}{2n+1} \cdot Y_n(\varphi,\lambda)_{P^*} \cdot 4\pi \; +$$

$$+ \frac{3}{8\pi} \sum_{n=0}^{\infty} d_n \cdot \frac{2}{2n+1} \cdot Y_n(\varphi,\lambda)_{P^*} \cdot 4\pi \; +$$

$$+ \sum_{n=0}^{\infty} c_n \cdot Y_n(\varphi,\lambda)_{P^*} \; . \qquad (99)$$

The orthogonality relations for Y_n are, (92), (93), (28) (29),

$$\iint_{l} Y_i(\varphi,\lambda) \cdot Y_j(\varphi,\lambda) \cdot dl = \begin{Bmatrix} 0, \text{ if } & i \neq j \\ 4\pi, \text{ if } & i = j \end{Bmatrix} . \qquad (100)$$

(99) and (100) yield

$$d_n = a_n \cdot \frac{2}{2n+1} + \frac{3}{2n+1} \cdot d_n + c_n \; , \qquad (101)$$

$$(n = 0, 1, 2, \ldots) \; . \qquad (102)$$

(101) leads to

$$0 = 2a_n + (2n+1)\, c_n - 2\,(n-1)\, d_n \; , \qquad (103)$$

$$(n = 0, 1, 2, \ldots) \qquad (104)$$

For $n = 0$ and $n = 1$ follows

$$0 = 2\, a_0 + c_0 + 2\, d_0 \; , \qquad (105)$$

$$0 = 2\, a_1 + 3\, c_1 \; . \qquad (106)$$

Thus, the identity of Green,(91), yields the condition equations (103) (104) for the Stokes constants of the developments for $\Delta g_T + D_T(1.1)$, for $F/(2\pi R')$, and for T/R'.

For a moment, the relation (107) is supposed to be the solution of the system (103). This supposition is verified below by the relations from (108) to (114).

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$$\frac{T}{R'} = \frac{1}{4\pi} \iint\limits_{1} \left[\alpha + \frac{3}{4\pi} \cdot \frac{F(T)}{R'} \right] S(p) \cdot dl \; + \frac{F(T)}{2\pi R'} \quad . \tag{107}$$

S(p) is the Stokes function, (32).
(107) has the character of an explicit representation of T in terms of α, since $F(T)$ on the right hand side of (107) comes from rough approximations of the T values, – for instance obtained by (31).

As to the verification of (107) by (103), the Legendre functions P_n (cos p) of (32) have the expression (108), according to the decomposition formula,

$$P_n (\cos p) = \frac{1}{2n+1} \sum_{m=0}^{n} \Big[R_{n.m}(\varphi,\lambda)_{P^*} \cdot R_{n.m} (\varphi,\lambda)_{Q^*} + S_{n.m} (\varphi,\lambda)_{P^*} \cdot S_{n.m} (\varphi,\lambda)_{Q^*} \Big] \; . \tag{108}$$

Hence, the here preferred brief manner of writing gives, (93),

$$S(p) = \sum_{n=2}^{\infty} \frac{1}{n-1} Y_n (\varphi,\lambda)_{P^*} \cdot Y_n (\varphi,\lambda)_{Q^*} \quad . \tag{109}$$

(103) is valid for the harmonics of all degrees, but (107) and (110) are valid for the harmonics of the degrees $n = 2,3,4, \ldots,$ only. The harmonics of the degree $n = 0$ and $n = 1$ will be discussed later on in the special chapter 6.

From (107) follows, with (94) (95) (96) (109),

$$\sum_{n=2}^{\infty} d_n \cdot Y_n = \sum_{n=2}^{\infty} a_n \frac{1}{n-1} Y_n + \sum_{n=2}^{\infty} \frac{3}{2} c_n \frac{1}{n-1} Y_n + \sum_{n=2}^{\infty} c_n \cdot Y_n \quad . \tag{110}$$

(110) gives

$$d_n(n-1) = a_n + c_n \left[\frac{3}{2} + n - 1 \right] , \tag{111}$$

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and further on ,

$$0 = 2a_n + (2n + 1)\, c_n - 2(n - 1)\, d_n \quad , \tag{112}$$

$$(n = 2, 3, 4, \ldots) \quad . \tag{113}$$

(112) is identical with (103), for $n = 2, 3, 4, \ldots$.

Consequently, the relation (107) is verified to be the unique solution of the problem formulated by the equation (103) and (85).

The final form of (107) is obtained by the introduction of (86). Further, by putting the surface functions T and $F(T)$ into parentheses { } , the fact is marked that the constituents represented by the spherical harmonics of 0 th and 1 st degree in the surface functions T and $F(T)$ are split off. Hence,

$$\{T\} = \frac{R'}{4\pi} \iint_1 \left[\Delta g_T + D_T(1.1) + \frac{3}{4\pi} \cdot \frac{F(T)}{R'} \right] \cdot S(p) \cdot dl + \frac{\{F(T)\}}{2\pi} . \tag{114}$$

dl is the surface element of the unit sphere.

With (A9), the relation (115) which is specified 3 lines below follows for $D_T(1.1)$, if the suffix T denotes the fact that the operator $D(1.1)$ is applied to the perturbation potential T,

$$D_T(1.1) = \frac{\partial T}{\partial n} \cdot \frac{1}{\cos(g',n)} + \frac{\partial T}{\partial r} \quad . \tag{115}$$

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6. The spherical harmonics of 0th and 1st degree

The perturbation potential T is the difference between the gravity potential W and the standard potential U , in the exterior space and on the Earth's surface u. This is the definition of T,

$$T = W - U \ . \tag{115a}$$

This harmonic perturbation potential T has the following uniform convergent series development in spatial spherical harmonics for test points in the exterior of the body of the Earth, [4] [5] ,

$$T = \sum_{n=o}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=o}^{n} \left[T_{1.n.m} \cdot R_{n.m}(\varphi,\lambda) + T_{2.n.m} \cdot S_{n.m}(\varphi,\lambda) \right], \quad \text{in } \Gamma \tag{116}$$

Γ denotes both the exterior space of the Earth and the surface of the Earth u. r, φ, λ are the spatial polar co-ordinates. The origin of this co-ordinate system is chosen in such a way that it does coincide with the gravity center of the Earth, (barycenter).
Hence, the Stokes constants of the spherical harmonics of the 1st degree are equal to zero, (116),

$$T_{1.1.0} = T_{1.1.1} = T_{2.1.1} = 0 \ . \tag{117}$$

Whole the gravitating scources which give rise to the standard potential U have a total mass which is equal to the mass of the Earth. Thus, also the Stokes constant of the spherical harmonic of 0th degree (n = 0) is equal to zero,

$$T_{1.0.0} = 0 \ . \tag{118}$$

Whether the T values obtained from the boundary value problem, (114), are compatible with the four conditions (117) (118) or not, that is the open question now to be discussed. It is intended to formulate certain criterions which make it possible to find out whether the conditions (117) and (118) are fulfilled by the T values of (114) or not. Furthermore, these criterions will make it possible to determine certain supplements to the harmonics of the 0th and 1st degree of the T values obtained by (114). Of course, in the surface values of T obtained by (114), the constituents described by the harmonics of the 0th and 1st degree are equal to zero, per definitionem. The addition of certain

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supplements to the T values of (114) completes the T values of (114) in such a way that (117) and (118) are observed, in the spatial representation of the T values given by (116).

If $\rho = \rho(\varphi,\lambda)$ is equal to the geocentric radius of the Earth's surface u, then, the series development (116) takes the following shape for test points situated on the surface u,

$$(T)_u = \sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^{n+1} \sum_{m=0}^{n} \left[T_{1.n.m} \cdot R_{n.m}(\varphi,\lambda) + T_{2.m.m} \cdot S_{n.m}(\varphi,\lambda)\right] \quad . \qquad (119)$$

All the functions of the manifold

$$\left(\frac{R}{\rho}\right)^{n+1} \cdot R_{n.m}(\varphi,\lambda) \quad \text{and} \quad \left(\frac{R}{\rho}\right)^{n+1} \cdot S_{n.m}(\varphi,\lambda) \quad , \qquad (120)$$

$$(n = 0, 1, 2, \ldots) \;, \quad (m = 0, 1, 2, \ldots, n) \;, \qquad (121)$$

are linear independent functions, [4], [5] pg. 162 and 163,

Henceforth, the functions of (120) get now a running numeration, as given by (122),

$$\omega_k = \omega_k(\varphi,\lambda) \;, \quad (k = 1, 2, 3, \ldots). \qquad (122)$$

Thus, the development (119) for the surface values of the potential T can be written in the following form,

$$(T)_u = \sum_{k=1}^{\infty} t_k \cdot \omega_k(\varphi,\lambda) \quad . \qquad (123)$$

t_k , (k = 1, 2, 3, ...), are the constant coefficients of this development.

The relations (117) (118) (123) give

$$t_1 = T_{1.0.0} \quad , \qquad (124)$$

$$t_2 = T_{1.1.0} \quad , \qquad (125)$$

$$t_3 = T_{1.1.1} \quad , \qquad (126)$$

$$t_4 = T_{2.1.1} \quad . \qquad (127)$$

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The Schmidt orthonormalization procedure leads from the functions $\omega_k(\varphi,\lambda)$, (122), to the system of the orthonormalized functions ω_k^*, $\omega_k^*(\varphi,\lambda)$, since the functions $\omega_k(\varphi,\lambda)$ are linear independent, [4] [5],

$$\begin{pmatrix} \omega_1^* \\ \omega_2^* \\ \omega_3^* \\ \dots \end{pmatrix} = \underline{\underline{B}} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \dots \end{pmatrix} , \qquad (128)$$

or, in short, in vector form,

$$\underline{\underline{\omega}}^* = \underline{\underline{B}} \cdot \underline{\underline{\omega}} \qquad (129)$$

The Gram determinants implied in (129) are never equal to zero. Consequently, (129) can be inverted,

$$\underline{\underline{\omega}} = \underline{\underline{B}}^{-1} \cdot \underline{\underline{\omega}}^* , \qquad (130)$$

with

$$\det \underline{\underline{B}} \neq 0 . \qquad (131)$$

The right hand side of (123) can be written in the form of a scalar product,

$$(T)_u = \underline{\underline{t}}^T \cdot \underline{\underline{\omega}} , \qquad (132)$$

$$\underline{\underline{t}}^T = (t_1, t_2, \dots) . \qquad (133)$$

In (132), the subscript u denotes the fact that the test point lies on the surface u, and the superscript T is the symbol for the transposition. (130) and (132) yield

$$(T)_u = \underline{\underline{t}}^T \cdot \underline{\underline{B}}^{-1} \cdot \underline{\underline{\omega}}^* \qquad (134)$$

The system of the base functions $\omega_k(\varphi,\lambda)$, (122), is complete, as so as the system of the functions $\omega_k^*(\varphi,\lambda)$, (129), at least in the space of the continuous functions; the proof is given in [5].

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Thus, the base functions $\omega_k(\varphi, \lambda)$ can be developed in surface spherical harmonics $\omega_k^{**}(\varphi, \lambda)$, and these relations have a well-defined inversion. The same is valid for the representation of the orthonormal functions $\omega_k^{*}(\varphi, \lambda)$ in terms of the functions $\omega_k^{**}(\varphi, \lambda)$. Hence,

$$\underline{\underline{\omega}}^{*} = \underline{\underline{A}} \cdot \underline{\underline{\omega}}^{**} , \tag{135}$$

$$\underline{\underline{\omega}}^{**} = \underline{\underline{A}}^{T} \cdot \underline{\underline{\omega}}^{*} , \tag{136}$$

$$\underline{\underline{\omega}}^{**} = \underline{\underline{A}}^{T} \cdot \underline{\underline{B}} \cdot \underline{\underline{\omega}} . \tag{136a}$$

The vector $\underline{\underline{\omega}}^{**}$ comprises the surface spherical harmonics $\omega_k^{**}(\varphi, \lambda)$ as its components. $\underline{\underline{A}}$ and $\underline{\underline{A}}^T$ are certain infinite orthonormal matrices, [5] pg. 166...170,

$$\det \underline{\underline{A}} = \det \underline{\underline{A}}^{T} = 1 , \tag{136b}$$

$$\underline{\underline{A}} \cdot \underline{\underline{A}}^{T} = \underline{\underline{E}} . \tag{136c}$$

$\underline{\underline{E}}$ is a unit matrix.

The combination of (134) and (135) gives

$$(T)_u = \underline{\underline{t}}^{T} \cdot \underline{\underline{B}}^{-1} \cdot \underline{\underline{A}} \cdot \underline{\underline{\omega}}^{**} . \tag{137}$$

Writing, abbreviating,

$$(\underline{\underline{t}}^{**})^{T} = \underline{\underline{t}}^{T} \cdot \underline{\underline{B}}^{-1} \cdot \underline{\underline{A}} , \tag{138}$$

the following form of (137) is obtained

$$(T)_u = (\underline{\underline{t}}^{**})^{T} \cdot \underline{\underline{\omega}}^{**} ; \tag{139}$$

$$(\underline{\underline{t}}^{**})^{T} = (t_1^{**}, t_2^{**}, \ldots.) , \tag{139a}$$

$$(T)_u = t_1^{**} \cdot \omega_1^{**} + t_2^{**} \cdot \omega_2^{**} + \ldots. \tag{139b}$$

(139) and (139b) is the development of the surface values of the potential T in terms of spherical harmonics. These T values come from (114), from the boundary value problem.

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Along the surface of the Earth u, the amounts of $\{T\}$ are known by the gravity anomalies Δg_T, using (114). From these $\{T\}$ values, the coefficients t_i^{**} of the surface spherical harmonics series development (139) (139b) can be computed, it is self-explanatory. Thus, the vector $\underline{\underline{t}}^{**}$ is known,

$$\underline{\underline{t}}^{**} = \begin{pmatrix} t_1^{**} \\ t_2^{**} \\ t_3^{**} \\ \cdots \end{pmatrix} , \qquad (140)$$

$$t_i^{**} = \frac{1}{4\pi} \iint_l (T)_u \cdot \omega_i^{**}(\varphi,\lambda) \cdot d\,l \qquad (140a)$$

$$\iint_l \omega_i^{**} \cdot \omega_j^{**} \cdot dl = 4\pi \cdot \delta_{i.j} \quad . \qquad (140b)$$

$\delta_{i.j}$ is the Kronecker symbol,(141 r),(28) (29).

l is the unit sphere, (92).

By definition, the $\{T\}$ values obtained by (114) are free of the spherical harmonics of 0 th and 1 st degree. Hence, the first four elements of (140) are equal to zero, (117)(118), (124)...(127),

$$t_1^{**} = t_2^{**} = t_3^{**} = t_4^{**} = 0. \qquad (141)$$

The relation (138) can be transformed, using the fact that the matrices $\underline{\underline{B}}$ and $\underline{\underline{A}}$ are non-singular; thus,

$$(\underline{\underline{t}}^{**})^T \cdot \underline{\underline{A}}^T \cdot \underline{\underline{B}} = \underline{\underline{t}}^T \quad . \qquad (141a)$$

By (129), $\underline{\underline{B}}$ is a subdiagonal matrix. The transposition of (141a) yields

$$\underline{\underline{t}} = \underline{\underline{B}}^T \cdot \underline{\underline{A}} \cdot \underline{\underline{t}}^{**} \quad . \qquad (141b)$$

The relation (141b) shows how to compute the vector $\underline{\underline{t}}$, (132), from the vector $\underline{\underline{t}}^{**}$, (140), and from the matrices $\underline{\underline{B}}$ and $\underline{\underline{A}}$. This vector $\underline{\underline{t}}$ is the vector of the coefficients t_i of the harmonics development (123).

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Simultaneonsly, these t_i values are also the coefficients of the spatial spherical harmonics series development (116).

Consequently, the relation (141b) gives automatically the amounts of the coefficients $T_{1.n.m}$ and $T_{2.n.m}$ which yield from the solution of the boundary value problem, (114). With that, the four amounts $T_{1.0.0}$ (for the 0th degree), (and for n=1) $T_{1.1.0}$, $T_{1.1.1}$, $T_{2.1.1}$ are known, (124) to (127). These four amounts have to satisfy the constraints (117) and (118).

The relation (141b) gives the desired criterion convenient to check whether the constraints (117) (118) are fulfilled or not. The conditions (117) (118) can be brought into the following form,

$$t_1 = t_2 = t_3 = t_4 = 0 \quad . \tag{141c}$$

In case, these equations (141c) are not fulfilled by the t_i values of (141b), (i = 1,2,3,4), the measure turns out to be necessary that the center of the reference ellipsoid has to be shifted in the three-dimensional space until the 3 condition equations for t_2, t_3, t_4 are satisfied, (141c) (117). Eventually, further on, the spherical symmetric constituent of the standard potential U has to be modified also until the condition equation for t_1 is fulfilled, (141c) (118) (115a).

In case, the four equations (141c) are not observed by the t_i values (i = 1,2,3,4), obtained from the $\underline{\underline{t}}$ vector deduced from {T}, (141b) (114), (140a), in this case, it is possible to reach the fulfillment of (141c) afterwards, by the subsequently described procedure of (141c) to (141 v). Here, the equation (141b) is in the fore. In (141b), the vectors $\underline{\underline{t}}$ and $\underline{\underline{t}}^{**}$ are amplified by the supplements $\delta\underline{\underline{t}}$ and $\delta\underline{\underline{t}}^{**}$, which have to bring about an adjustment of the T potential with intent to observe the constraints (141c).
Thus,

$$\underline{\underline{t}} + \delta\underline{\underline{t}} = \underline{\underline{B}}^T \cdot \underline{\underline{A}} \cdot (\underline{\underline{t}}^{**} + \delta\underline{\underline{t}}^{**}) \quad . \tag{141d}$$

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$\delta \underline{\underline{t}}$ obeys the following conditions, a priori valid,

$$\delta t_i = - t_i, \ (i = 1,2,3,4). \qquad (141e)$$

The relations (141e) make the left hand side of (141d) equal to zero, for i = 1,2,3,4, in accordance with (141c). (t_i values of (141e) taken from (141b)).

$\delta \underline{\underline{t}}^{**}$ fulfills the subsequent conditions, a priori valid. (see (114)),

$$\delta t_i^{**} = 0, \ (i = 5,6,7, \ldots). \qquad (141f)$$

The following amounts of (141g) and (141h) have to be determined, a posteriori,

$$\delta t_i \ , \ (i = 5,6,7, \ldots), \qquad (141g)$$

and

$$\delta t_i^{**}, \qquad (i = 1,2,3,4). \qquad (141h)$$

These values are a priori unknown, they have to be determined in such a way that (141d) (141e) (141f) are valid.

Putting, abbreviating,

$$\delta \underline{\underline{t}} = \underline{\underline{l}}_1 \ , \qquad (141i)$$

$$\delta \underline{\underline{t}}^{**} = \underline{\underline{l}}_2 \ , \qquad (141j)$$

$$\underline{\underline{B}}^T \cdot \underline{\underline{A}} = \underline{\underline{L}} \ , \qquad (141k)$$

denoting the vector of the four a priori known components of (141e) by

$$\underline{\underline{l}}_{1.1} \ , \qquad (141l)$$

and denoting the vector of the four a priori unknown components of (141h) by

$$\underline{\underline{l}}_{2.1} \ , \qquad (141m)$$

then the equation (141n) follows from (141b) (141d) (141i) (141j) (141k) (141m), - since the relations (141b) and (141k) yield

$$\underline{\underline{t}} = \underline{\underline{L}} \cdot \underline{\underline{t}}^{**} \ .$$

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$$\underline{\underline{l}}_1 = \underline{\underline{L}} \cdot \underline{\underline{l}}_2 \quad , \qquad (141n)$$

or, by a self-explanatory rearrangement,

$$\begin{pmatrix} \underline{\underline{l}}_{1.1} \\ \\ \underline{\underline{l}}_{1.2} \end{pmatrix} = \begin{pmatrix} \underline{\underline{L}}_{1.1} & \underline{\underline{L}}_{1.2} \\ \\ \underline{\underline{L}}_{2.1} & \underline{\underline{L}}_{2.2} \end{pmatrix} \cdot \begin{pmatrix} \underline{\underline{l}}_{2.1} \\ \\ \underline{\underline{l}}_{2.2} \end{pmatrix} \quad . \qquad (141o)$$

The determinant of $\underline{\underline{L}}_{1.1}$ is the minor in principal position (covering the indices $i = 1,2,3,4$) of the matrix $\underline{\underline{L}}$.

(141f) is introduced into (141o); thus the relation

$$\underline{\underline{l}}_{2.2} = 0 \qquad (141p)$$

has to be considered, treating the matrix equation (141o). Obviously, (141q) is the result,

$$\underline{\underline{l}}_{1.1} = \underline{\underline{L}}_{1.1} \cdot \underline{\underline{l}}_{2.1} \quad . \qquad (141q)$$

$\underline{\underline{L}}_{1.1}$ is about a unit matrix, in close approximation. This fact can be evidenced by the structure of the terms (120) which are identical with the functions ω_i of (122). Putting the radius $\rho(\varphi,\lambda)$ of the surface of the Earth u equal to the radius R by the neglection of relative errors of the order of Z/R, the following relations for the first four functions of $\omega_i(\varphi,\lambda)$, $\omega_i^{**}(\varphi,\lambda)$, and $R_{n.m}(\varphi,\lambda)$, and $S_{n.m}(\varphi,\lambda)$ are obtained,

$$\begin{aligned} \omega_1 &\cong R_{o.o} = \omega_1^{**} \quad , \\ \omega_2 &\cong R_{1.o} = \omega_2^{**} \quad , \\ \omega_3 &\cong R_{1.1} = \omega_3^{**} \quad , \\ \omega_4 &\cong S_{1.1} = \omega_4^{**} \quad . \end{aligned} \qquad (141r)$$

A comparison of (141r) with (136a) (141k) (141o) yields the following relation,

$$\underline{\underline{L}}_{1.1} \cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \qquad (141s)$$

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Consequently,

$$\det \underline{\underline{L}}_{1.1} \neq 0 \quad . \qquad (141t)$$

(141t) shows that the matrix $\underline{\underline{L}}_{1.1}$ has a well-defined inverse. Thus, the inverse of (141q) is

$$\underline{\underline{l}}_{2.1} = (\underline{\underline{L}}_{1.1})^{-1} \cdot \underline{\underline{l}}_{1.1} \quad . \qquad (141u)$$

(141u) allows the computation of $\underline{\underline{l}}_{2.1}$ from $\underline{\underline{l}}_{1.1}$.

The vector $\underline{\underline{l}}_{1.2}$ is the other vector, which is to be determined, besides of $\underline{\underline{l}}_{2.1}$, (141u). $\underline{\underline{l}}_{1.2}$ is obtained by (141o) (141p) (141u),

$$\underline{\underline{l}}_{1.2} = \underline{\underline{L}}_{2.1} \cdot \underline{\underline{l}}_{2.1} = \underline{\underline{L}}_{2.1} \cdot (\underline{\underline{L}}_{1.1})^{-1} \cdot \underline{\underline{l}}_{1.1} \quad . \qquad (141v)$$

The relations (141u) and (141v) solve the here discussed problem.

The surface potential $\{T\}$ along the surface u, according to (114), has to be amended by an alteration that consists in the addition of the constituents formed by the spherical harmonics of o th and 1 st degree. The Stokes constants of these harmonics are well-defined by the relation (141u).

Further on, in the harmonics series development for the spatial potential T, (116), the Stokes constants of the degree $n \geqslant 2$ undergo certain amendments and alterations by the values of (141v).

But, the surface harmonics of the degree $n \geqslant 2$ in the T potential of (114) remain unchanged. They conserve the values obtained (in terms of the gravity anomalies) by the computations according to (114).

Furthermore, in the spatial development (116), the Stokes constants of 0 th and 1 st degree fulfill after these amendments the required constraint that they have to be equal to zero, finally, (117) (118), as demanded in our applications.

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7. The superposition of the perturbation potential T upon the potential B of the mountain masses with standard density

The here discussed mountain masses are the masses which are situated above the mean Earth-ellipsoid in the domain of the continents. Thus, they lie between the mean ellipsoid of the Earth and the surface of the Earth u. In the here discussed boundary value problem, the flattening of the Earth is neglected; the ellipsoid is replaced by the globe v, Fig. 2. Consequently, in this context, the mountains and the heights H rise above the globe v, but not above the mean ellipsoid of the Earth. Further facilities and computation reliefs, connected with this model of the mountains, consist in the fact that these masses have the standard density $\vartheta = 2\ 650$ kg m^{-3} and not the real density.

The gravitational potential B of these mountain masses can be expressed by the following integral, Fig. 5,

$$B = f\vartheta \iiint_V \frac{1}{\bar{e}} \, dV \quad . \tag{142}$$

V is the volume of the mountain masses considered above; the integral (142) covers the continental domain only. f is the gravitational constant.
$\bar{e}$ represents the straight distance between the running volume element dV and the test point $\bar{P}$ in the exterior of the body of the Earth, Fig. 5.

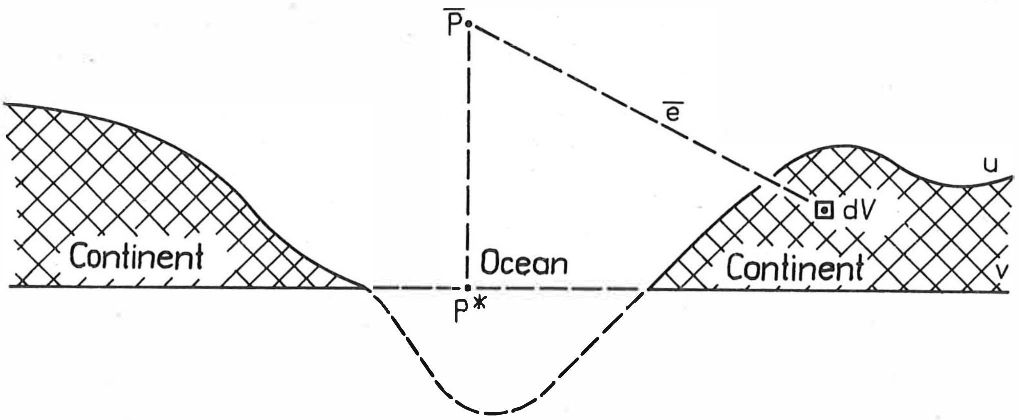

Fig. 5.

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The here considered mountain masses fill the crosswise hatched domain, shown in Fig. 5. Perpendicular below the test point $\overline{P}$ and, moreover, in the level of the globe v, the point P^* is situated, Fig. 5. The volume element dV has the equation

$$dV = r^2 \cdot \sin p \cdot dr \cdot dp \cdot dA \quad . \qquad (143)$$

In (143), r is the distance which the volume element dV has to the barycenter of the globe v (v having the radius R) . p is here the spherical distance between the volume element dV and the point P^*, Fig. 5. A is the clockwise counted azimuth. It is defined as the angle, which has the point P^* as the vertex, and which measures clockwise the direction the volume element dV shows with regard to the north.

The height of the surface of the Earth u above the globe v is H, (see Fig. 2 and Fig. 4). Hence, the integral (142) turns to

$$B = f \vartheta \int_{p=0}^{\pi} \int_{A=0}^{2\pi} \int_{r=R}^{R+H} \frac{1}{\bar{e}} \cdot r^2 \cdot \sin p \cdot dr \cdot dp \cdot dA \quad . \qquad (144)$$

Now, the potential M is introduced by the equation

$$M = T - B \quad . \qquad (145)$$

M is a harmonic and continuous potential in the exterior space of the body of the Earth. The potential M has about the same structure as the potential T. The amounts of $|M/G|$ will not be greater than about ten times the amounts of $|T/G|$, at least in the global average. G is here the global mean of the gravity. If it is intended to compute M by the relation (145), the potential B on the right hand side of (145) comes from (144). But, the integration according to (144) does not imply the isostatic compensation masses, situated below the isostatic compensation depth of 30 km, in case of the Airy - Heiskanen system.
Because (144) does not imply the compensating mountain roots, the amount of $|M|$ will generally be greater than the amount of $|T|$, (145). $|M/G|$ can amount up to 1000 m, about.

At this place, before a further discussion about the potential M, it should be stressed that only the coming equation (151) on the next page defines the term Δg_M ! Especially in (6), it is not allowed simply to substitute T by M, and g by g''' , without any inclusion of any additives. Replacing Δg_M by g''' - g' , this procedure will be wrong, or, more precisely, it will not be sufficient precise. A term quadratic in (M/g') has to be added as a more or less important additive. The deeper reason is the fact, that (M/g') is in its absolute amount 5 time or 10 time greater than $|(T/g')|$. In the subsequent deliberations and deductions about the boundary value problem, these additives do not occur. The following deductions make no mention of these additives. - But in (6), the analogous term quadratic in (T/g') can be neglected, according to common use. The reason is, that $|(T/g')|$ is generally considerably smaller than $|(M/g')|$.

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It is possible to apply the equations (114) and (115) to the potential M, defined by (145). Thus, if M serves as substitute for T,

$$\{M\} = \frac{R'}{4\pi} \iint_l \left[\Delta g_M + D_M\,(1.1) + \frac{3}{4\pi}\,\frac{F(M)}{R'} \right] S(p) \cdot dl \;+ \frac{\{F(M)\}}{2\pi} \quad . \tag{146}$$

The relation (115) turns to

$$D_M(1.1) = \frac{\partial M}{\partial n} \cdot \frac{1}{\cos\,(g',\, n)} + \frac{\partial M}{\partial r} \quad . \tag{147}$$

The fundamental equation of Physical Geodesy gives, (1),

$$\Delta g_T = -\frac{\partial T}{\partial r} - \frac{2}{r}\, T \quad , \tag{148}$$

$$\Delta g_B = -\frac{\partial B}{\partial r} - \frac{2}{r}\, B \quad . \tag{149}$$

Thus, (145),

$$\Delta g_M = \Delta g_T - \Delta g_B \quad , \tag{150}$$

$$\Delta g_M = -\frac{\partial M}{\partial r} - \frac{2}{r}\, M \quad , \tag{151}$$

$$\Delta g_M = -\frac{\partial T}{\partial r} + \frac{\partial B}{\partial r} - \frac{2}{r} \cdot (T - B) \quad . \tag{152}$$

The transformations of (147) happen along the same way as those of (115); but, considering (147), the fact has to be in view that the amounts of $|M/G|$ can reach about 1000 m, whereas the amounts of $|T/G|$ hardly reach 100 m. The concerned rearrangements of (115) can be found in the appendix, by the equations from (A9) up to (A 21). Some hints at the amounts of $|M/G|$ can be found in the following publication: Veröff, d. Bayerischen Kommission f. d. Intern. Erdmessung, Astr.-Geod. Arbeiten, Heft Nr. 48, München 1986, S. 153.

As to the rearrangements of (147), taken in the potential field M, μ_1 is the plumb - line deflection component at the surface of the Earth u, taken in the north - south direction. μ_2 is the corresponding east - west component, (71) (72) (73).

$$\mu_1 = -\frac{1}{R' + Z} \cdot \frac{1}{g'''} \cdot \frac{\partial M}{\partial \varphi} \quad , \tag{153}$$

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$$\mu_2 = -\frac{1}{R' + Z} \cdot \frac{1}{g'''\cos\varphi} \cdot \frac{\partial M}{\partial \lambda} , \tag{154}$$

μ: the absolute amount of this plumb - line deflection,

$$\mu^2 = \mu_1^2 + \mu_2^2 . \tag{155}$$

μ, μ_1, μ_2 have smoothed values, but the functions of Θ, ξ, η are not smoothed, (71) (72) (73).

The standard gravity g' is the amount of the gradient of the standard potential U. The amount of g''' is the intensity of the gravity in the potential field U + M; U and U + M are rotating potentials. Thus, by the gradient, (Fig. A 1),

$$g''' = \left| \nabla (U + M) \right| . \tag{156}$$

A''' is the azimuth of the plumb - line deflection μ . As it is found in the above mentioned Bavarin publication Nr. 48, the horizontal alteration of the amounts of |M/G| is maximal about 0,5 km for a distance of 2 000 km. Here, the fact has to be regarded, that, in this Bavarian publication, the mountains have the density - surplus of 2 670 kg m^{-3} and the ocean basins the density-defect of - 1 640 kg m^{-3}. But, in the here discussed calculation of the M values, the mass deficiency in the domain of the oceans has to be discarded. Consequently, the amounts of | M/G | will be a little greater, in reality, than the values taken from the above cited publication Nr. 48. Summarizing, the maximal amount of μ can be characterized by

$$\mu_{max} = 0.5 \text{ km}/2000 \text{ km} = 2.5 \cdot 10^{-4} . \tag{157}$$

Obviously, the relation (157a) is valid,

$$M = (U + M) - U . \tag{157a}$$

Introducing M as a substitute for T , and U + M as a substitute for W (being equal to U + T), the relation (A 14) turns to, (156),

$$\frac{\partial M}{\partial n} = g''' \cdot \cos (g''', n) - g' \cdot \cos (g', n) . \tag{158}$$

Analogously as (A 15), the equation (159) can be obtained,

$$\cos (g''', n) = \cos (g', n) \cdot \cos\mu + \sin (g', n) \cdot \sin\mu \cdot \cos (A''' - A') . \tag{159}$$

This above relation (159) can be obtained also from Fig. A 1 and the cosine formula of a spherical triangle. In Fig. A 1 on the page 95, the vector $(-\underline{\underline{g}}''')^{o}$ is the normalized vector of $-\underline{\underline{g}}'''$; thus, $\left|(-\underline{\underline{g}}''')^{o}\right| = 1$. In Fig. A 1, the normalized vectors $(-\underline{\underline{g}}')^{o}$, $(-\underline{\underline{n}})^{o}$, $(-\underline{\underline{g}})^{o}$, and $(-\underline{\underline{g}}''')^{o}$ are the radii of a unit sphere with the surface point Q as center .

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Considering

$$\sin\mu = \mu - \frac{1}{6}\mu^3 + - \ldots \quad , \qquad (160)$$

$$\cos\mu = 1 - \frac{1}{2}\mu^2 + - \ldots \quad , \qquad (161)$$

the relation (162) follows from (159),

$$\cos(g''', n) = \cos(g', n) - \frac{1}{2}\cdot\mu^2\cdot\cos(g', n) +$$

$$+ \mu\cdot\sin(g', n)\cdot\cos(A''' - A') -$$

$$- \frac{1}{6}\cdot\mu^3\cdot\sin(g', n)\cdot\cos(A''' - A'). \qquad (162)$$

The relation (157) yields

$$(\mu_{max})^2 = 6\cdot 10^{-8} \quad . \qquad (163)$$

The relation (163) makes it clear, that the 4 th term on the right hand side of (162) is insignificant in comparison with the 3rd term. Thus,

$$\cos(g''', n) = \cos(g', n) - \frac{1}{2}\cdot\mu^2\cdot\cos(g', n) +$$

$$+ \mu\cdot\sin(g', n)\cdot\cos(A''' - A') \quad . \qquad (164)$$

The combination of (158) with (164) gives (165),

$$\frac{\partial M}{\partial n}\cdot\frac{1}{\cos(g', n)} = g''' - g' - \frac{1}{2}\cdot\mu^2\cdot g''' +$$

$$+ \mu\cdot g'''\cdot\tan(g', n)\cdot\cos(A''' - A') \quad . \qquad (165)$$

μ is the angle the direction of g''' makes with the radius r ; thus, (166) follows

$$g'''\cdot\cos\mu = -\frac{\partial(M + U)}{\partial r} \quad , \qquad (166)$$

or, with (161),

$$g''' = -\frac{\partial(M + U)}{\partial r}\left[1 + \frac{1}{2}\mu^2\right] . \qquad (167)$$

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Further, (A 13),

$$g' = - \frac{\partial U}{\partial r} \tag{168}$$

The difference of (167) and (168) is, in sufficient approximation,

$$g''' - g' = -\frac{\partial M}{\partial r} - \frac{\partial U}{\partial r} \cdot \frac{1}{2} \cdot \mu^2 \quad , \tag{169}$$

or, (168),

$$g''' - g' = -\frac{\partial M}{\partial r} + \frac{1}{2} \cdot g' \cdot \mu^2 \quad . \tag{170}$$

(165) and (170) are combined to

$$\frac{\partial M}{\partial n} \cdot \frac{1}{\cos (g', n)} = -\frac{\partial M}{\partial r} + \mu \cdot g''' \cdot \tan (g', n) \cdot \cos (A''' - A') \quad . \tag{171}$$

In (171), the amount of

$$\frac{1}{2} \mu^2 \cdot (g' - g''') \tag{172}$$

was neglected, since it is considerably smaller than 1 μ gal. The 2nd term on the right hand side of (171) contains the gravity value of g'''. Replacing here g''' by the standard gravity g', a relative error of (g''' - g')/ g' is the consequence. Putting the amount of $|g''' - g'|$ equal to 0.3 gal and g' equal to 10^3 gal (i. e. 10^3 cm/sec^2), this relative error amounts to

$$\frac{g''' - g'}{g'} \cong 3 \cdot 10^{-4} \quad . \tag{173}$$

The neglection of such a small relative error of the order of (173) can always be tolerated in the second term on the right hand side of (171). Obviously, the admissibility of this neglection is due to mere the fact that the plumb-line deflection can never be determined empirically better than within a relative error of $3 \cdot 10^{-4}$. Thus, (171) turns to

$$\frac{\partial M}{\partial n} \cdot \frac{1}{\cos(g', n)} = - \frac{\partial M}{\partial r} + G \cdot \mu \cdot \tan(g', n) \cdot \cos (A''' - A') \quad . \tag{174}$$

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In (174), g' (or better g''') was put equal to G. G is the global mean value of the gravity. (174) and (147) yield

$$D_M\ (1.1) = G \cdot \mu \cdot \tan\,(g',\ n) \cdot \cos\,(A''' - A') \quad . \tag{175}$$

Summarizing the details of the three mathematical expressions representing the three symbols Δg_M, D_M (1.1) and F(M) on the right hand side of (146), the following is found: In (146), Δg_M is obtained by (151). D_M (1.1) comes from (175). F(M) is represented by (64), and by (74), (74a) to (74h), (replacing Δg_T by Δg_M , T by M, and further ξ , η by μ_1 , μ_2).

8. Gauss' integral theorem

The term D_M (1.1) exerts the following impact on the integral on the right hand side of (146), in the computation of $\{M\}$, (175),

$$J = \frac{R'}{4\pi} \iint_l D_M\ (1.1) \cdot S(p) \cdot dl \quad . \tag{176}$$

This expression for J undergoes now some rearrangements using the Gauss' integral theorem, in order to bring the expression for J into a shape more convenient for numerical routine calculations.

(175) and (176) yield

$$J = \frac{G}{4\pi R'} \iint_w \mu \cdot \tan\,(g',\ n) \cdot \cos\,(A''' - A') \cdot S(p) \cdot d\,w \quad . \tag{177}$$

A' is the azimuth of the slope of the terrain, A''' that of the plumb - line deflection μ , Fig. A1. The north - south and the east - west component of the plumb - line deflection μ are denoted by μ_1 and μ_2 , (153) (154) ,

$$\mu_1 = \mu \cdot \cos A''' \quad , \tag{178}$$

$$\mu_2 = \mu \cdot \sin A''' \quad . \tag{179}$$

The expressions for μ , μ_1 and μ_2 are functions which represent these values along the surface of the Earth u.

$$\mu^2 = \mu_1^2 + \mu_2^2 \quad . \tag{179a}$$

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Thus, the values μ, μ_1, and μ_2 have two - parametric functions of φ and λ. They can be understood as functions the values of which are distributed along the unit sphere or along the sphere w, having the radius R'.

In the point Q situated at the surface of the Earth u, in the direction of growing p - values, (i. e. in the direction the great circle connecting P and Q is heading for, in the point Q), the component of the plumb - line deflection μ has the following relations (153) (154),

$$\mu_p = - \frac{1}{R' + Z} \cdot \frac{1}{g'''} \cdot \frac{\partial M}{\partial p} = - \left[\frac{1}{R' + Z} \cdot \frac{1}{g'''} \cdot \frac{\partial M}{\partial p} \right]_u \cong - \frac{1}{R' \cdot G} \cdot \left[\frac{\partial M}{\partial p} \right]_u \quad (180)$$

p is here again the spherical distance from the test point P (fixed within one integration) and the point Q, which is variable within one integration covering whole the sphere w, (177). Thus, also μ_p is a two - parametric function, similarly as μ, μ_1, and μ_2. Hence,

$$\mu = \mu(\varphi, \lambda) \quad , \quad (181)$$
$$\mu_1 = \mu_1 (\varphi, \lambda) \quad , \quad (182)$$
$$\mu_2 = \mu_2 (\varphi, \lambda) \quad , \quad (183)$$
$$\mu_p = \mu_p (\varphi, \lambda) \quad . \quad (184)$$

In (153) (180), the derivations of M have to be taken in horizontal direction; that is to say, these derivations happen along the horizontal plane of the considered Earth's surface point, in north - south or east - west, or in radial direction. The values of (181) (182) (183) (184) refer to points situated on the surface of the Earth u.

The values μ_1 and μ_2 can be considered as the components of a vector $\underline{\underline{\mu}}$ which is tangential to the sphere w (having the radius R'). Thus,

$$\underline{\underline{\mu}} = \mu_1 \cdot \underline{\underline{e}}_1 + \mu_2 \cdot \underline{\underline{e}}_2 \quad . \quad (185)$$

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$\underline{\underline{e}}_1$ and $\underline{\underline{e}}_2$ are orthogonal unit vectors, Fig. A1.
Each point on the sphere w has a vector $\underline{\underline{e}}_1$ which is tangential to the sphere w and which is heading to the north. The same is valid for the vector $\underline{\underline{e}}_2$ which is heading to the east. Hence,

$$\underline{\underline{e}}_1^2 = \underline{\underline{e}}_2^2 = 1 \quad , \tag{186}$$

$$\underline{\underline{\mu}}^2 = \mu^2 = \mu_1^2 + \mu_2^2 \quad . \tag{187}$$

The slope of the terrain is described by tan (g', n), Fig. 4, Fig. A1. This expression allows certain developments which are similar to the above developments for μ, from (178) to (187). The north - south and the east - west component of the slope of the terrain are denoted by s_1 and s_2 , they have the following expressions, (see Fig. A1 of the appendix),

$$s_1 = \tan(g', n) \cdot \cos A' \quad , \tag{188}$$
$$s_2 = \tan(g', n) \cdot \sin A' \quad . \tag{189}$$

The height difference Z is equal to H_Q minus H_P , (57a), whereat H_P is fixed because P is the fixed test point, but, whereat H_Q is variable because the point Q varies over the whole globe. s_1 and s_2 can be obtained by the derivation of the height H_Q of the point Q in the north - south and in the east - west direction.
Therefore, it is possible to find s_1 and s_2 also by derivations of this kind but with regard to the height difference Z, instead of H_Q. Hence, for the point Q,

$$s_1 = - \frac{1}{R' + Z} \cdot \frac{\partial Z}{\partial \varphi} \quad , \tag{190}$$

$$s_2 = - \frac{1}{R' + Z} \cdot \frac{1}{\cos \varphi} \cdot \frac{\partial Z}{\partial \lambda} \quad . \tag{191}$$

The integral J is a relative small supplementary term, (177). Thus, in the integrand of J and, consequently, also in the expressions for s_1 and s_2 , a relative error of the order of Z/R' or Z/R can be tolerated. Z/R reaches not more than about 10^{-3} to 10^{-4}, (see also the appendix, (A386) to (A 387b) . Consequently,

$$s_1 \cong - \frac{1}{R'} \cdot \frac{\partial Z}{\partial \varphi} \tag{192}$$

$$s_2 \cong - \frac{1}{R' \cos \varphi} \cdot \frac{\partial Z}{\partial \lambda} \quad . \tag{193}$$

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(192) and (193) are valid for the point Q.

A vector $\underline{\underline{s}}$ can be constructed,

$$\underline{\underline{s}} = s_1 \cdot \underline{\underline{e}}_1 + s_2 \cdot \underline{\underline{e}}_2 \quad . \tag{194}$$

The height differences Z, taken with regard to the fixed test point P, construct a scalar field of two - parametric values along the sphere w (having the radius R'). Obviously, the vector $\underline{\underline{s}}$ can be represented by the gradient of the scalar Z field, taken along the sphere w, (192) (193),

$$\underline{\underline{s}} = - \nabla \cdot H_Q = - \nabla \cdot Z \quad . \tag{195}$$

Or,

$$s_1 = - (\nabla \cdot Z) \cdot \underline{\underline{e}}_1 \quad , \tag{196}$$

$$s_2 = - (\nabla \cdot Z) \cdot \underline{\underline{e}}_2 \quad . \tag{197}$$

(196) and (197) follow from (194) and (195) .

For any scalar function q, defined on the surface w, the gradient has the subsequent shape,

$$\nabla \cdot q = \frac{1}{R'} \frac{\partial q}{\partial \varphi} \underline{\underline{e}}_1 + \frac{1}{R' \cos\varphi} \frac{\partial q}{\partial \lambda} \underline{\underline{e}}_2 \quad . \tag{198}$$

φ, λ: the geocentric latitude and longitude. (199) is self-explanatory, (188) (189) ,

$$\underline{\underline{s}}^2 = s^2 = (\tan (g', n))^2 = s_1^2 + s_2^2 \quad . \tag{199}$$

s is the slope of the terrain.

The decomposition formula for the cosine function gives for (175)

$$D_M(1.1) = G \cdot \mu \cdot \tan (g', n) \cdot \left[\cos A''' \cos A' + \sin A''' \sin A' \right] \quad . \tag{200}$$

with (178) (179) and (188) (189), the relation (200) turns to

$$D_M(1.1) = G (\mu_1 \cdot s_1 + \mu_2 \cdot s_2) \quad . \tag{201}$$

The inner product of the two vectors $\underline{\underline{\mu}}$ and $\underline{\underline{s}}$ leads to , (185) (194),

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$$D_M(1.1) = G \cdot \underline{\underline{\mu}} \cdot \underline{\underline{s}} \tag{202}$$

Thus, the integral expression for J takes the following shape, (176) (202),

$$J = \frac{G}{4\pi R'} \iint_w \underline{\underline{\mu}} \cdot \underline{\underline{s}} \cdot S(p) \cdot dw \quad . \tag{203}$$

With intend to rearrange the integrand of (203), a new vector $\underline{\underline{a}}_o$ is introduced,

$$\underline{\underline{a}}_o = Z \cdot S(p) \cdot \underline{\underline{\mu}} \quad . \tag{204}$$

In (204), the scalar value Z and the expression S(p), and the components of the vector $\underline{\underline{\mu}}$ are all continuous functions of φ and λ, (182) (183). They can be considered as functions distributed along the sphere w. In this context, they are understood that they are functions of the variable co-ordinates of the point Q, only. But, in this context, the co-ordinates of the point P are constant.

The gradient of a scalar function q has the relation (198), for a function q distributed along the surface of the sphere w. Further,

$$dw = R'^2 \cdot dl \quad . \tag{205}$$

Now, a vector $\underline{\underline{q}}$ is introduced; it has the components q_1 and q_2, in the direction of $\underline{\underline{e}}_1$ and $\underline{\underline{e}}_2$. The divergence of this vector $\underline{\underline{q}}$, defined for points on the surface of the sphere w, can be described by q_1 and q_2 (in the spherical co-ordinates φ, λ; for points with the radius R' of the sphere w). (206) follows,

$$\operatorname{div} \underline{\underline{q}} = \nabla \cdot \underline{\underline{q}} = \frac{1}{R'} \cdot \frac{\partial q_1}{\partial \varphi} + \frac{1}{R' \cos\varphi} \frac{\partial q_2}{\partial \lambda} - \frac{\tan\varphi}{R'} q_1 \quad . \tag{206}$$

Thus, the divergence of the vector field $\underline{\underline{a}}_o$, distributed over the sphere w, has the form, (204) (205),

$$\nabla \cdot \underline{\underline{a}}_o = \operatorname{div} \underline{\underline{a}}_o = \operatorname{div} (Z \cdot S(p) \cdot \underline{\underline{\mu}}) = \nabla \cdot (Z \cdot S(p) \cdot \underline{\underline{\mu}}) =$$

$$= (\nabla \cdot Z) \cdot S(p) \cdot \underline{\underline{\mu}} + Z \cdot (\nabla \cdot S(p)) \underline{\underline{\mu}} + Z \cdot S(p) \cdot (\nabla \cdot \underline{\underline{\mu}}) \quad . \tag{207}$$

The function S(p) has a peculiarity. In case of approaching the test point P, the parameter p does tend to zero and S(p) does tend to infinity: If $p \to 0$, follows $S(p) \cong (2/p) \to \infty$. Since only continuous functions are tolerated in (207), the close neighborhood of the point P is separated, avoiding the above discussed singularity of S(p), Fig. 6.

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This near environment of the point P has the shape of a spherical cap, named w_0. w_0 is concentric to the point P, it has the spherical radius $R'p_0$ measured along the sphere w, and the circular bounds of w_0 are denoted by c_0. That part of w which is complementary to w_0 is denoted by w_{00}. Thus, the sum of w_0 and w_{00} is equal to w. In the domain w_{00}, Z and S(p) and the components of $\underline{\underline{\mu}}$ are continuous functions. Consequently, the vector $\underline{\underline{a}}_0$ of (204) is continuous in the domain w_{00}. Therefore, it is allowed to apply the integral theorem of Gauss (for the domain w_{00}) to the vector field $\underline{\underline{a}}_0$. The equation (208) follows,

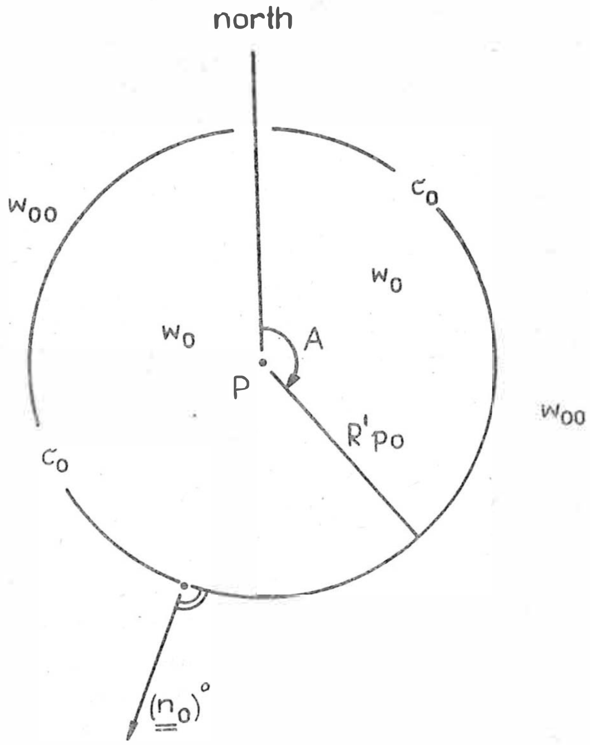

Fig. 6.

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$$\iint_{w_{oo}} (\nabla \cdot \underline{\underline{a}}_o)\ dw = - \oint_{c_o} \underline{\underline{a}}_o \cdot (\underline{\underline{n}}_o)^o \cdot dc_o \qquad (208)$$

$(\underline{\underline{n}}_o)^o$ is the unit vector of the normal of the circle c_o which is the boundary of the spherical cap w_o, the positive direction of $(\underline{\underline{n}}_o)^o$ is heading to the exterior of w_o, Fig. 6.

The circulatory integral on the right hand side of (208) does vanish, if the radius of the spherical cap does vanish, $R'p_o \longrightarrow 0$. This transition behavior is easily proved along the following lines. The definition of the inner product leads to

$$\underline{\underline{a}}_o \cdot (\underline{\underline{n}}_o)^o \leqslant \left| \underline{\underline{a}}_o \right| \cdot \left| (\underline{\underline{n}}_o)^o \right| \quad . \qquad (209)$$

$\underline{\underline{a}}_o$ comes from (204). In (204), the height difference Z does vanish in case the point Q approaches the point P. Further, the quotient $Z/(R' \cdot p_o)$ has a finite value if $R' \cdot p_o$ does vanish approaching the point P, whatever the azimuth A of the approach may be, Fig. 6. But, the Stokes function S(p) of (204) has another transition behaviour. For small values of p, S(p) can be approximated by 2/p. Thus, the Stokes function tends to infinity as 2/p, if p tends to zero. Consequently, for small values of p, the product $Z \cdot S(p)$ has the limit (for a star-shaped Earth with finite slopes of the terrain)

$$\lim_{p \to 0} \left[Z \cdot S(p) \right] = \lim_{p \to 0} \left[Z \ \frac{2}{p} \right] = \lim_{p \to 0} \left[\frac{Z}{R'p} \ 2\,R' \right] \quad . \qquad (209a)$$

Since $Z/(R'p)$ has a finite value if p tends to zero, the limit value of (209a) is a finite amount, also. Further on, the amount of the vector $\underline{\underline{\mu}}$ is also always finite, obviously. Consequently, approching the test point P, the amount of the vector $\underline{\underline{a}}_o$ of (204) is always finite. As to the inequality (209) and the vector $(\underline{\underline{n}}_o)^o$ of this relation, the amount of the vector $(\underline{\underline{n}}_o)^o$, being $\left|(\underline{\underline{n}}_o)^o\right|$, is by definition always equal to the unity. Thus, the absolute amount of the integrand of the circulatory integral on the right hand side of (208) has an upper limit if p_o tends to zero,

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$$\lim_{p_o \to 0} \left| \underline{\underline{a}}_o \cdot (\underline{\underline{n}}_o)^o \right| < N \qquad \text{(210)}$$

Hence, (208) (210), for p_o tending to zero,

$$\lim_{p_o \to 0} \left| \int_{c_o} \underline{\underline{a}}_o \cdot (\underline{\underline{n}}_o)^o \cdot dc_o \right| < 2\pi R'Np_o \qquad \text{(211)}$$

In case, p_o tends to zero, the right hand side of (211) tends to zero and, consequently, the left hand side of (211), too.

Thus, finally, (211) and (208) yield,

$$\lim_{p_o \to 0} \left[\iint_{w_{oo}} (\nabla \cdot \underline{\underline{a}}_o) \cdot dw \right] = 0, \qquad \text{(211a)}$$

or, abbreviating the denotation, Fig. 6,

$$\iint_{w} (\nabla \cdot \underline{\underline{a}}_o) \cdot dw = 0 \qquad \text{(212)}$$

The integrand of (212) comes from (207). In (207), the gradient of the Stokes function is equal to

$$\nabla \cdot S(p) = \frac{d\,S(p)}{R' \cdot dp} \cdot \underline{\underline{e}}_p \qquad \text{(213)}$$

$\underline{\underline{e}}_p$ is a unit vector, distributed over the sphere w as a tangent vector of it. It is heading into the direction of growing values of the parameter p. Thus, the combination of (207) (195) (213) leads to,

$$\text{div } \underline{\underline{a}}_o = - \underline{\underline{s}} \cdot \underline{\underline{\mu}} \cdot S(p) + \frac{Z}{R'} \cdot \frac{d\,S(p)}{d\,p} \cdot \underline{\underline{e}}_p \cdot \underline{\underline{\mu}} + Z \cdot S(p) \cdot (\nabla \cdot \underline{\underline{\mu}}) \; ; \qquad \text{(213a)}$$

further, with (203) (212), and with

$$\mu_p = \underline{\underline{e}}_p \cdot \underline{\underline{\mu}} \qquad \text{(213b)}$$

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$$\iint_W \underline{\underline{e}} \cdot \underline{\underline{\mu}} \cdot S(p) \cdot dw = \iint_W Z \circ (\nabla \underline{\underline{\mu}}) \cdot S(p) \cdot dw + \iint_W \frac{Z}{R'} \frac{dS(p)}{dp} \cdot \mu_p \cdot dw \quad . \tag{214}$$

μ_p is the component of the vector $\underline{\underline{\mu}}$ pointing into the direction of the unit vector $\underline{\underline{e}}_p$.

Hence, the here needed integral J turns to, (203),

$$J = \frac{G}{4\pi R'} \iint_W Z \cdot (\nabla \cdot \underline{\underline{\mu}}) \cdot S(p) \cdot dw + \frac{G}{4\pi R'^2} \iint_W Z \cdot \frac{dS(p)}{dp} \mu_p \cdot dw \quad . \tag{215}$$

With (206) and (A 444), the expression for $\nabla \cdot \underline{\underline{\mu}}$ takes on the following shape,

$$\Phi(\mu_1, \mu_2) = \frac{\partial \mu_1}{R' \, \partial\varphi} + \frac{\partial \mu_2}{R' \cdot \cos\varphi \cdot \partial\lambda} - \frac{1}{R'} \cdot \mu_1 \cdot \tan\varphi \cdot \tag{216}$$

$$\Phi(\mu_1, \mu_2) = \nabla \cdot \underline{\underline{\mu}} \quad . \tag{217}$$

In the here discusssed applications, μ_1 and μ_2 are understood that they are the components of the plumb-line deflection at the Earth's surface u, i. e. $\mu_{1.u}$ and $\mu_{2.u}$, (153) (154).

Thus, more precisely written than in (216),

$$\Phi(\mu_1, \mu_2) = \Phi(\mu_{1.u}, \mu_{2.u}) =$$

$$= \frac{1}{R'} \cdot \frac{\partial \mu_{1.u}}{\partial \varphi} + \frac{1}{R' \cdot \cos\varphi} \cdot \frac{\partial \mu_{2.u}}{\partial \lambda} - \frac{1}{R'} \cdot \tan\varphi \cdot \mu_{1.u} \quad . \tag{217a}$$

$$\Phi(\mu_1, \mu_2) = \Phi(\mu_{1.u}, \mu_{2.u}) = \nabla \cdot \underline{\underline{\mu}}_u \quad . \tag{217b}$$

The value of μ_p can be transformed in the following way, (180) (215), if μ_p is understood that it is the radial component of the plumb-line deflection for the potential M taken at the Earth's surface u,

$$\mu_p = \mu_{p.u} \quad , \tag{217c}$$

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$$\mu_p = - \left[\frac{1}{G \cdot R'} \cdot \frac{\partial M}{\partial p} \right]_u \quad . \tag{218}$$

Introducing the relations from (216) to (218) into (215), the expression for J turns to,

$$J = \frac{G}{4 \pi R'} \iint_w Z \cdot \Phi(\mu_1, \mu_2) \cdot S(p) \cdot dw -$$

$$- \frac{1}{4 \pi R'^2} \iint_w Z \cdot \frac{dS(p)}{dp} \cdot \frac{1}{R'} \cdot \frac{\partial M}{\partial p} \cdot dw \quad . \tag{219}$$

With (176) and (219), the relation (146) for the potential M takes on the following shape,

$$\{M\} = \frac{1}{4 \pi R'} \iint_w \left[\Delta g_M + GZ \cdot \Phi(\mu_1, \mu_2) + \frac{3}{4\pi} \cdot \frac{F(M)}{R'} \right] S(p) \cdot dw +$$

$$+ \frac{1}{2\pi} \cdot \{F(M)\} - \frac{1}{4 \pi R'^2} \iint_w Z \cdot \frac{dS(p)}{dp} \cdot \frac{1}{R'} \cdot \frac{\partial M}{\partial p} \cdot dw \quad . \tag{220}$$

9. The model potential M represented by the Stokes integral and the supplementary topographical terms

9.1. The formula for test points in high mountains

With regard to the further developments, the equation for M of the form (220) undergoes some rearrangements. The topographical terms of (220) are now denominated by new symbols.
They are given by (221) and (222),

$$C_1(M) = G \cdot Z \cdot \Phi(\mu_1, \mu_2) \quad , \tag{221}$$

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$$\Omega_1 (M) = \frac{1}{4\pi R'} \iint_w \frac{3}{4\pi} \cdot \frac{F(M)}{R'} \cdot S(p) \cdot dw + \frac{1}{2\pi} \cdot \{F(M)\} -$$

$$- \frac{1}{4\pi R'^2} \iint_w Z \cdot \frac{dS(p)}{dp} \cdot \frac{1}{R'} \cdot \frac{\partial M}{\partial p} \cdot dw \quad . \tag{222}$$

(220) (221) (222) yield the final expression,

$$\{M\} = \frac{1}{4\pi R'} \iint_w \left[\Delta g_M + C_1(M) \right] S(p) \cdot dw + \{\Omega_1 (M)\} \quad . \tag{223}$$

Ω_1 (M) has the following explicit expression, convenient for numerical routine calculations, (74) (222), (75) to (78), (80) to (84),

$$\Omega_1 (M) = \frac{3}{(4\pi R')^2} \iint_w F(M) \cdot S(p) \cdot dw +$$

$$+ \frac{1}{2\pi} \iint_w \Delta g_M \cdot \frac{Z}{R} \left[2 - \frac{1}{y + y^2} \right] \frac{1}{e'} \cdot dw +$$

$$+ \frac{1}{2\pi} \iint_w \frac{M}{R} \cdot \frac{Z}{R} \left[1 - \frac{2}{y + y^2} \right] \frac{1}{e'} \cdot dw +$$

$$+ \frac{1}{2\pi} \iint_w \frac{M}{R} \cdot \frac{v_1}{R} \cdot dw +$$

$$+ \frac{1}{2\pi} \iint_w \frac{\partial M}{R \, \partial p} \cdot \left[- \frac{1}{R} \cdot \frac{(\cos p/2)^2}{\sin p} \cdot b_7 - \frac{Z}{2 R'^2} \cdot \frac{dS(p)}{dp} \right] \cdot dw +$$

$$+ \frac{1}{2\pi} \iint \Delta g_M \cdot \frac{-x^2}{y + y^2} \cdot de' \cdot dA +$$

$$+ \frac{1}{2\pi} \iint \frac{M}{R} \cdot \left[\frac{-2x^2}{y + y^2} + v_3 \right] de' \cdot dA +$$

$$+ \frac{1}{2\pi} \iint \frac{\partial M}{\partial e'} \cdot (v_2 - b_{11}) \cdot de' \cdot dA +$$

$$+ \frac{1}{2\pi} \iint (-GZ) \cdot \Phi (x^* \cdot \mu_1, \; x^* \cdot \mu_2) \cdot de' \cdot dA \quad . \tag{224}$$

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In (224), – if, there, dw is used as integration element – , the integration has to cover whole the globe. But, – if the product $de' \cdot dA$ is the integration element – , the integration can be limited to the near surroundings of the test point P, up to a distance of some tens of kilometers, only.

As to the function F(M) in the integrand of the first term on the right hand side of (224), the values of F(M) can be computed by (74) and by the relations from (74a) up to (74h). But now, T has to be replaced by M, and Δg_T by Δg_M , furthermore, ξ and η have to be replaced by μ_1 and μ_2 . These modifications lead to the relations (225), (225a) to (225h) ,

$$F(M) = \sum_{i=1}^{8} f_i(M) \quad ; \tag{225}$$

$$f_1(M) = \iint_w \Delta g_M \cdot \frac{Z}{R} \cdot \left[2 - \frac{1}{y + y^2} \right] \cdot \frac{1}{e'} \cdot dw \quad , \tag{225a}$$

$$f_2(M) = \iint_w \frac{M}{R} \cdot \frac{Z}{R} \cdot \left[1 - \frac{2}{y + y^2} \right] \cdot \frac{1}{e'} \cdot dw \quad , \tag{225b}$$

$$f_3(M) = \iint_w \frac{M}{R} \cdot \frac{v_1}{R} \cdot dw \quad , \tag{225c}$$

$$f_4(M) = - \iint_w \frac{\partial M}{R \, \partial p} \cdot \frac{1}{R} \cdot \frac{(\cos p/2)^2}{\sin p} \cdot b_7 \cdot dw \quad , \tag{225d}$$

$$f_5(M) = - \iint \Delta g_M \cdot \frac{x^2}{y + y^2} \cdot de' \cdot dA \quad , \tag{225e}$$

$$f_6(M) = \iint \frac{M}{R} \cdot \left[\frac{-2x^2}{y + y^2} + v_3 \right] \cdot de' \cdot dA \quad , \tag{225f}$$

$$f_7(M) = \iint \frac{\partial M}{\partial e'} \cdot (v_2 - b_{11}) \cdot de' \cdot dA \quad , \tag{225g}$$

$$f_8(M) = - \iint G \cdot Z \cdot \Phi(x^* \cdot \mu_1 , x^* \cdot \mu_2) \cdot de' \cdot dA \quad . \tag{225h}$$

The expressions x^* , x , y , v_1 , v_2 , v_3 , b_7 , b_{11} are explained by (75)(76) (78), (80) up to (84). Again, the symbol dw stands for the global coverage by the

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integration, de'·dA for the coverage of the near surroundings, only.

The universally valid formulas, from (223) to (225h), can be applied wherever the test point P may be situated, even in high mountains. The relations (223) to (225h) can be handled without any complication, they have no singularity and no divergences.

9.2. The formula for test points in low mountain ranges or in the lowlands

The detailed universal formulas (224) and (225) for $\Omega_1(M)$ and for F(M) will find an application in seldom and extreme situations, only. They will be of use if the cliffs in the surroundings of the test point will reach an inclination of 30° or 45°, and more. They are valid for all finite inclination values, since a star-shaped Earth was presupposed.

Exterior of such regions, the formulas (224) and (225) can be simplified enormously. Such a simplification, often permitted, was already discussed in connection with the transition from the formula (74) to the formula (79), (i. e. from F(T) to $F^{*}(T)$). These simplifications are governed by the constraint, that the inequality

$$x^2 \ll 1 \qquad (225i)$$

has to be fullfilled, (66). Only in high mountains, the inequality (225i) will be violated. Besides of (225i), these simplifications imply also the neglection of a relative error of the order of Z/R in the small topographical supplements (i. e. F(M) and Ω_1 (M)). A relative error of 10^{-3} to 10^{-4} is permitted in these supplements, which do not reach an amount of about 1 m. An error smaller than 10^{-3} m can be tolerated in any case.

In the course of these simplifications, caused by the transition from the high mountains to the lowlands, F(M) of (225) can be replaced by the simple lowland expression $F^{*}(M)$, described by (227). Further on, $\Omega_1(M)$ of (224) turns to the lowland expression $\Omega_1^{*}(M)$, described by (226) (230). Thus, accounting for (225i) and neglecting relative errors of the order of Z/R, (225) and (224) change to the simple shape of (227) and (226) for the lowland expressions,

$$\Omega_1(M) \longrightarrow \Omega_1^{*}(M) \quad , \qquad (225j)$$

$$F(M) \longrightarrow F^{*}(M) \quad . \qquad (225k)$$

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Thus, (222) turns to

$$\Omega_1^*(M) = \frac{1}{4\pi R} \iint_w \frac{3}{4\pi} \cdot \frac{F^*(M)}{R} \cdot S(p) \cdot dw \; +$$

$$+ \frac{1}{2\pi} \cdot \left\{ F^*(M) \right\} \; -$$

$$- \frac{1}{4\pi R^2} \iint_w Z \cdot \left\{ \frac{d\,S(p)}{d\,p} \right\} \cdot \frac{1}{R} \cdot \frac{\partial M}{\partial p} \cdot dw \; . \qquad (226)$$

Further on, by (79),

$$F^*(M) = \sum_{i=1}^{3} f_i^*(M) \; , \qquad (227)$$

$$f_1^*(M) = \iint_w \Delta g_M \cdot \frac{Z}{R} \cdot \frac{3}{2} \cdot \frac{1}{e_o} \cdot dw \; , \qquad (227a)$$

$$f_2^*(M) = \iint_w \frac{M}{R} \cdot \frac{Z}{R} \cdot \frac{1}{e_o} \cdot dw \; , \qquad (227b)$$

$$f_3^*(M) = - \iint_w \frac{\partial M}{R\,\partial p} \cdot \frac{Z}{4R^2} \cdot \frac{\cos p/2}{(\sin p/2)^2} \cdot dw \; . \qquad (227c)$$

$$e_o = 2 \cdot R \cdot \sin p/2 \; . \qquad (228)$$

The third term on the right hand side of (226) and the term of (227c), multiplied with $(1/2\pi)$, can be combined to the following expression, (229),

$$- \frac{1}{8\pi R^2} \iint_w \frac{\partial M}{R\,\partial p} \cdot Z \cdot \left[\frac{\cos p/2}{(\sin p/2)^2} + 2\,\frac{d\,S(p)}{dp} \right] \cdot dw \; . \qquad (229)$$

The relations (227), (227a) to (227c), and (229) are introduced into (226). Along these lines, the final form of $\Omega_1^*(M)$ is reached, (230).

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$$\Omega_1^*(M) = \frac{3}{(4\pi R)^2} \cdot \iint_w F^*(M) \cdot S(p) \cdot dw \quad +$$

$$+ \frac{1}{2\pi} \iint_w \Delta g_M \cdot \frac{Z}{R} \cdot \frac{3}{2} \cdot \frac{1}{e_o} \cdot dw \quad +$$

$$+ \frac{1}{2\pi} \iint_w \frac{M}{R} \cdot \frac{Z}{R} \cdot \frac{1}{e_o} \cdot dw \quad -$$

$$- \frac{1}{8\pi R^2} \iint_w \frac{\partial M}{R \partial p} \cdot Z \cdot \left[\frac{\cos p/2}{(\sin p/2)^2} + 2 \frac{d\, S(p)}{dp} \right] \cdot dw \quad , \qquad (230)$$

In the integrand of the first term on the right hand side of (230), the value of $F^*(M)$ can be computed by the formulas described by (227), (227a) to (227c).

Consequently, in the most frequent cases of our applications, if the test point P is not situated in the peak area of the high mountains: about the following form, it is emphasized that it is convenient for routine calculations, (223) (221) (227) (230); it is the lowland form,

$$\{M\} = \frac{1}{4\pi R'} \iint_W \left[\Delta g_M + C_1 (M) \right]' S(p) \cdot dw + \left\{ \Omega_1^*(M) \right\} \quad . \qquad (231)$$

Later on, this formula undergoes a rearrangement, transforming the left hand side back, from the potential M to the potential T, (see chapter 11).

10. The Helmert condensation method

Now, the mountain masses situated above the mean globe v having the radius R (or above the mean ellipsoid of the Earth, to be more precise) are condensed along this sphere v. The real mountain masses of the real density cannot be considered here, since the precise values of these real density values are unknown. But, for the here discussed problem, it is possible to substitute the real density of the mountain masses by the standard density having the amount of $\vartheta = 2650$ kg m^{-3}, (142), (see Fig. 5). As to the use of the standard density, this easy substitution is opportune, and it makes no trouble. The crucial point for the introduction of the potential B of the visible mountain masses is the fact that, in the main, the gravitational force caused by the difference potential T - B has no perceptible correlation with

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the topographical heights. This peculiarity is right, may the potential B be computed in terms of the real density values, or in terms of the standard density. The here executed derivations make use of the letter version.
If the density of these masses changes over from the real values to the standard value, the accompanying alteration of the gravitational force is relative small, it has no clear correlation with the heights. A long wave residual correlation of this kind is discussed by the relations (289)(290).

For a test point P^* situated on the spherical surface v, the gravitational potential B^* of these condensed mountain masses has the following representation, (condensed at the sphere v; R : Radius),

$$B^* = (L_1 + L_2)_{P^*} \quad . \tag{232}$$

For this potential B^*, the derivative with regard to the radius r has the following expression, if approaching the test point P^* at the surface v from the exterior space of the globe v, Fig. 5, Fig. 2,

$$\frac{\partial B^*}{\partial r} = (L_3 + L_4)_{P^*} \quad . \tag{233}$$

The symbols L_1, L_2, L_3, L_4 of (232) and (233) have the following equations,

$$(L_1)_{P^*} = 4 \cdot \pi \cdot f \cdot \vartheta \cdot R \cdot H_P \quad , \tag{234}$$

$$(L_2)_{P^*} = f \vartheta \iint_v Z \cdot \frac{1}{e_o} \cdot dv \quad , \tag{235}$$

$$(L_3)_{P^*} = - 4 \cdot \pi \cdot f \cdot \vartheta \cdot H_P \quad , \tag{236}$$

$$(L_4)_{P^*} = - f \vartheta \iint_v Z \cdot (\sin p/2) \cdot \frac{1}{(e_o)^2} \cdot dv \quad . \tag{237}$$

f is again the gravitational constant, R is the radius of the sphere v, Fig. 2,

$$e_o = 2R \cdot \sin p/2 \quad . \tag{238}$$

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As it is evident from Fig. 2, H_P is the height attached to the test point P^*, within the scope of the condensation method. Obviously, the density of the surface distribution underlying the potential B^* is equal to $\vartheta \cdot H_Q$. The equations (234) (235) (236) (237) represent the values L_1, L_2, L_3, L_4, taken for the test point P^*.

For the moving point Q^* at the sphere v, the following relations are valid, analogous to the above relations for P^*, Fig. 2,

$$(L_1)_{Q^*} = 4 \cdot \pi \cdot f \cdot \vartheta \cdot R \cdot H_Q \quad , \tag{239}$$

$$(L_2)_{Q^*} = f\,\vartheta \iint_v (H_Y - H_Q) \cdot \frac{1}{e_{oo}} \cdot dv \quad , \tag{240}$$

$$(L_3)_{Q^*} = -\,4 \cdot \pi \cdot f \cdot \vartheta \cdot H_Q \quad , \tag{241}$$

$$(L_4)_{Q^*} = -\,f\,\vartheta \iint_v \frac{H_Y - H_Q}{(e_{oo})^2} \cdot (\sin\ (p/2)_{oo}) \cdot dv \quad . \tag{242}$$

The values e_o and sin p/2 refer to the distance between the two points Q^* and P^*. But, the values e_{oo} and $\sin\,(p/2)_{oo}$ relate to the distance between the points Y^* and Q^*, Fig. 2,

$$e_{oo} = 2 \cdot R \cdot \sin\ (p/2)_{oo} \quad . \tag{243}$$

In chapter 3, a detailed solution was derived for the problem of a spherical boundary surface. This solution is rigorously valid. It can be applied to the potential B^* which has a spherical surface distribution as the underlying gravitating scource, (31), (232) (234) (235). The potential B^* causes certain gravity anomalies in the exterior of the sphere v. Along the spherical surface v, these gravity anomalies are represented in terms of the potential B^*, by the relation (244), (see also (22)).

$$\Delta g_{B^*} = -\,\frac{\partial B^*}{\partial r} - \frac{2}{R} \cdot B^* \quad . \tag{244}$$

The integral relation (31) leads to

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$$B^{*} = \frac{1}{4\pi R} \iint\limits_{v} \Delta g_{B^{*}} \; S(p) \cdot dv \quad , \tag{245}$$

or, writing it with a more clear distinction of the different points the various values refer to,

$$(B^{*})_{P^{*}} = \frac{1}{4\pi R} \iint\limits_{v} (\Delta g_{B^{*}})_{Q^{*}} \; S(p) \cdot dv \; . \tag{246}$$

The relations from (232) to (244) are introduced into (246). Hence,

$$\left\{ (L_1)_{P^{*}} \right\} + \left\{ (L_2)_{P^{*}} \right\} =$$

$$= \frac{1}{4\pi R} \iint\limits_{v} \left[- (L_3)_{Q^{*}} - (L_4)_{Q^{*}} - \frac{2}{R} (L_1)_{Q^{*}} - \frac{2}{R} (L_2)_{Q^{*}} \right] \cdot S(p) \cdot dv. \tag{247}$$

In the relation (247), the parentheses stand for the direction that the constituents described by the surface spherical harmonics of the 0 th and 1 st degree are split off.

11. The retransformation from the model potential M back to the potential T

The essential property and the very important advantage of the relations (223) (231) is the fact that, in the integrand of (223) (231), the smoothed and small term $C_1(M)$ does appear.
Whereas in (114), the relative great and rugged term D_T (1.1) gives rise to a lot of trouble, if it is intended to compute this term.
The transition from D_T (1.1) to C_1 (M), that is the main reason for the introduction of the model potential M. However, not M is the required potential, but T is the potential to be determined.
Consequently, in (223), a retransformation from M back to T is necessary. But, in the course of this retransformation, the term $C_1(M)$ keeps to be unchanged. it is not retransformed.

In this context, the equations (145) and (150) are introduced into (223). Hence,

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$$\{M\} = \{T\} - \{B\} =$$

$$= \frac{1}{4\pi R'} \iint_w \left[\Delta g_T - \Delta g_B + C_1\ (M) \right] \cdot S(p) \cdot dw +$$

$$+ \left\{ \Omega_1\ (M) \right\} . \tag{248}$$

The equation (142) gives the possibility to compute the potential B for the test point P at the Earth's surface u, as it is needed in (248).

In case of the condensed masses, the potential B^* can be computed by (232) (234) (235) for the test point P^* situated at the spherical surface v , Fig. 2. On condition that P^* lies perpendicular below the point P, the difference between B in the point P and B^* in the point P^* is introduced by $[B]''$, (248a),

$$(B)_P - (B^*)_{P^*} = [B]'' . \tag{248a}$$

In an analogous way, the radial derivatives of these potentials B and B^* have the following equations, (233) (236) (237),

$$\left(\frac{\partial B}{\partial r}\right)_P - \left(\frac{\partial B^*}{\partial r}\right)_{P^*} = \left[\frac{\partial B}{\partial r}\right]'' . \tag{248b}$$

Taking the liberty to omit the suffix P at both the term B and the radial derivative of B, further, omitting also the suffix P^* which appears at B^* and the derivative of it (or at L_1, L_2, L_3, L_4), the subsequent relations are obtained,

$$B = L_1 + L_2 + [B]'' , \tag{249}$$

$$\frac{\partial B}{\partial r} = L_3 + L_4 + \left[\frac{\partial B}{\partial r}\right]'' . \tag{250}$$

The combination of (249) and (250) with (149) gives

$$\Delta g_B = - L_3 - L_4 - \frac{2}{r} L_1 - \frac{2}{r} L_2 - \left[\frac{\partial B}{\partial r}\right]'' - \frac{2}{r}[B]'' . \tag{251}$$

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r is the geocentric radius of the surface of the Earth u,

$$r = R + H_P + Z \quad . \qquad (252)$$

(252) leads to

$$\frac{2}{r} \cong \frac{2}{R} \cdot \left[1 - \frac{H_P + Z}{R} \right] = \frac{2}{R} - 2 \cdot \frac{H_P + Z}{R^2} \quad . \qquad (253)$$

(253) is introduced into (251),

$$\Delta g_B = - L_3 - L_4 - \frac{2}{R} \cdot L_1 - \frac{2}{R} \cdot L_2 - \left[\frac{\partial B}{\partial r} \right]'' - \frac{2}{r} \cdot [B]'' + 2 \cdot \frac{H_P + Z}{R^2} \cdot (L_1 + L_2) \quad . \qquad (254)$$

Now, the relations (249) and (254) are put into (248). The amount of (2/r) $\cdot$ [B]'' does not surmount some microgal (10^{-6} cm sec $^{-2}$); thus, a relative error of the order of H/R or Z/R can be neglected there, [4] [5] . Consequently, (248) turns to (255),

$$\{M\} = \{T\} - \{(L_1)_{P*}\} - \{(L_2)_{P*}\} - \{[B]''\} = \frac{1}{4\pi R'} \iint_w X' \, S(p) \, dw + \{\Omega_1(M)\} \quad , \qquad (255)$$

with

$$X' = \Delta g_T + (L_3)_{Q*} + (L_4)_{Q*} + \frac{2}{R} \cdot (L_1)_{Q*} + \frac{2}{R} \cdot (L_2)_{Q*} + \left[\frac{\partial B}{\partial r} \right]'' + \frac{2}{R} \cdot [B]'' - 2 \cdot \frac{H_P + Z}{R^2} \cdot (L_1 + L_2)_{Q*} + C_1(M) \quad . \qquad (255a)$$

The transition from R' to R , and from the surface w to the surface v, has the following equations, Fig. 2,

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$$\frac{1}{R'} = \frac{1}{R + H_P} \cong \frac{1}{R} - \frac{H_P}{R^2} , \tag{256}$$

and

$$dw = \left(\frac{R'}{R}\right)^2 \cdot dv \cong dv + 2 \cdot \frac{H_P}{R} \cdot dv . \tag{257}$$

The relations (256) and (257) are introduced into (255), neglecting a relative error of the order of H_P/R in the amount of $\{\Omega_1(M)\}$. These rearrangements lead to the equation (258),

$$\{T\} - \{(L_1)_{P*}\} - \{(L_2)_{P*}\} - \{[B]''\} =$$

$$= \frac{1}{4\pi R} \iint\limits_v X \cdot S(p) \cdot dv + \{\Omega_1(M)\} -$$

$$- \left\{\frac{H_P}{R} \cdot M\right\} + 2 \cdot \left\{\frac{H_P}{R} \cdot M\right\} , \tag{258}$$

with, (255a) , replacing in (254) the multiplier (2/r) by (2/R), at $[B]''$,

$$X = X' + 2 \cdot \frac{H_P + Z}{R^2} \cdot \left[(L_1 + L_2)_{Q*} - B\right] \cong X' . \tag{258a}$$

In the transition from X' to X , the errors of the kind already discussed by the lines between the equations (254) and (255) are neglected.

The potential B in the brackets on the right hand side of (258a) refers to the point Q at the Earth's surface u, Q lies vertical above Q^*, Fig. 2.

The relation (247) of the condensation method and the equation (258) yield

$$\{T\} = \frac{1}{4\pi R} \iint\limits_v X_1 \cdot S(p) \cdot dv + \{\Omega_1(M)\} + \left\{\frac{H_P}{R} \cdot M\right\} + \{[B]''\} , \tag{259}$$

This above equation (259) is important. As to the topographical additives appearing in (259) completing the original shape of the Stokes integral, these additives are now expressed in terms of the smoothed M potential and the smoothed anomalies Δg_M , instead of the T potential, and instead of the anomalies Δg_T which are not smoothed in the mountains. The term X_1 appearing in (259), this term has the following expression (259a) ,

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$$X_1 = \Delta g_T + \left[\frac{\partial B}{\partial r}\right]'' + \frac{2}{R}\left[B\right]'' + C_1 (M) - 2B \frac{H_Q}{R^2} \quad . \qquad (259a)$$

In case the test point P at the surface of the Earth u is not situated in high mountain ranges, the relation (225j) and (225 k) can be applied in (259). Then, Ω_1 (M) can be replaced by Ω_1^* (M), according to (230). The computation of Ω_1^*(M) is much more easy than that of Ω_1 (M).

12. The final formula for the perturbation potential T in terms of the gravity anomalies

12.1. The perturbation potential T expressed by the Stokes integral and the topographical supplements

In the expression for X_1 described by (259a), the second and the third term on the right hand side depend on the potential B. These two terms can be expressed by the plane terrain reduction of the gravity which is generally denoted by the symbol C, (see [4] page 38, equation (97)). The following relation is valid

$$\left[\frac{\partial B}{\partial r}\right]'' + \frac{2}{R}\left[B\right]'' = C + \delta C \quad , \qquad (260)$$

with

$$\delta C = \delta_1 C + \delta_2 C + \delta_3 C + \delta_4 C \quad . \qquad (261)$$

In seldom cases only, the first three terms on the right hand side of (261) will surmount the amount of 1 μgal, [4].
Therefore, these terms can be neglected. As to $\delta_4 C$, it has the rather simple formula

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$$\delta_4 C = 4 \pi f \vartheta H_Q \frac{H_Q}{R} , \qquad (262)$$

as can be taken from [4] .
For H_Q = 2 km, the expression of (262) leads to an amount for $\delta_4 C$ which is equal to 0.1 mgal (i. e. 10^{-4} cm sec^{-2}).
Thus, also the term $\delta_4 C$ seems to be within the noise of the method (gravity data noise) in the routine applications, generally. To be complete, $\delta_4 C$ is taken along; with (260), and with (262), we have the following relation, thus,

$$\left[\frac{\partial B}{\partial r}\right]'' + \frac{2}{R}\left[B\right]'' \cong C + 4 \pi f \vartheta H_Q \frac{H_Q}{R} . \qquad (263)$$

Further on, the last term on the right hand side of (259a) undergoes a rearrangement and a combination with (263).
Considering (232) (239) (240), the following development is found,

$$- 2 B \frac{H_Q}{R^2} \cong - 2 (L_1 + L_2)_{Q^*} \frac{H_Q}{R^2} =$$

$$= - 8 \pi f \vartheta H_Q \frac{H_Q}{R} - 2 (L_2)_{Q^*} \frac{H_Q}{R^2} . \qquad (264)$$

In (264), the term

$$- \frac{2}{R} \left[B\right]'' \frac{H_Q}{R} \qquad (264a)$$

was neglected, since it will not be greater than about 10^{-3} μgal (i. e. 10^{-9} cm sec^{-2}), (see [4], page 36).

The combination of the 2 nd, the 3 rd, and the 5 th term on the right hand side of (259a) gives (265), accounting for (263) (264),

$$\left[\frac{\partial B}{\partial r}\right]'' + \frac{2}{R}\left[B\right]'' - 2B \frac{H_Q}{R^2} = C + C_2 , \qquad (265)$$

with

$$C_2 = - 4 \pi f \vartheta H_Q \frac{H_Q}{R} - 2 (L_2)_{Q^*} \frac{H_Q}{R^2} . \qquad (266)$$

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The relations (259) (259a) (265) (266) lead to the following final result for the solution of the boundary value problem

$$\{T\} = \frac{1}{4\pi R} \iint\limits_{v} \left[\Delta g_T + C + C_1(M) \right] S(p) \cdot dv + \{\Omega(M)\}. \tag{267}$$

The topographical supplement of (267) has the following expression

$$\Omega(M) = \Omega_1(M) + M \cdot \frac{H_P}{R} + [B]'' + \frac{1}{4\pi R} \cdot \iint\limits_{v} C_2 \cdot S(p) \cdot dv . \tag{268}$$

As to (267), $C_1(M)$ comes from (221) and (216), (217a) and (217b) ,

$$C_1(M) = GZ \cdot \left[\frac{\partial \mu_1}{R' \partial \varphi} + \frac{\partial \mu_2}{(R' \cdot \cos \varphi) \partial \lambda} - \frac{\tan \varphi}{R'} \mu_1 \right] . \tag{269}$$

As to (268), $\Omega_1(M)$ is described by (224), valid also in the high mountain ranges. In (269), μ_1 and μ_2 stand for the surface values $\mu_{1.u}$, $\mu_{2.u}$,(217a).

The potential M is computed by (145), with approximative values of T and with B according to (142) (144) .
The M values along the surface of the Earth u are computed by

$$M = T - f\vartheta \iiint\limits_{V} \frac{1}{e} \cdot dV . \tag{270}$$

The 1st and the 2nd term on the right hand side of (268) depend on the M values of (270), valid for points along the surface u. In (270), e is the straight distance between the test point P at the surface of the Earth u and the volume element dV. The potential M influences the expression (268) after multiplication with the very small factor (H_P/R'). Thus, in (270), approximative values can be accepted not only for T, but also for B. Hence, B is replaced by the potential B^* of the condensed masses. (270) turns to, (232) (234) (235),

$$M \cong T - f\vartheta \iint\limits_{v} H_Q \cdot \frac{1}{e_o} \cdot dv . \tag{271}$$

A precision of ± 10 m to ± 50 m in the computed amount of M/G will suffice, in any case, computing the amount of (M/G) by the formula (271); - since later on, in the relation (268), this amount of (M/G) comes to be multiplied by the factor (H_P/R) the amount of which reaches about (1/1000) or (1/10000) , only.

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As to the 3rd term of (268), the amount of $[B]''/G$ will seldom surmount some centimeters. $[B]''$ can be computed by the formulas given in [4] , page 35, 36; (see also [5], chapter B).

The 4th term on the right hand side of (268) can easily be calculated by (266).

The relation (267) is the high mountain variant of the solution of the boundary value problem. The much more simple lowland variant of the solution has the following shape,

$$\{T\} = \frac{1}{4\pi R} \iiint\limits_{v} \left[\Delta g_T + C + C_1(M)\right] \cdot S(p) \cdot dv + \left\{ \Omega^*(M) \right\}, \qquad (272)$$

(225j) (225k) (226) (227) and (230) , with

$$\Omega^*(M) = \Omega_1^*(M) + M \cdot \frac{H_P}{R} + [B]'' + \frac{1}{4\pi R} \cdot \iiint\limits_{v} C_2 \cdot S(p) \cdot dv. \qquad (273)$$

The expression (267) should be applied for test points situated in high mountains. For test points situated in the lowlands, the simple shape (272) will bring a computation relief.

The relation (273) is derived from (268) under consideration of the substitutions described by (225j) (225k), and applying (230); (see also [6]).

12.2. The supplementary term $C_1(M)$

Beforehand, the structure and the main properties of the term $C_1(M)$, appearing in (267) (269) (272), should be sketched. Seldom only, the amount of $C_1(M)$ will be greater than 1 mgal (i. e. 10^{-3} · cm/sec^2); further, it will be positive and negative. Thus, the $C_1(M)$ values will generally not surmount the noise of the free-air anomalies Δg_T . Further, the $C_1(M)$ values will generally not exel the noise of the errors committed in the determination of the C values, obtained by numerical computations in terms of the heights. Consequently, in most cases, the neglection of $C_1(M)$ in the brackets of (267) and (272) will be justified, before the background of the noise of the Δg_T and C values.

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Now, the details of the computation of the $C_1(M)$ values are to be discussed.

The formula (269) representing the $C_1(M)$ values in terms of the deflections μ_1 and μ_2 was applied in the Austrian Alps. Along the lines of (269), the following results were obtained, if

$$H_Q - H_P = Z = 1 \text{ km} :$$

a) The mean value of $|C_1(M)|$ over a distance of 300 km was about 0.1 mgal.

b) The mean value of $|C_1(M)|$ over a distance of 200 km was about 0.1 mgal.

c) The mean value of $|C_1(M)|$ over a distance of 40 km was about 0.8 mgal.

d) The mean value of $|C_1(M)|$ over a distance of 20 km was about 0.5 mgal.

These above results can be found in: [4], page 42, 43, 44, 45 of chapter B.

As to the relation which connects the radial derivative of the M potential with the Bouguer anomalies, the investigations of [5], chapter D, section 5, contain all the needed deliberations. In [5], the following equation was obtained, (eq. (67) in another place),

$$\frac{\partial M}{\partial r} = - \Delta g_{Bouguer} + \sigma , \tag{274}$$

with

$$\sigma = 2 \pi f \vartheta \zeta - \frac{2G}{R} \zeta + \frac{1}{2} f \vartheta R \iint\limits_{l} H_Q \frac{1}{e_o} dl. \tag{275}$$

In [5], chapter D, section 5 and 6, it was shown that σ has a small amplitude and a great wave length. Thus, the height gradient of $\partial M/\partial r$ can be identified with the height gradient of the Bouguer anomalies, in sufficient approximation, (see eq. (131), page 140, at another place, [5]).

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The determination of $C_1(M)$ by the plumb-line deflections μ_1, μ_2 according to (269) allows to get an idea of the amounts of $C_1(M)$. But, this method is not convenient for a general application in the routine determinations of $C_1(M)$, since there is not a sufficient dense net of the global μ_1, μ_2 values. Therefore, (269) is now rearranged expressing $C_1(M)$ in terms of the Bouguer anomalies (the refined Bouguer anomalies are here considered implying also the plane terrain reduction of the gravity, C). The μ_1 values are understood that they are distributed along the surface of the Earth u, Fig. 7. In the derivations of μ_1 (resp. μ_2), the way from P_1 to P_2 conducts via $P_{1.2}$. The two points P_1 and P_2 are situated on the oblique surface of the Earth u; ν_x is the inclination of the terrain in the vertical plane through P_1 and P_2. In Fig. 7, these two surface points are situated in the north-south direction.

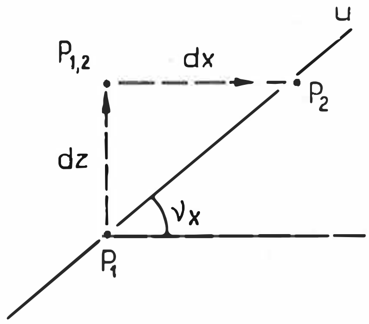

Fig. 7.

The following lines are self-explanatory, (153) (154) (269), (217a) and (217b),

$$\frac{\partial \mu_{1.u}}{R\,\partial\varphi} = \lim_{\Delta x \to 0} \left[\frac{(\mu_1)_{P_2} - (\mu_1)_{P_1}}{\Delta x} \right] =$$

$$= \frac{\partial \mu_1}{\partial z} \cdot \frac{dz}{dx} + \frac{\partial \mu_1}{\partial x} \quad . \tag{276}$$

$$\frac{dz}{dx} = \tan \nu_x \ , \qquad \frac{dz}{dy} = \tan \nu_y \quad . \tag{277}$$

The arc element dx has horizontal direction, Fig. 7. Thus, dx is equal to the value of $R \cdot \partial\varphi$. Analogously, the other arc element is horizontal in the east-west direction, i.e. the arc element dy. Hence, dy is equal to the amount of $R \cdot \cos\varphi \cdot \partial\lambda$.

In (276), the deflection $\mu_{1.u}$ (resp. $\mu_{2.u}$) is the value of the deflection of the plumb-line μ_1 (resp. μ_2) taken on points situated on the oblique surface of the Earth u.

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Neglecting the 3rd term in the brackets of (269), (it amouts to not more than some tens microgals), the subsequent relations yield,

$$C_1(M) = C_{1.a} + C_{1.b} \tag{278}$$

$$C_{1.a} = GZ\left[\frac{\partial \mu_1}{\partial x} + \frac{\partial \mu_2}{\partial y}\right] , \tag{279}$$

$$C_{1.b} = GZ\left[\frac{\partial \mu_1}{\partial z} \cdot \tan \gamma_x + \frac{\partial \mu_2}{\partial z} \tan \gamma_y\right] . \tag{280}$$

$$\mu_1 = - \frac{1}{G} \cdot \frac{\partial M}{\partial x} , \tag{281}$$

$$\mu_2 = - \frac{1}{G} \cdot \frac{\partial M}{\partial y} . \tag{282}$$

$$C_{1.a} = Z\left[- \frac{\partial^2 M}{\partial x^2} - \frac{\partial^2 M}{\partial y^2}\right] , \tag{283}$$

and, with the Laplace equation,

$$C_{1.a} = Z \cdot \frac{\partial^2 M}{\partial z^2} . \tag{284}$$

And with (274), considering the fact that the vertical gradient of σ can be neglected (see [5], chapter D, section 6 ,page 139,14o;eq.(124)...(133)),

$$C_{1.a} = - Z \cdot \frac{\partial}{\partial z} (\Delta g_{Bouguer}) . \tag{285}$$

$$C_{1.b} = Z\left[- \frac{\partial^2 M}{\partial x \partial z} \cdot \tan \gamma_x - \frac{\partial^2 M}{\partial y \partial z} \cdot \tan \gamma_y\right] , \tag{286}$$

$$C_{1.b} = C_{1.b.1} + C_{1.b.2} , \tag{287}$$

$$C_{1.b.1} = Z\left[\frac{\partial}{\partial x} \Delta g_{Bouguer}\right] \tan \gamma_x , \tag{288}$$

$$C_{1.b.1} = Z\left[\frac{(\Delta g_{Bouguer})_o - (\Delta g_{Bouguer})_u}{\Delta x}\right]\left[\frac{(H)_o - (H)_u}{\Delta x}\right] \tag{289}$$

$C_{1.b.2}$ follows in a similar way, as $C_{1.b.1}$, exchanging x and y .

In (289), the differential quotient was replaced by the difference quotient; this procedure is allowed, since the Bouguer anomalies have

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the advantage to have not a pronounced correlation with the heights, in any case if short distances are considered. Over longer distances, in the areas of isostatic mountain roots, a certain correlation of these values can be observed, possibly. It is brought to bear by the formula (289).
As to (289), the parameters $Z = 1$ km, $(H)_o - (H)_u = 1$ km, $\Delta x = 50$ km, and a value of 60 mgal for the difference of the Bouguer anomalies in the first nominator of (289) (these Bouguer anomalies, perhaps, are caused by the isostatic mountain roots of the Alps) lead to a value of

$$\left| C_{1.b.1} \right| = 0.02 \text{ mgal.} \tag{290}$$

For $\left|C_{1.b.2}\right|$, a similar amount can be awaited. Consequently, $\left|C_{1.b}\right|$ will be smaller than 0.04 mgal, for the here underlying parameters. $C_{1.b}$ can be neglected, therefore. $C_1(M)$ can be replaced by $C_{1.a}$, (285).

$$C_1(M) = - Z \cdot \frac{\partial}{\partial H} \left(\Delta g_{Bouguer} \right) . \tag{291}$$

The above equation is equivalent to the relations (122) (132) of [5], chapter D, section 6. As demonstrated in [5], (291) leads to, (eq. (123c) at another place),

$$C_1(M) = - Z \cdot \frac{R^2}{2\pi} \iint\limits_{l} \frac{(\Delta g_{Bouguer})_Y - (\Delta g_{Bouguer})_Q}{e_{oo}^3} \cdot dl . \tag{292}$$

It may be stressed that in (291) a neglection of terms with higher powers of Z, (i. e. Z^2, Z^3, ...), did not take place. The right hand side of (291) comes not from a truncation of any series development of rising powers of Z.

The impact that $C_1(M)$ exerts on T can be found by (267) and (291). It is denoted by K.

$$K = - \frac{1}{4\pi R} \iint\limits_{v} Z \cdot \left[\frac{\partial}{\partial H} \left(\Delta g_{Bouguer} \right) \right] \cdot S(p) \cdot dv . \tag{293}$$

The following very useful and instructive deliberation should be added to the relation (293).

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The Bouguer anomalies are caused by certain density anomalies in the crust. The deviation of the real mass density from the standard density ϑ, that is the underlying gravitational source. In reality, these underlying mass anomalies δm have the depth t below the surface of the Earth u. The impact that δm exerts in reality on the T value of the test point P can be approximated by the consideration of a spherical model.

A globe with the radius R is introduced. The test point lies on the surface of this globe. The mass anomaly δm lies below the surface of this globe, in a depth of t. The spherical distance between δm and the test point P has in the spherical model the same value as in reality. Thus, the impact of δm on T is about, Fig. 8,

$$K_1 = f \cdot \frac{1}{e_1} \cdot \delta m \quad . \tag{294}$$

e_1 is the straight distance between the test point P and the mass anomaly. Vertical above δm, at the surface of the globe, δm (or its potential) causes a gravity anomaly of about $(\Delta g_{Bouguer})_1$. Hence,

$$K_1 = \frac{1}{4\pi R} \iint_v (\Delta g_{Bouguer})_1 \cdot S(p) \cdot dv \quad . \tag{295}$$

A second variant of this spherical model is now considered. The test point has the same position as before, but δm is shifted downwards to a depth of $t + |Z|$. For this second variant, the relation (296) follows, instead of (294), - (t is positive, always),

$$K_2 = f \cdot \frac{1}{e_2} \cdot \delta m \quad . \tag{296}$$

(295) turns to

$$K_2 = \frac{1}{4\pi R} \iint_v (\Delta g_{Bouguer})_2 \cdot S(p) \cdot dv. \tag{297}$$

As to the gravity anomalies in (295) and (297), they follow in a self-explanatory way by the surface values of K_1 and K_2 for the test point vertical above δm,

$$K_1' = f \cdot (1/t) \cdot \delta m \quad , \quad K_2' = f \cdot (1/(t + |Z|)) \cdot \delta m \quad . \tag{297a}$$

These potentials K_1' and K_2' are inserted into the fundamental equation of the physical geodesy. We find, (Fig. 8),

$$(\Delta g_{Bouguer})_1 = -(\partial K_1' / \partial r) - (2/R) K_1' \cong -(\partial K_1' / \partial r) \quad . \tag{297b}$$

A similar formula is valid for $(\Delta g_{Bouguer})_2$.

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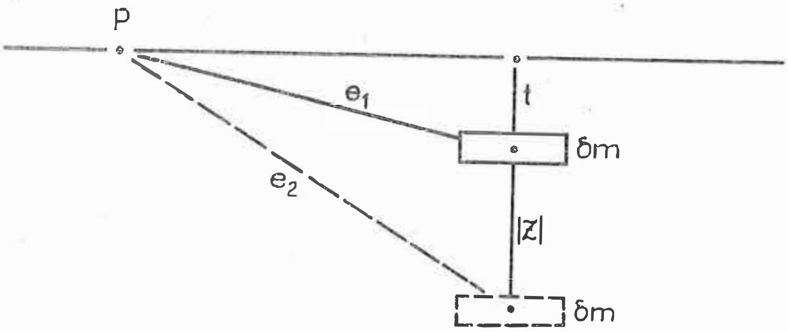

Fig. 8.

Obviously,

$$(\Delta g_{Bouguer})_2 \cong (\Delta g_{Bouguer})_1 + |Z| \cdot \frac{\partial}{\partial H} (\Delta g_{Bouguer})_1 \quad . \qquad (298)$$

Thus, (295) (297),

$$K_2 - K_1 = \frac{1}{4\pi R} \iiint_v |Z| \cdot \left[\frac{\partial}{\partial H} (\Delta g_{Bouguer})_1 \right] \cdot S(p) \cdot dv \quad . \qquad (299)$$

Whereas, the relations (294) and (296) give

$$K_2 - K_1 = f \cdot \left(\frac{1}{e_2} - \frac{1}{e_1} \right) \cdot \delta m \quad . \qquad (300)$$

The oblique distances e_1 and e_2 have the following equations (see [4] page 35, [5] [6]), (the t value is always positive, here) ,

$$e_1^2 = e_o^2 + t^2 - e_o^2 \cdot \frac{t}{R} \quad . \qquad (301)$$

$$e_2^2 = e_o^2 + (t + |Z|)^2 - e_o^2 \cdot \frac{t + |Z|}{R} \quad . \qquad (302)$$

Generally, the amount of e_o is here much more great than t or |Z| . Hence,

$$\frac{1}{e_2} - \frac{1}{e_1} \cong + \frac{|Z|}{2 e_o R} \quad . \qquad (303)$$

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Consequently, (293) (300) (303), [6],

$$\left| K_2 - K_1 \right| = \left| K \right| \cong \frac{f \cdot |\delta m| \cdot |Z|}{2 \cdot e_o \cdot R} . \tag{304}$$

Finally, the order of the amount of $|K|$ is to be estimated approximatively. The isostatic mountain roots, compensating the mountain masses situated above sea level, are the underlying sources of a great part of the Bouguer anomalies. These mountain roots have always a density defect of about -600 kg m^{-3}; the sign of this value is always negative, thus, it can give rise to an accumulating effect which can cause biases.

The model computations may use the following parameters. The mountain roots have a horizontal extension of a square of 100 km x 100 km side length. The vertical extension of the mountain roots is 10 km. The amount of $|Z|$ is equal to 2 km. For the value of e_o, in the denominator of (304), the amount of 2 000 km is introduced. If one single mountain root of the above parameters is the underlying source, an amount of

$$\delta\zeta = \frac{|K|}{G} = 3 \cdot 10^{-3} \text{ cm} \tag{305}$$

is obtained for the effect exerted on the height anomaly at the test point P.

In case of a global extension of the considerations, for the total number of these mountain roots, a total number of N = 1 000 of such mountain roots seem to be a plausible basis. In our applications,here discussed, the amount of δm has always the same sign; the same property can be valid also for Z . Consequently, the amount of (305) has to be multiplied by N and not by the square root of N, in order to obtain the global effect. The amount of $N \cdot 3 \cdot 10^{-3}$ cm = 3 cm follows for the global effect.

Thus, summarizing the above considerations, the share that $C_1(M)$ exerts on the height anomaly T/G of the test point P is not more than about 3 cm, as long as the integration by (267) covers areas which are more than about 1 000 km distant from the test point P, ($e_o > 1$ 000 km in (304)).

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But, for the estimation of this $C_1(M)$ effect resulting by the integration over the surroundings of the test point P up to a distance of 1 ooo km, a special and individual computation appears to be desirable.

It may be stated that the publication [6] does contain a discussion of the impact that the short wave constituents (or, better, the constituents having short wave lengthes) of $C_1(M)$ exert on the height anomaly ζ of the test point P. In [6], it is shown that this impact will not reach the amount of 1 cm in the height anomalies ζ . There, for the global distribution of the Bouguer anomalies, a convenient model with plausible parameters was introduced (See [6] , page 25... 27, equations (38)...(41)).

12.3. The supplementary term C_2

Some lines about the term C_2 of (266) (268) and (273) should be added. The relations (240) and (266) give

$$C_2 = - f \cdot \vartheta \cdot \frac{H_Q}{R} \cdot \left[4 \cdot \tilde{\pi} \cdot H_Q + \frac{2}{R} \cdot \iint_v \frac{H_Y - H_Q}{e_{oo}} \cdot dv \right] . \qquad (3o6)$$

In the brackets of (3o6), the potential B can be represented by B^* in sufficient approximation, (232) (239) (24o). Thus,

$$C_2 = C_{2.1} - 2 \cdot B \cdot \frac{H_Q}{R^2} , \qquad (3o7)$$

with

$$C_{2.1} = 4 \pi f \vartheta H_Q \cdot \frac{H_Q}{R} . \qquad (3o8)$$

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12.4. The supplementary term Ω (M)

As to the term Ω(M) of (267) and (268), the amount of this term should now be considered. The first term of (268) is Ω_1(M), it has the development (224).

The 4. term on the right hand side of (224) is, with (80), ($v_1 \cong x = Z/e'$), (for $x^2 = o$ and $y^2 = 1$), (see also the term $(1/2\pi)\cdot f_2^*(M)$ of equation (227b)),

$$\frac{1}{2\pi} \iint_w \frac{M}{R} \cdot \frac{Z}{R} \cdot \frac{1}{e'} \cdot dw \ , \tag{309}$$

(309) requires an integration over whole the globe. Therefore, x^2 is put equal to zero. Consequently, y^2 is equal to the unity, (76) (78). The integral (309) is transformed into the shape of a sum,

$$\frac{1}{2\pi} \ \Delta w \ \sum_{i=1}^{I} \left(\frac{M}{GR} \cdot \frac{Z}{R} \cdot \frac{1}{e'} \right)_i \ . \tag{310}$$

In (310), the multiplication with 1/G transforms from the perturbation potential T to the height anomalies. The following parameters are introduced: Δw = 2 000 km x 2 000 km, M/G = 0.3 km, Z = 1 km, R = 6 000 km, e' = 3 000 km, I = 130 .
For the above parameters, a single summand of the sum described by (310) is computed. This summand is multiplied with the square root of I, being the total number of the members of the sum given by (310). Along these lines, for the global average of the amount of (310), a value of about 0.02 m is computed. It approximates the average amount of the integral (309). Thus, the amount of 0.02 m, found above, is a good estimation of the impact which the 4. term on the right hand side of (224) exerts on the final height anomaly ζ of the test point.

The corresponding impact, which the 2nd, the 3rd, and the 5th term on the right hand side of (224) exert on the final height anomaly, can be computed in a similar way. Similar amounts will result for them, but the 3rd term will be considerably smaller since the value in its brackets is very small.

As to the 6th term on the right hand side of (224), it has the subsequent form,

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$$\frac{1}{2\pi}\iint \Delta g_M \cdot \frac{-x^2}{y+y^2} \cdot de' \cdot dA \quad . \qquad (311)$$

This integration requires not a global coverage, an integration over the near surroundings suffices. In the evaluation of the amount of (311), — or better, of the order of this amount —, x^2 may be equal to the unity and y^2 equal to two, (76) (78). Thus, cliffs of extreme inclinations are considered in the near surroundings of the test point P. Integrating in (311) up to a radius of 3 km, Δg_M can be introduced as a constant value of 100 mgal (i. e. 0.1 cm sec^{-2}).
With these presuppositions, (311) turns to (312), considering the absolute amount,

$$\frac{1}{G} \cdot \frac{1}{2\pi} \cdot \Delta g_M \cdot 0.3 \iint de' \cdot dA = 0.09 \text{ m} \quad . \qquad (312)$$

The division through the mean global gravity G gives the impact which the 6th term of (224) exerts on the height anomaly of the test point P. It will not be more than about 0.09 m.

The 7th, 8th, and the 9th term on the right hand side of (224) have an amount that can be estimated in a similar way; a similar amount will yield.

The first term on the right hand side of (224) is, in a rough approximation, the global average of such values as given by (309) and (311). Thus, probably, this term will not be greater than some centimeters, integrating globally over F(M) according to (225).

The lowland variant of the expression in the brackets of the 5th term on the right hand side of (224) was discussed already in [4]; (χ_6/G): pg. 45; page 29. There, a graph shows the dependence of the kernel function S^* on the spherical distance. This term of the lowland variant is equal to the 4th term on the right hand side of the development (230). This 5th term yields about 2 cm.

After the above discussion of the term $\Omega_1(M)$ in the expression for $\Omega(M)$, the second term of this expression is now in the fore. It is equal to $(M H_P)/R$, (268). With $M/G = 0.5$ km, $H_P = 2$ km, $R = 6\,000$ km, the following value is obtained,

$$\frac{M}{G} \cdot \frac{H_P}{R} = 0.17 \text{ m} \quad . \qquad (313)$$

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The effect, which the 3rd term on the right hand side of (268) takes on the height anomaly of the test point P, can be estimated by

$$\frac{[B]''}{G} = 0.03 \text{ m} \qquad (314)$$

for an extreme topographical situation , as can be found in [4] , page 36.

At last, the 4th term on the right hand side of (268) is to be considered. It has the shape of a Stokes integral, C_2 stands here for a kind of gravity anomalies which covers whole the globe v. The height anomalies which are obtained from the field of the C_2 values, this are the values here to be estimated. The C_2 values are in the vicinity of the following value,

$$\frac{2}{R} B \cdot \frac{H_Q}{R} \qquad (315)$$

(see (232) (239) (240) (306) (307)).

With $B = G \cdot 0.5$ km, $H_Q = 0.8$ km, the amount of (315) is 0.02 mgal (i. e. $0.02 \cdot 10^{-3}$ cm sec^{-2}).
Since a global field of gravity anomalies of about 20 mgal gives rise to height anomalies of about 30 m, the above obtained field of global values of 0.02 mgal exerts an effect on the height anomalies by about

$$30 \text{ m} \cdot \frac{0.02}{20} = 0.03 \text{ m} \qquad (316)$$

This is a very small amount.
By (308), the share of $C_{2.1}$ has about the same amount as C_2 , by (316).

12.5. On the superposition with the potential of the isostatic masses

By the relation (145), the superposition of the perturbation potential T with the potential B of the visible mountain masses was introduced into the mathematical developments, in order to represent the additive to the Stokes integral by a functional depending on smoothed arguments, only.

Following up this idea, it is also interesting to take into consideration the superposition of the perturbation potential T with the potential I, being the potential of the isostatic masses. In the course of these developments about the isostatic potential, the Faye-anomalies in the Stokes integral change over to the smoothed isostatic anomalies; furthermore, the topographical additive of the Stokes integral comes out to be expressed in terms of smoothed arguments, but, to be sure, these additives have to be supplemented by the I potential of the test point P computed from the isostatic

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masses.

These isostatic masses are understood that they consist of the following parts:

a.) The mass surplus of the mountains situated above the sea level; here, the masses have the standard density 2 650 kg m^{-3}.
b.) The mass defect of the ocean basins; this density defect is the density of the water minus the standard density 2 650 kg m^{-3}.
c.) The mass defect of the compensating mountain roots situated below the depth of 30 km, if the Airy-Heiskanen isostatic model is applied.
d.) The mass surplus of the anti-roots in the area of the ocean basins.

In a way similar as that followed up by the introduction of (145), we have

$$N = T - I. \tag{317}$$

Further on, in the expression representing the T potential, (114), the T potential can be replaced by the N potential given by (317). This thus obtained version of (114) undergoes some rearrangements which lead, finally, to a representation of the T potential in terms of the isostatic gravity anomalies and of the isostatic potential I.

Also in this case, applying the isostatic superposition, the finally obtained T values have the property to be situated on the Earth's surface u, /see also: Arnold, K.: Die Methoden der Freiluftreduktion und der isostatischen Reduktion in ihren gegenseitigen Beziehungen. Gerlands Beitr. z. Geophysik, 70 (1960), 131-136; cf. also: Bulletin Géodésique, 65 (1962), 259-264/.

Before the background of the above chapters, the details of this superposition with the isostatic masses is intended to be dealt with anew, later on, at another place.

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13. References

[1] Arnold, K.: Zur Bestimmung der Geoidundulationen aus Freiluftanomalien. Veröffl. Geod. Inst. Potsdam, Nr. 12, Berlin, Akademie - Verlag 1959.

[2] Arnold, K.: Zur strengen Theorie der Figur der Erde. Gerlands Beiträge z. Geophysik 68 (1959), 257 - 262.

[3] Arnold, K.: A closed solution for the boundary value problem of geodesy. Gerlands Beiträge z. Geophysik 94 (1985), 83 - 101.

[4] Arnold, K.: Geodetic boundary value problems I. Veröff. Zentralinst. Physik d. Erde, Nr. 84, Potsdam, 1986.

[5] Arnold, K.: Geodetic boundary value problems II. Veröff. Zentralinst. Physik d. Erde, Nr. 89, Potsdam, 1987.

[6] Arnold, K.: The solution of the geodetic boundary value problem by the Runge - Krarup theorem. Krarup - Festschrift, Dan. Geod. Inst., Meddelelse No.58, Copenhagen, 1989.

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14. Appendix

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14. Appendix

14.1. The expression for the term $D_T(1.1)$

The equation (46) of the section 4 is the starting point,

$$2\pi T = \iint_w \left[- \frac{\partial T}{\partial r} + D(1.1) \right] \frac{1}{e'} \, dw + \iint_w T \frac{\partial (1/e')}{\partial r} dw + D(2.1) \; . \qquad (A\ 1)$$

The equation (45) gives the expression for the term D(2.1) of (A 1). The fundamental equation of the physical geodesy is

$$\Delta g_T = - \frac{\partial T}{\partial r} - \frac{2}{r} T \; , \qquad (A\ 2)$$

it leads to

$$\frac{\partial T}{\partial r} = - \Delta g_T - \frac{2}{r} T \; . \qquad (A\ 3)$$

By means of (A 3), it is possible to substitute the radial derivative of the perturbation potential by the free-air anomalies. Hence,

$$2\pi T = \iint_w \left[\Delta g_T + \frac{2}{r} T + D\,(1.1) \right] \frac{1}{e'} \, dw +$$

$$+ \iint_w T \frac{\partial (1/e')}{\partial r} dw + D(2.1). \qquad (A\ 4)$$

In the second term on the right hand side of (A 3), the term r is replaced by R,

$$\frac{\partial T}{\partial r} = - \Delta g_T - \frac{2}{R} T + D(2.2) \qquad (A\ 5)$$

Further, for abbreviation, the suffix T affixed to the free-air anomalies is no more taken along; hence, we have this subsequent substitution by (A 5a), – (In our applications, the slopes of the terrain are considered to have continuous functions; this property is found in the topographical maps, of course. Thus, each point at the surface of the Earth u has a clearly defined tangential plane.) –

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$$\Delta g_T = \Delta g \quad . \tag{A 5a}$$

For D(2.2), the difference of (A 3) and (A 5) yields the relation (A 6),

$$D(2.2) = -\left(\frac{2}{r} - \frac{2}{R}\right) T \; . \tag{A 6}$$

(A 4) and (A 6) are combined to (A 7), accounting for (A 5a),

$$2\pi T = \iint_w \left[\Delta g + \frac{2}{R} T + D(1.1)\right] \frac{1}{e'} \, dw + \iint_w T \frac{\partial 1/e'}{\partial r} dw + D(3.1) , \tag{A 7}$$

$$D(3.1) = - \iint_w D(2.2) \frac{1}{e'} \, dw + D(2.1) \; . \tag{A 8}$$

The equation (36) of the section 4 gives

$$D(1.1) = D_T(1.1) = \frac{\partial T}{\partial n} \frac{1}{\cos (g',n)} + \frac{\partial T}{\partial r} \; . \tag{A 9}$$

Here, the suffix T is affixed calling special attention to the fact that $D_T(1.1)$ has to be computed for the potential T.

W is the real gravity potential,
U is the standard potential. The perturbation potential T has the equation

$$T = W - U, \tag{A 10}$$

In the exterior of the body of the Earth, T obeys the Laplace differential equation.
By means of the gradient operation, (A 10) leads to (A 11),

$$\frac{\partial T}{\partial n} = (\text{grad } T) \cdot \underline{\underline{n}} = (\text{grad } W) \cdot \underline{\underline{n}} - (\text{grad } U) \cdot \underline{\underline{n}} \; . \tag{A 11}$$

$\underline{\underline{n}}$ is the unit vector of the normal of the Earth's surface, **u** , heading into the interior, (see: section 2, Fig. 2). Accounting

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for (A 12) and (A 13),

$$(\text{grad } W)^2 = g^2 , \tag{A 12}$$

$$(\text{grad } U)^2 = g'^2 , \tag{A 13}$$

the relation (A 11) turns to

$$\frac{\partial T}{\partial n} = g \cdot \cos (g,n) - g' \cdot \cos (g', n) \quad . \tag{A 14}$$

(g,n) and (g', n) symbolize the angles spanned by the two vectors within the concerned braces, i. e. the vectors grad W and $\underline{\underline{n}}$, resp. grad U and $\underline{\underline{n}}$.

Now, the angle (g,n) is expressed in terms of the inclination angle of the terrain, which is denoted by (g', n).
At the surface of the Earth, the three vectors $-\underline{\underline{g}}$, $-\underline{\underline{g}}'$, $-\underline{\underline{n}}$ can be defined. They are heading into the mass - free space, and they construct the spherical triangle which is shown by Fig. A 1.

As to this spherical triangle of Fig. A 1, a unit sphere is constructed having the surface point Q as center, Fig. 2. Then, the vectors $-\underline{\underline{g}}'$, $-\underline{\underline{n}}$, and $-\underline{\underline{g}}$ at the point Q are plotted from the center Q of this unit sphere. In Fig. A 1, the points at the normed vectors $(-\underline{\underline{g}}')^0$, $(-\underline{\underline{n}})^0$ being equal to $-\underline{\underline{n}}$, and $(-\underline{\underline{g}})^0$, they mark the places where these three vectors, or these three normed vectors, pierce this above defined unit sphere. They are the projections of these three vectors on this unit sphere .

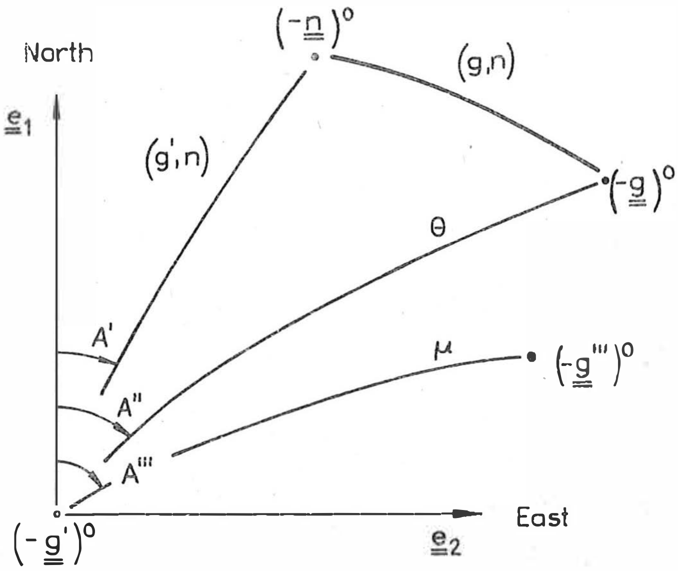

Fig. A 1

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In Fig. A 1, A' is the azimuth of the slope of the terrain, and A'' is that of the plumb - line deflection. Both of them are measured clockwise from the north. But, Θ denotes the full absolute amount of the plumb-line deflection, taken at the surface of the Earth. ξ and η are the north - south and the east - west component of this deflection (in the potential field T). The cosine law for the side of a spherical triangle leads to the relation (A 15), Fig. A 1,

$$\cos(g, n) = \cos(g', n) \cdot \cos\Theta + \sin(g', n) \cdot (\sin\Theta) \cdot \cos(A'' - A'). \quad \text{(A 15)}$$

In case, Θ has an amount of about $10''$, the following approximations are valid,

$$\sin\Theta \cong \Theta = 10''/\rho'' = 0.5 \cdot 10^{-4},$$

$$\cos\Theta \cong 1 - (1/2) \cdot \Theta^2 \cong 1 - (1/8) \cdot 10^{-8}. \quad \text{(A 16)}$$

(A 15) and (A 16) are combined to (A 17),

$$\cos(g,n) = \cos(g',n) + \Theta \cdot \sin(g',n) \cdot \cos(A''-A') - (1/2) \cdot \Theta^2 \cdot \cos(g',n). \quad \text{(A 17)}$$

Thus, the relation (A 14) turns to

$$\frac{\partial T}{\partial n} = (g - g') \cdot \cos(g',n) + \Theta \cdot g \cdot \sin(g',n) \cdot \cos(A'' - A') -$$

$$- (1/2) \cdot \Theta^2 \cdot g \cdot \cos(g',n). \quad \text{(A 18)}$$

Neglecting some microgals only

(1 to 2 μgal, i. e. 1 to $2 \cdot 10^{-6}$ gal),

the relation (A 18) leads to

$$\frac{\partial T}{\partial n} = \left[g - g' + \Theta \cdot g \cdot \tan(g',n) \cdot \cos(A'' - A') \right] \cdot \cos(g',n). \quad \text{(A 19)}$$

In (A 19), g and g' refer to the same moving point Q at the surface of the Earth u, Fig. A 2, Fig. 2. It is emphasized, that the amount of g' in (A 19) is not the standard gravity at the telluroid t, point P_t; (see: section 1, Fig. 1).

Neglecting the flattening of the best-fitting ellipsoid of the Earth, the relation (A 10) gives ,(considering (A 16), and considering that

$$(g,g') = \Theta \quad \text{and} \quad \cos(g,g') = \cos\Theta \cong 1 - (1/2) \cdot \Theta^2,$$

and considering that the direction of **r** is the direction of $-\underline{\underline{g}}'$, since we have a sphere as reference figure),

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$$-\frac{\partial T}{\partial r} = -\frac{\partial W}{\partial r} + \frac{\partial U}{\partial r} = |\text{grad } W| \cdot \cos(g,g') - |\text{grad } U| =$$

$$= g \cdot \cos(g,g') - g' = g - g' - \frac{1}{2} g \cdot \Theta^2 \cong g - g' \quad . \qquad \text{(A 19a)}$$

Here, in (A 19a), the fact is considered that the angle (g,g') is equal to the deflection Θ , Fig. A 1.
Accounting for (A 16), the developments of (A 19a) are easily understood.
(A 19a) yields,

$$g - g' = -\frac{\partial T}{\partial r} \quad . \qquad \text{(A 20)}$$

The reader is asked to compare also the deductions given by the equations from (158) to (174) of the section 7.

With (A 19) (A 20), the relation (A 20a) is obtained,

$$\frac{\partial T}{\partial n} \cdot \left[1 / \cos(g',n) \right] = -\frac{\partial T}{\partial r} + \Theta \cdot g \cdot \tan(g',n) \cdot \cos(A'' - A') \,, \qquad \text{(A 20a)}$$

(see also (165) (169) (170) of the section 7).

The two equations (A 9) and (A 20a) are combined to

$$D_T\,(1.1) = \Theta \cdot g \cdot \tan(g',n) \cdot \cos(A'' - A') . \qquad \text{(A 21)}$$

In (A 21), the neglection of terms smaller than about 1 microgal took place.

14.2. The impact of the term D(1.2) and the representation of it by the expression for E(1)

Now, an expression for the term D(1.2) of the relation (37) of the section 4 is intended to be found. Further, an expression for E(1) will be found. E(1) depends on D(1.2) by the relation (45a) of the section 4. The relation (37) and Fig. A 2 yield,

$$D(1.2) = 1/e - 1/e' = (e' - e) / e \cdot e' \quad , \qquad \text{(A 22)}$$

$$e' = 2 \cdot R' \cdot \sin p/2 = 2 \cdot (R + H') \cdot \sin p/2 \quad . \qquad \text{(A 23)}$$

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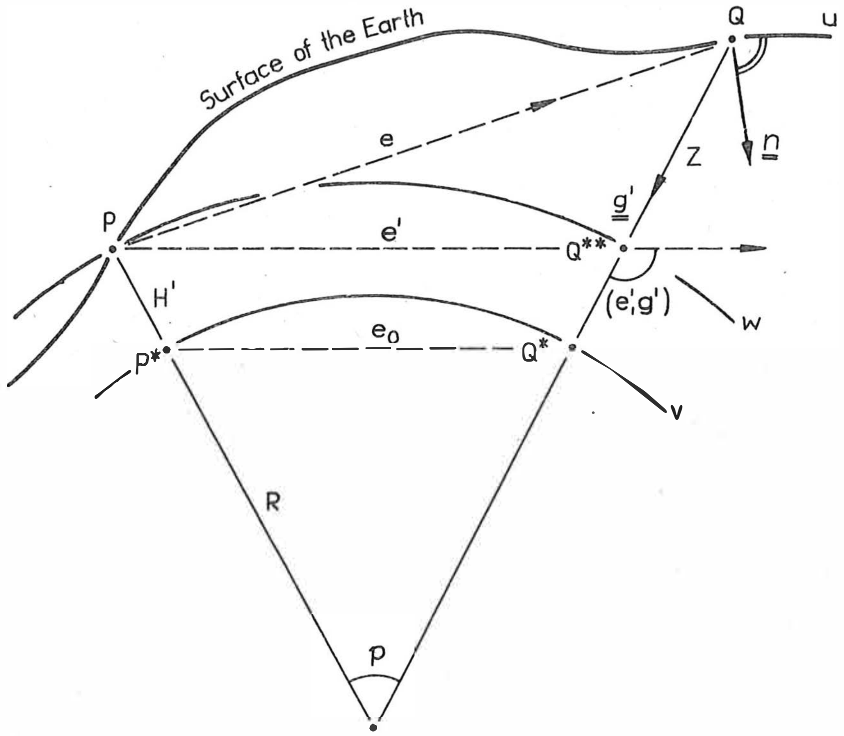

Fig. A 2.

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The oblique distance e is understood as the distance between the two points P and Q, Fig. A 2. Z is the difference of the heights of the two points Q and P, $Z = H_Q - H'$. ($H' = H_P$, $Z = H_Q - H_P$).
The cosine law gives, Fig. A 2,

$$e^2 = e'^2 + Z^2 - 2 \cdot e' \cdot Z \cdot \cos(e', g') \quad . \qquad \text{(A 24)}$$

Further, from Fig. A 2 and with

$$R' = R + H' ,$$

$$\cos(180^\circ - (e', g')) = -\cos(e', g') = e'/(2 \cdot R') \quad . \qquad \text{(A 25)}$$

(A 24) and (A 25) are combined to

$$e^2 - e'^2 = Z^2 + e'^2 \cdot Z/R' \quad . \qquad \text{(A 26)}$$

Abbreviating, the symbol x denotes the quotient Z/e',

$$x = Z/e' \quad . \qquad \text{(A 27)}$$

(A 27) and (A 26) give

$$(e^2 - e'^2)/e'^2 = x^2 + Z/R' \quad . \qquad \text{(A 28)}$$

From (A 26) follows

$$(e' - e)\cdot(e' + e) = - Z^2 - e'^2 \cdot Z/R' \quad , \qquad \text{(A 29)}$$

$$e' - e = -(Z^2 + e'^2 \cdot Z/R') \cdot (e' + e)^{-1} \quad . \qquad \text{(A 30)}$$

The symbol x' is introduced now, it has the following meaning,

$$x' = 1 + x^2 + Z/R' \quad . \qquad \text{(A 31)}$$

Thus, combining (A 26) and (A 31),

$$e^2 = e'^2 \cdot x' \quad , \qquad \text{(A 32)}$$

$$e = e' \cdot (x')^{1/2} \quad , \qquad \text{(A 33)}$$

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$$e \cdot e' = e'^2 \cdot (x')^{1/2} \quad , \qquad \text{(A 34)}$$

$$e + e' = e' \cdot \left\{ 1 + (x')^{1/2} \right\} \quad , \qquad \text{(A 35)}$$

$$e \cdot e' \cdot (e + e') = e'^3 \cdot \left[(x')^{1/2} + x' \right] \quad . \qquad \text{(A 36)}$$

(A 22), (A 30) and (A 36) are combined to

$$D(1.2) = -(e')^{-3} \cdot (Z^2 + e'^2 \cdot Z/R') \cdot \left\{ x' + (x')^{1/2} \right\}^{-1} \quad . \qquad \text{(A 37)}$$

In the expression for D(2.1), in the first integral on the right hand side of (45), (in the section 4), the term D(1.2) does appear. Therefore, it is necessary to develop a convenient expression for E(1) , see (45a) ,

$$E(1) = - \iint_W \frac{\partial T}{\partial r} \cdot D(1.2) \cdot dw. \qquad \text{(A 38)}$$

For the sake of abbreviation, the symbol y is introduced; it has the following meaning,

$$y^2 = 1 + x^2 \geqslant 1 \quad . \qquad \text{(A 39)}$$

Thus, (A 31) turns to

$$x' = y^2 + Z/R' \quad . \qquad \text{(A 40)}$$

For the inverse of $x' + (x')^{1/2}$ appearing in (A 37), it is intended, now, to find a series development of rising powers of Z/R'. Because the inequality (A 41) is always fulfilled,

$$\left| Z/R' \right| \ll 1 \quad , \qquad \text{(A 41)}$$

the binominal series leads to

$$x' = y^2 + Z/R' = y^2 \cdot \left\{ 1 + Z/\left(R' \cdot y^2\right) \right\} \quad , \qquad \text{(A 42)}$$

$$(x')^{1/2} \cong y \cdot \left\{ 1 + Z/(2\,R\,y^2) \right\} \quad . \qquad \text{(A 43)}$$

(A 42) and (A 43) yield

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$$x' + (x')^{1/2} = y \cdot \left[1 + y + Z/(2 \cdot R \cdot y^2) + Z/(R \cdot y) \right] . \qquad (A\ 44)$$

Hence, if $1 + y$ is put before the brackets,

$$x' + (x')^{1/2} = (y + y^2) \cdot \left[1 + \frac{1}{y + y^2} \cdot (1 + \frac{1}{2 \cdot y}) \cdot Z/R \right] . \qquad (A\ 45)$$

Neglecting a relative error, which is smaller than

$$\left| Z/R \right| \cong 3\ \text{km} / 6\ 000\ \text{km} = 1/2\ 000, \qquad (A\ 46)$$

the following relation is obtained,

$$x' + (x')^{1/2} \cong y + y^2 . \qquad (A\ 47)$$

Now, the equation (A 38) for E(1) is considered. For $\partial T/\partial r$ appearing in the integrand of (A 38) follow the subsequent lines, with (A 3) (A 5) (A 6), in a self - explanatory way, (**Fig. A 2**),

$$\partial T/\partial r = - \Delta g_T - (2/r) \cdot T . \qquad (A\ 47a)$$

$$r = R + Z + H' ,$$

$$r = R(1 + \frac{Z + H'}{R}) ,$$

$$1/r \cong (1/R) \cdot \left\{ 1 - (Z + H') / R \right\} ,$$

$$- \left[(2/r) - (2/R) \right] \cong (2/R) \cdot \left[(Z + H')/ R \right] . \qquad (A\ 47b)$$

(A 47a) and (A 47b) yield

$$\frac{\partial T}{\partial r} = - \Delta g_T - \frac{2}{R} \cdot T + \frac{2}{R} \cdot \frac{Z + H'}{R} \cdot T + - \cdots . \qquad (A\ 47c)$$

The relations (A 37), (A 45), and (A 47c) are combined to (A 47d),

$$- \frac{\partial T}{\partial r} \cdot D(1,2) = a \cdot b , \qquad (A\ 47d)$$

$$a = - \Delta g_T - \frac{2}{R} \cdot T + \frac{2}{R} \cdot \frac{Z + H'}{R} \cdot T ,$$

$$b = \frac{1}{e'} \cdot (x^2 + Z/R) \cdot \frac{1}{y + y^2} \cdot \left[1 - \frac{1 + 2 \cdot y}{2 \cdot y^2 + 2 \cdot y^3} \cdot (Z/R) \right] .$$

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In the term represented by (A 47d), relative errors of the order 1/2 000 can be neglected, (A 46). Thus, if in (A 47d) the gravity anomalies and the T values are multiplied with coefficients of the order of

$$x^2 \cdot (Z/R) \qquad \text{(A 47e)}$$

or with

$$(Z/R)^2 \quad , \qquad \text{(A 47f)}$$

it is allowed to neglect these amounts. Hence,

$$-\frac{\partial T}{\partial r} \cdot D(1.2) = c \cdot d \quad , \qquad \text{(A 47g)}$$

with

$$c = -\Delta g_T - \frac{2}{R} T \quad ,$$

$$d = \frac{1}{e'} \cdot (x^2 + Z/R) \cdot \frac{1}{y + y^2} \quad .$$

Thus, finally, the expression for E(1) follows to be, see (A 38),

$$E(1) = -\iint\limits_w \left(\Delta g + 2\,T/R\right) \cdot \frac{1}{e'^3} \left[Z^2 + e'^2 \cdot Z/R' \right] \cdot \frac{1}{y + y^2} \cdot dw \quad . \qquad \text{(A 48)}$$

And with (A 27), neglecting the suffix T affixed to the gravity anomalies,

$$E(1) = -\iint\limits_w \left[\Delta g + 2\,T/R\right] \cdot (x^2 + Z/R) \cdot \frac{1}{y + y^2} \cdot (e')^{-1} \cdot dw . \qquad \text{(A 49)}$$

It is convenient to divide E(1) into two parts,

$$E(1) = E(1.1) + E(1.2) \quad , \qquad \text{(A 50)}$$

$$E(1.1) = -\iint\limits_w \left(\Delta g + 2\,T/R\right) \cdot x^2 \cdot \frac{1}{y + y^2} \cdot (e')^{-1} \cdot dw \qquad \text{(A 51)}$$

$$E(1.2) = -\iint\limits_w \left(\Delta g + 2\,T/R\right) \cdot (Z/R) \cdot \frac{1}{y + y^2} \cdot (e')^{-1} \cdot dw \quad . \qquad \text{(A 52)}$$

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14.3. The representation of cos (e, n)

The relation (38) of the section 4 determines the term D(1.3),

$$D(1.3) = \frac{\partial 1/e}{\partial n} \left[1/\cos(g', n) \right] + \frac{\partial 1/e'}{\partial r} . \qquad (A\ 53)$$

The normal derivative of the inverse of the oblique distance e is equal to,(Fig.2,page 15; Fig.3,page 16; Fig.A 2,page 98) ,

$$\left[\frac{\partial}{\partial n} (1/e) \right] = - \cos(e, n) \cdot \frac{1}{e^2} . \qquad (A\ 54)$$

In (A 54), first of all, the term cos (e,n) has to be developed in terms of the slope of the terrain.

In this context, a spherical triangle is considered. It is constructed by the following 3 vectors, (see Fig. A 3). The first vector is the negative vector of the standard gravity in the surface point Q, i. e. $-\underline{\underline{g}}'$. The second vector is the negative vector of the normal of the surface of the Earth in the surface point Q, i. e. $-\underline{\underline{n}}$. Since the vector $\underline{\underline{n}}$ is heading into the interior of the Earth, the vector $-\underline{\underline{n}}$ points into the exterior of the body of the Earth. The third vector has the spatial direction of the oblique straight line e which connects the two surface points P and Q, Fig. A 2. This vector $\underline{\underline{e}}$ is heading from the point P to the point Q. Fig. A 3 shows this spherical triangle spanned by the 3 vectors $-\underline{\underline{g}}'$, $-\underline{\underline{n}}$, and $\underline{\underline{e}}$.
In Fig.A 3, the 3 vectors $(-\underline{\underline{g}}')^0$, $(-\underline{\underline{n}})^0$, and $(\underline{\underline{e}})^0$ are the normalized vectors of our 3 vectors $-\underline{\underline{g}}'$, $-\underline{\underline{n}}$, and $\underline{\underline{e}}$. They meet our unit sphere (having the point Q at the surface of the Earth u as center point) at the dots plotted in Fig.A 3 .

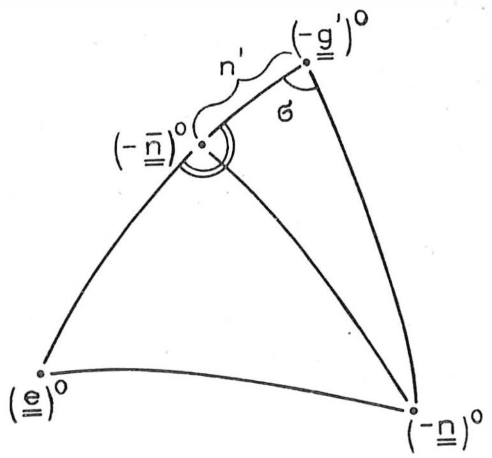

Fig. A 3 .

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Some simple goniometric equations and some well-known relations from the spherical trigonometry lead to the following developments, (A 55), (A 56a ... e), and (A 57), Fig. A 3,

$$\cos(e, -n) = \cos(e, -g') \cos(g', n) + \sin(e, -g') \sin(g', n) \cos \sigma , \quad \text{(A 55)}$$

$$\cos(e, -n) = - \cos(e, n) , \quad \text{(A 56a)}$$

$$\cos(e, -g') = - \cos(e, g') , \quad \text{(A 56b)}$$

$$\sin(e, -g') = \sin(e, g') , \quad \text{(A 56c)}$$

$$\cos(-g', -n) = \cos(g', n) , \quad \text{(A 56d)}$$

$$\sin(-g', -n) = \sin(g', n) . \quad \text{(A 56e)}$$

The relations (A 55), and (A 56a) to (A 56e), yield

$$\cos(e,n) = \cos(e,g') \cdot \cos(g', n) - \sin(e, g') \cdot \sin(g', n) \cdot \cos \sigma . \quad \text{(A 57)}$$

The center of the Earth and the points P and Q at the Earth's surface determine a certain plane, (see Fig. A 2, Fig. A 4, Fig. A 5). The 3 vectors $-\underline{\underline{g}}'$, $\underline{\underline{e}}$, $\underline{\underline{e}}'$ are situated in even this plane. Thus, also these three vectors $-\underline{\underline{g}}'$, $\underline{\underline{e}}$, $\underline{\underline{e}}'$, or at least two of them, define this plane here considered. Consequently, the vector $\underline{\underline{e}}'$ having the direction of the straight line e' of Fig. A 2, (the positive direction of $\underline{\underline{e}}'$ is shown by Fig. A 5 and Fig. A 2), is situated in the plane spanned by the vectors $-\underline{\underline{g}}'$ and $\underline{\underline{e}}$.
Further, the spherical representation of the vector $\underline{\underline{e}}'$ is situated on the great circle spanned by the two vectors $\underline{\underline{e}}$ and $-\underline{\underline{g}}'$, Fig. A 3. The two added figures show this situation, Fig. A 4, Fig. A 5.

As to the spherical representation of a vector, this representation is defined in the following way : The vector (e.g. $\underline{\underline{e}}'$) is translated in such a way that the starting point of this vector coincides with the center of the unit sphere. In this case, the vector $\underline{\underline{e}}'$ (or the prolongation of it) pushes through the surface of the unit sphere in a certain point; this point is the spherical representation of the vector $\underline{\underline{e}}'$.

In Fig. A 4 , the vectors $(\underline{\underline{e}}')^o$, $(\underline{\underline{e}})^o$, and $(-\underline{\underline{g}}')^o$ are the unit vectors of the vectors $\underline{\underline{e}}'$, $\underline{\underline{e}}$, and $-\underline{\underline{g}}'$.

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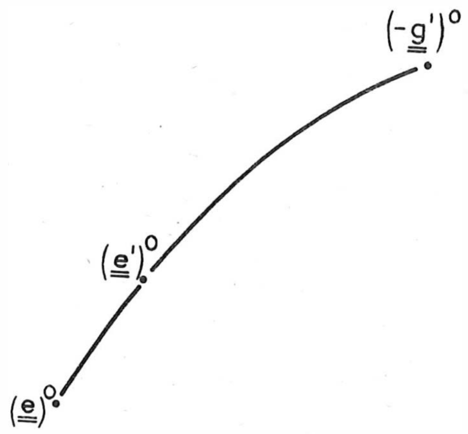

Fig. A 4.

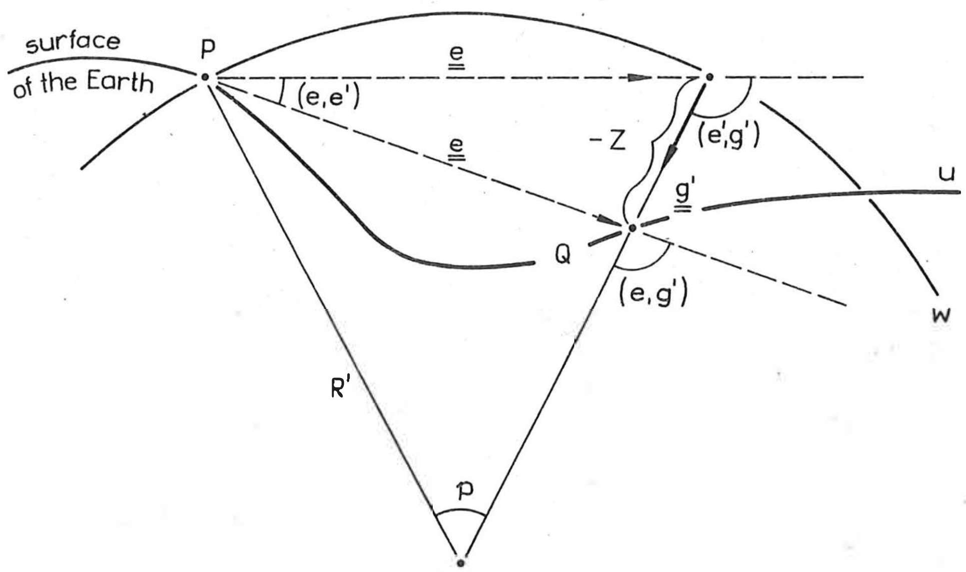

Fig. A 5.

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In the here considered case, the test point P lies higher than the moved point Q; Z is negative; Fig. A 4, Fig. A 5.
Thus, for all spherical distances p between the two points P and Q, (A 58) is valid,

$$(e, -g') = (e', -g') + (e, e') . \qquad \text{(A 58)}$$

(A 56b) and (A 58) yield

$$\cos (e, g') = \cos (e', g') \cdot \cos (e, e') + \sin (e', g') \cdot \sin (e, e') , \qquad \text{(A 59)}$$

and, further on,

$$\sin (e, g') = \sin (e', g') \cdot \cos (e, e') - \cos (e', g') \cdot \sin (e, e') . \qquad \text{(A 60)}$$

The combination of (A 57) (A 59), and (A 60) gives

$$\frac{\cos (e, n)}{\cos (g', n)} = a + b + c , \qquad \text{(A 61a)}$$

with

$$a = \cos (e', g') \cdot \cos (e, e') + \sin (e', g') \cdot \sin (e, e'), \qquad \text{(A 61b)}$$

$$b = - \sin (e', g') \cdot \cos (e, e') \cdot \tan n', \qquad \text{(A 61c)}$$

$$c = \cos (e', g') \cdot \sin (e, e') \cdot \tan n' . \qquad \text{(A 61d)}$$

In the above equations, (A 61c) and (A 61d), the following relation is valid from the rules of the spherical trigonometry (see: Fig. A 3, Fig. A 5),

$$\tan n' = \tan (g', n) \cdot \cos \sigma . \qquad \text{(A 62)}$$

The meaning of n' is shown in Fig. A 3 and Fig. A 6.
n' is a component of the slope of the terrain. n' is understood that it is taken for the moved point Q; n' is the component of the slope of the terrain measured in the direction of the line $\overrightarrow{PQ}$, in radial direction, for growing distances from the test point P; Fig. A 3, Fig. A 6. If, at the point Q, the topographical heights diminish for growing distances to the point P, the amount of n' is positive. Further, in this case, the angle σ of Fig. A 3 is smaller than 90° ; this fact is also evidenced from the equation (A 62) . The latter fact is also corroborated by the following deliberation : Per definitionem, the angle (g', n) is always smaller than 90°, since we have a star-shaped Earth. Consequently, $\tan (g', n)$ is always positive. Thus, in (A 62), the sign of $\tan n'$ is the same as that of $\cos \sigma$.

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14.4. The development of sin (e, e') and cos (e, e') in terms of the heights

On the right hand side of (A 61a), the terms cos (e', g'), sin (e', g'), cos (e,e'), and sin (e,e') appear. They have to be expressed as functions which depend on the spherical distance p from the test point P and, further, on the height difference Z.

From Fig. A 5, it is learnt that

$$(e', g') = 90^{\circ} + p/2 \quad . \tag{A 63}$$

Hence,

$$\sin (e', g') = \cos p/2 = 1 - (1/8) \cdot p^{2} + - \ldots \quad . \tag{A 64}$$

If the distances e' are small, if e' is smaller than about 50 km, the following relation is valid

$$p \cong \frac{e'}{R'} \leqslant \frac{50 \text{ km}}{6\ 000 \text{ km}} = \frac{5}{600} \quad .$$

Thus,

$$\sin (e', g') \cong 1 - 10^{-5}, \quad (e' = 50 \text{ km}). \tag{A 65}$$

Furthermore, from (A 63),

$$\cos (e', g') = - \sin p/2 = - (p/2) + - \ldots \quad . \tag{A 66}$$

From Fig. A 5 and from the sine law of plane trigonometry, and from (A 27), it is learnt that

$$- (Z/e') = - x = \sin (e, e') / \sin (e, g') \; . \tag{A 67}$$

Further on, Fig. A 4, Fig. A 5,

$$(e, e') + (e, g') = (e', g') \; . \tag{A 68}$$

(A 67) and (A 68) are combined to

$$\sin (e, e') = - x \cdot \sin \left[(e', g') - (e, e') \right] \quad . \tag{A 69}$$

Thus,

$$\sin (e, e') = - x \cdot \sin (e', g') \cdot \cos (e, e') + x \cdot \cos (e', g') \cdot \sin (e, e'), \tag{A 70}$$

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$$\tan(e, e') = -x \cdot \sin(e', g') + x \cdot \cos(e', g') \cdot \tan(e, e'),$$

$$\tan(e, e')\left[1 - x \cdot \cos(e', g')\right] = -x \cdot \sin(e', g') .$$

With

$$x'' = x \cdot \cos p/2 , \tag{A 70a}$$

and with (A 64) and (A 66), the relation (A 71) yields,

$$\tan(e, e') \cdot (1 + x \cdot \sin p/2) = -x'' . \tag{A 71}$$

(A 23) and (A 27) give for an expression on the left hand side of (A 71)

$$x \cdot \sin p/2 = (Z/e') \cdot \sin p/2 = Z/(2 \cdot R') . \tag{A 72}$$

(A 71) and (A 72) are combined to

$$\tan(e, e') \cdot \left[1 + Z/(2 \cdot R')\right] = -x'' , \tag{A 73}$$

or,

$$\tan(e, e') = -x'' \cdot \left[1 + Z/(2 \cdot R')\right]^{-1} , \tag{A 74}$$

or,

$$\tan(e, e') = -x'' \cdot \left[1 - Z/(2 \cdot R') + - \cdots\right] . \tag{A 75}$$

After $\tan(e, e')$ is represented in terms of the heights, by (A 75), the function of $\sin(e, e')$ in terms of the heights is easily found by $\tan(e, e')$. In the interval

$$-90^{\circ} < (e, e') < +90^{\circ} ,$$

the subsequent formula is valid,

$$\sin(e, e') = \tan(e, e') \cdot \left[1 + \tan^2(e, e')\right]^{-1/2} . \tag{A 75a}$$

Neglecting terms of the order of $(Z/R')^2$, the combination of (A 75) and (A 75a) leads to

$$\sin(e, e') \cong -x'' \cdot \left\{1 - Z/(2 \cdot R')\right\} \cdot \left[1 + (x'')^2 \cdot \left\{1 - Z/R'\right\}\right]^{-1/2} . \tag{A 76}$$

As to (A 76), since $-x''$ has the sign of $-Z$ (because we have the following equation : $-x'' = -(Z/e') \cdot \cos p/2$), since the term in the parentheses { } of (A 76) is always positive, and since the term in the brackets [ ] of (A 76) is always positive, too, therefore, $\sin(e,e')$ has always the sign of $-Z$. The same is valid for the function $\tan(e,e')$.

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Some self-explanatory rearrangements yield, in the brackets of (A 76), the ensuing form,

$$\left[1 + (x'')^2 \cdot \{1 - Z/R'\}\right]^{-1/2} =$$

$$= \left[\{1 + (x'')^2\} \cdot \{1 - (x'')^2 \cdot \left(1 + (x'')^2\right)^{-1} \cdot (Z/R)\}\right]^{-1/2} =$$

$$= \left[1 + (x'')^2\right]^{-1/2} \cdot \left[1 + (x'')^2 \cdot \left(1 + (x'')^2\right)^{-1} \cdot (Z/(2 \cdot R))\right] . \qquad (A\ 77)$$

(A 77) is introduced into (A 76), the equation (A 78a) follows,

$$\sin(e, e') = a \cdot b \quad , \qquad (A\ 78a)$$

$$a = -x'' \cdot \left[1 + (x'')^2\right]^{-1/2} \quad , \qquad (A\ 78b)$$

$$b = 1 + \left\{(x'')^2 \cdot \left[1 + (x'')^2\right]^{-1} \cdot (Z/(2 \cdot R))\right\} - (Z/(2 \cdot R)) . \qquad (A\ 78c)$$

The second and the third term on the right hand side of (A 78c) can be substituted by one term, only. We have

$$\frac{(x'')^2}{1 + (x'')^2} - 1 = - \frac{1}{1 + (x'')^2} .$$

Thus, instead of (A 78a, b, c),

$$\sin(e, e') = -x'' \cdot \left[1 + (x'')^2\right]^{-1/2} \cdot \left[1 - \{1 + (x'')^2\}^{-1} \cdot \{Z/(2 \cdot R)\}\right] . \qquad (A\ 79)$$

As to the function cos (e, e') of the relations (A 61b) and (A 61c), it can be obtained from tan (e, e') by , ($90^\circ > |(e,e')|$) ,

$$\cos(e, e') = \left[1 + \tan^2 (e, e')\right]^{-1/2} . \qquad (A\ 80)$$

(A 75) and (A 80) yield

$$\cos(e, e') = \left[1 + (x'')^2 \cdot \{1 - Z/(2 \cdot R)\}^2\right]^{-1/2} ,$$

or

$$\cos(e, e') = \left[1 + (x'')^2 \cdot \{1 - Z/R\}\right]^{-1/2} .$$

The last expression on the right hand side is already known from (A 77). Consequently,

$$\cos(e, e') = c \cdot d \quad , \qquad (A\ 81a)$$

$$c = \left[1 + (x'')^2\right]^{-1/2} \quad , \qquad (A\ 81b)$$

$$d = 1 + (x'')^2 \cdot \left[1 + (x'')^2\right]^{-1} \cdot \{Z/(2 \cdot R)\} . \qquad (A\ 81c)$$

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A special discussion about the sign of tan (e, e') and that of sin (e, e') obtained by (A 75) and (A 79) is necessary. The sign of tan (e, e') and that of sin (e, e') is positive if the straight line e' lies above the straight line e - , if Z is negative, i. e. if the point Q is deeper than the point P_s (see Fig.A 5).

(In case, the reader prefers the more mnemonic conception that tan (e, e') and sin (e, e') should have the same sign as Z, the following formulas yield, (A 75),

$$\tan (e, e') = x'' \cdot \left[1 - Z/(2 R) + - \cdots \right] ,$$

and, (A 79),

$$\sin (e, e') = x'' \cdot \left[1 + (x'')^2 \right]^{-1/2} \cdot \left[1 - \left\{ 1 + (x'')^2 \right\}^{-1} \cdot \left\{ Z/(2 R) \right\} \right].$$

In this latter case, further on, (A 58) turns to

$$(e, - g') = (e', -g') - (e, e') .$$

But, the coming formula (A 82) is not changed, may the first or the second variant for the sign of (e, e') be introduced).

14.5. The terms X_1, X_2, X_3, X_4

Now the relation (A 61a) is in the fore. The equations (A 64), (A 66), (A 79), and (A 81a) are introduced into (A 61a). The following relation is found,

$$\frac{\cos (e, n)}{\cos (g', n)} = X_1 + X_2 + X_3 + X_4 \quad ; \tag{A 82}$$

$$X_1 = \cos (e', g') \cdot \cos (e, e') \quad , \tag{A 82a}$$

$$X_2 = \sin (e', g') \cdot \sin (e, e') \quad , \tag{A 82b}$$

$$X_3 = - \sin (e', g') \cdot \cos (e,e') \cdot \tan n' \quad , \tag{A 82c}$$

$$X_4 = \cos (e', g') \cdot \sin (e,e') \cdot \tan n' \quad . \tag{A 82d}$$

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By (A 64), (A 66), (A 79), and (A 81a), the terms X_i, (i = 1, 2, 3, 4), turn into the following shape,

$$X_1 = - \sin p/2 \left[1 + (x'')^2\right]^{-1/2} \cdot X_{1.1} \quad , \qquad \text{(A 83)}$$

$$X_{1.1} = 1 + (x'')^2 \cdot \left[1 + (x'')^2\right]^{-1} \cdot \left\{Z/(2R)\right\} \quad , \qquad \text{(A 83a)}$$

$$X_2 = - \cos p/2 \left[1 + (x'')^2\right]^{-1/2} \cdot x'' \cdot X_{2.1} \quad , \qquad \text{(A 84)}$$

$$X_{2.1} = 1 - \left[1 + (x'')^2\right]^{-1} \cdot \left\{Z/(2R)\right\} \quad , \qquad \text{(A 84a)}$$

$$X_3 = - \cos p/2 \left[1 + (x'')^2\right]^{-1/2} \cdot X_{1.1} \cdot \tan n' \quad , \qquad \text{(A 85)}$$

$$X_4 = \sin p/2 \left[1 + (x'')^2\right]^{-1/2} \cdot x'' \cdot X_{2.1} \cdot \tan n' \quad . \qquad \text{(A 86)}$$

Later on, in the coming investigations, in the integrations over the globe, it will be convenient to distinguish between the integration over the whole globe and that over the near surroundings of the test point P, up to a distance of about 100 km or 1 000 km from P.

Now, the terms X_i, (i = 1, 2, 3, 4), are brought into a special form which does suffice for the integrations over the near surroundings of the test point. In the near surroundings, the following inequality is right,

$$e' \ll R \quad . \qquad \text{(A 86a)}$$

Further, the integrations over the near surroundings are governed by the fact that such integrands are taken along which are proportional to x^2, x^3, ... Indeed, if $e' > 100$ km or $e' > 1\,000$ km, the amounts of x^2 and x^3 are extremely small. They are so small that they can be neglected in the domain beyond the near surroundings of the test point P. This is the underlying mechanism which allows a restriction of the integrations to the near surroundings, only.
Later on, it will be found that the integrations over the near surroundings of the test point P have to be executed only for test points situated in the higher mountains; and they will share to the height anomaly at the point P by not more than about a decimeter.
For lowland points, the impact of these integrands proportional to x^2 or x^3 will be smaller than a centimeter , - negligibly small amounts in most cases. Therefore, it is allowed to neglect relative errors of the order of Z/R, in any case, in these above discussed integrations over the near surroundings.
Thus, introducing the approximations

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$$1 + (Z/R) \cong 1\,,\quad p^2 \cong (e'/R)^2 \cong 0\,, \tag{A 86b}$$

the expressions of (A 83), (A 84), (A 85), and of (A 86) turn to

$$X_1 = -\sin p/2 \left[1 + (x'')^2\right]^{-1/2}, \tag{A 87}$$

$$X_2 = -x''\cdot\left[1 + (x'')^2\right]^{-1/2}, \tag{A 88}$$

$$X_3 = -\left[1 + (x'')^2\right]^{-1/2}\cdot \tan n'\,, \qquad \text{(A 83)}$$

$$X_4 = \sin p/2 \left[1 + (x'')^2\right]^{-1/2}\cdot x''\cdot \tan n'\;; \tag{A 90}$$

The above 4 equations are valid if

$$(e'/R)^2 \ll 1\,,\quad |Z/R| \ll 1\,. \tag{A 90a}$$

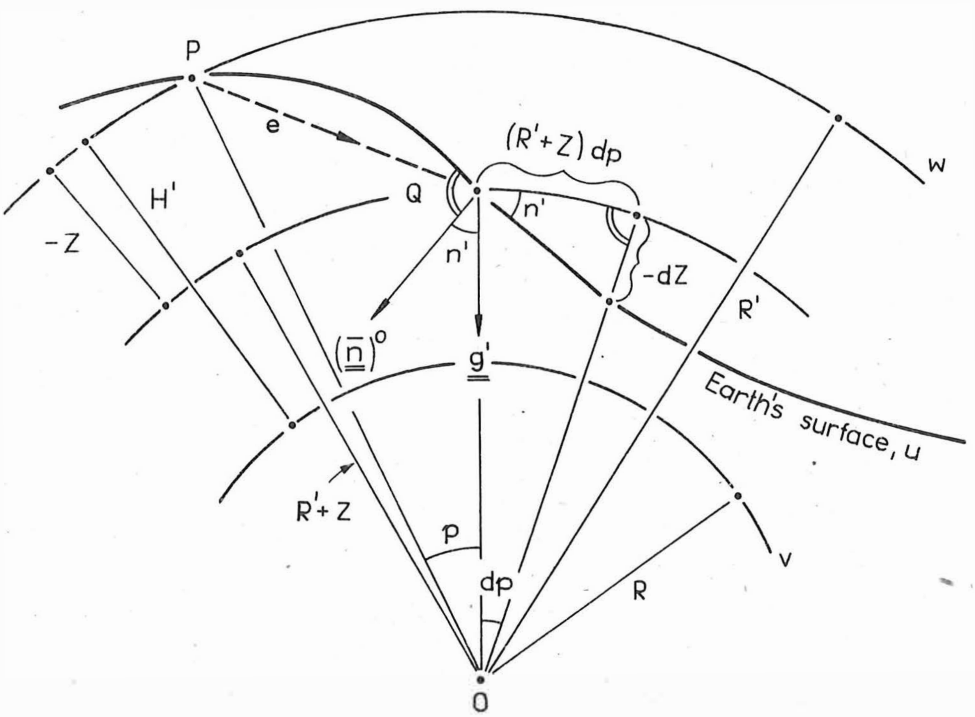

Fig. A 6

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The function $\overline{\tan n'}$ and the sign of $\tan n'$ are defined by (A 62). For a star-shaped Earth, $\tan (g', n)$ is always positive. Thus, the sign of $\tan n'$ is that of $\cos \sigma$: If the height of the terrain diminishes for rising values of p at the point Q, in this case, $\tan n'$ is positive as $\cos \sigma$, also. (See Fig. A 3, Fig. A 6).

The unit vector $(\overline{\underline{n}})^{o}$ of Fig. A 6 and of Fig. A 3 is the projection of the unit vector of the normal of the Earth's surface (pointing into the interior of the body of the Earth) into the plane constructed by the points P, Q, O.

Now, the term $\tan n'$, appearing in (A 85), (A 86) and (A 89), (A 90), is expressed by a formula depending on x and (Z/R'). The following differential relation is easily obtained, Fig. A 6,

$$\tan n' = - dZ \,/\, \left[(R' + Z)\cdot dp\right] \quad . \tag{A 91}$$

Here, R' has to be considered as a constant value. (A 91) can be brought in the form of an integral,

$$Z = - \int_{p=o}^{p} (R' + Z)\cdot(\tan n') \cdot dp \quad . \tag{A 92}$$

The spherical distance p and the straight line e' are connected by a one - one mapping. Thus, in (A 91), dp can be expressed by de', and inverse. It is easily found that

$$e' = 2 \cdot R' \cdot \sin p/2 \quad ,$$

$$de' = R' \cdot (\cos p/2) \cdot dp,$$

$$dp = \left[R' \cdot (\cos p/2) \right]^{-1} \cdot de' \quad . \tag{A 93}$$

The combination of (A 91) and (A 93) yields

$$d\,Z/\,de' = - (1 + Z/R') \cdot (\cos p/2)^{-1} \cdot (\tan n') \quad . \tag{A 94}$$

(A 27) gives, for $x = x\,(H', e')$, $H' = $ const., and for $Z=Z(e')$, (for $x= Z(e')\,/e'$),

$$\partial x/\partial e' = - (Z/e'^2) + (1/e')\,(\partial Z/\,\partial e'), \tag{A 95}$$

and further, (A 94) (A 95),

$$\partial x/\,\partial e' = - x/e' - (1/e')\cdot(1 + Z/R')\cdot(\cos p/2)^{-1}\cdot(\tan n') \quad . \tag{A 96}$$

(A 96) leads to the following expression for $\tan n'$,

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$$\tan n' = -(\cos p/2)\cdot(1 - Z/R')\cdot\Big[\, e'\cdot(\partial x/\partial e') + x \,\Big] \;. \qquad \text{(A 97)}$$

Now, regarding the simplified formulas (A 88) and (A 89) for X_2 and X_3, they offer to get combined to one single expression. With the constraints (A 90a), this fusion of X_2 and X_3 is, considering (A 70a),

$$X_2 + X_3 = -\Big[\, 1 + (x'')^2 \Big]^{-1/2}\cdot (x'' + \tan n') \;. \qquad \text{(A 98)}$$

The following rearrangements of (A 97) are self-explanatory,

$$\tan n' + x'' = -(\cos p/2)\cdot(1 - Z/R')\cdot e'\cdot(\partial x/\partial e') + x''\cdot Z/R',$$

$$(\tan n' + x'')\cdot(1 + Z/R') \cong -(\cos p/2)\cdot e'\cdot(\partial x/\partial e') + x''\cdot Z/R' \;,$$

and, neglecting relative errors of the order of Z/R' in the two expressions $(\tan n' + x'')$ and x'', (A 90a), the following relation is obtained

$$\tan n' + x'' \cong -(\cos p/2)\cdot e'\cdot(\partial x/\partial e').$$

This above equation turns (A 98) to

$$X_2 + X_3 = \Big[1 + (x'')^2 \Big]^{-1/2}\cdot e'\cdot(\cos p/2)\cdot(\partial x/\partial e'), \qquad \text{(A 99)}$$

for the constraints (A 90a).

With (A 87), (A 99), (A 90), and (A 97), the simplified form of (A 82) turns to (A 100), - observing the range of validity of the constraints (A 90a) - ,

$$\Big[X_1 + X_2 + X_3 + X_4 \Big]_0 =$$

$$= \Big[1 + (x'')^2 \Big]^{-1/2}\cdot\Big[-\sin p/2 + e'\cdot(\cos p/2)\cdot(\partial x/\partial e') -$$

$$- (\sin p/2)\cdot(\cos p/2)\cdot x''\Big\{ e'\cdot(\partial x/\partial e') + x \Big\} \Big] \;. \qquad \text{(A 100)}$$

In (A 100), the suffix $[\ ]_0$ behind the brackets denotes that the simplified form of the sum $X_1 + X_2 + X_3 + X_4$ is specified here.
The relation (A 100) is allowed to be applied only if it appears as a factor which is multiplied with x^n, (n = 2, 3, ...). Thus, (A 100) has the restriction to appear only within the forms

$$x^n\cdot\Big[X_1 + X_2 + X_3 + X_4 \Big]_0 \;, \qquad \text{(A 100a)}$$

for

$$n = 2, 3, \ldots \qquad \text{(A 100b)}$$

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x^2 diminishes quickly for growing distances from the test point P, since x= Z/e'. This is the reason why integrations over integrands of the form (A 100a) need to be extended over the near surroundings of the test point P, only, up to a distance of about 1000 km, perhaps.

The integrations over this near surroundings, up to 1000 km distance from P, are accompanied by the following approximation (A 102). Further, (A 101) is applied.

$$\sin p/2 = e' / (2 \cdot R'), \qquad \text{(A 101)}$$

$$\cos p/2 \cong 1, \quad (e' < 1000 \text{ km}) . \qquad \text{(A 102)}$$

(A 70a), (A 101), and (A 102) are introduced into (A 100). The relation (A 103) is the consequence,

$$\left[X_1 + X_2 + X_3 + X_4\right]_0 = a \cdot b , \qquad \text{(A 103)}$$

$$a = \left[1 + (x'')^2\right]^{-1/2} , \qquad \text{(A 103a)}$$

$$b = -\left\{e'/(2 \cdot R')\right\} + \left\{e' \cdot \left[1 - x \cdot e'/(2 \cdot R')\right] \cdot (\partial x/\partial e')\right\} - \left\{x^2 \cdot e'/(2 \cdot R')\right\}, \qquad \text{(A 103b)}$$

valid for

$$e' < 1000 \text{ km} . \qquad \text{(A 103c)}$$

In the brackets of the second term on the right hand side of (A 103b), a rearrangement leads to

$$1 - \left\{x \cdot e'/(2 \cdot R')\right\} = 1 - \left\{Z/(2 \cdot R')\right\} . \qquad \text{(A 103d)}$$

A relative error of the order of Z/ (2·R') can be neglected in the second term on the right hand side of (A 103b),(for (A 103c)). Hence,

$$\left[X_1 + X_2 + X_3 + X_4\right]_0 =$$

$$= \left[1 + (x'')^2\right]^{-1/2} \cdot \left[-1 + 2 \cdot R' \cdot (\partial x/\partial e') - x^2\right] \cdot \left\{e'/(2 \cdot R')\right\} . \qquad \text{(A 104)}$$

In the above equation (A 104), the terms of the relation (A 103d) are inserted after they are put equal to the unity, neglecting relative errors of the order of Z/R' in the relations (A 103d) as well as in the second term in the second brackets on the right hand side of (A 104). Sure, these approximations are allowed before the background of the constraints (A 90a) as well as before the background constructed by the fact that, in the course of the coming investigations, the expression of (A 104) will come to be treated after multiplication with the factors x^2, x^3, ... , in any case, (A 100a) (A 100b). Here, it is essential that the amounts of x^2, x^3, ... diminish rapidly for growing distances e' to the test point P at the oblique surface of the Earth u.

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14.6. The representation of E(2) by a sum of 3 terms

(A 53) and (A 54) yield for D(1.3)

$$D(1.3) = -(1/e^2)\cdot\frac{\cos(e,n)}{\cos(g',n)} + \frac{\partial(1/e')}{\partial r} \quad . \qquad \text{(A 105)}$$

This form is inserted into the relation (45b) of section 4,

$$E(2) = -\iint_w T\cdot D(1.3)\cdot dw \quad . \qquad \text{(A 106)}$$

(A 105) is divided into the spherical and into the topographical part, (Fig.A 2, A 5),

$$1/e^2 = 1/e'^2 + (e'^2 - e^2)\,/\,(e^2\cdot e'^2) \quad . \qquad \text{(A 107)}$$

(A 105), (A 106), (A 107), and (A 82) result

$$E(2) = E(2.1) + E(2.2) + E(2.3) \;, \qquad \text{(A 108)}$$

with

$$E(2.1) = \iint_w T\cdot(e'^2 - e^2)\cdot(e\cdot e')^{-2}\cdot(X_1 + X_2 + X_3 + X_4)\cdot dw \;, \qquad \text{(A 108a)}$$

$$E(2.2) = \iint_w T\cdot(1/e'^2)\cdot(X_1 + X_2 + X_3 + X_4)\cdot dw \;, \qquad \text{(A 109)}$$

$$E\,(2.3) = -\iint_w T\cdot\frac{\partial(1/e')}{\partial r}\cdot dw \quad . \qquad \text{(A 110)}$$

14.6.1. The developments and decompositions of the expression for E(2.1)

E(2.1) is given by (A 108a). From (A 28), (A 31), (A 32) follows

$$(e'^2 - e^2)\,/\,e'^2 = -x^2 - Z/R', \qquad \text{(A 110a)}$$

$$e^2 = e'^2\cdot x' \;,$$

$$x' = 1 + x^2 + Z/R' \quad .$$

Thus,

$$(e'^2 - e^2)\cdot(e\cdot e')^{-2} = -(x^2 + Z/R')\,/\,(e'^2\cdot x'). \qquad \text{(A 111)}$$

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The following decomposition of E(2.1) is recommended,

$$E(2.1) = E(2.1.1) + E(2.1.2) \quad , \qquad \text{(A 112)}$$

with

$$E(2.1.1) = -\iint_W T\cdot(X_1 + X_2 + X_3 + X_4)\cdot\left[x^2/\,(e'^2\cdot x')\right]\cdot dw \quad , \qquad \text{(A 113)}$$

and

$$E(2.1.2) = -\iint_W T\cdot(X_1 + X_2 + X_3 + X_4)\cdot\left[Z/(R'\cdot e'^2\cdot x')\right]\cdot dw \quad . \qquad \text{(A 114)}$$

14.6.1.1. The formula for E(2.1.1)

In the integrand of (A 113), the term x^2 stands in the numerator of the fraction in the brackets. x^2 diminishes quickly for growing distances e' from the test point P. For e' = 1000 km, x^2 will be of the order of 10^{-6}, for instance. Thus, in (A 113), the integration has to cover only the area with

$$e' < 1000 \text{ km} \quad , \qquad \text{(A 115)}$$

(see also: (A 100a), (A 100b)).
Consequently, in (A 113), the sum over the four X_i values can be replaced by the simplified expression (A 104). With (A 104), (A 70a), (A 39), (A 40), neglecting relative errors of the order Z/R' and $(e'/R')^2$,

$$x'' \cong x \quad ,$$

$$x' \cong y^2 = 1 + x^2 \quad ,$$

and, with the constraint (A 115), we find the subsequent equation (A 115a),

$$\left[X_1 + X_2 + X_3 + X_4\right]_0\cdot\left[x^2/\,(e'^2\cdot x')\right] \cong$$
$$\cong x^2\cdot(1 + x^2)^{-3/2}\cdot\left[2\cdot R'\cdot(\partial x/\partial e') - 1 - x^2\right]\cdot\left\{1/(2\cdot e'\cdot R')\right\} \quad . \qquad \text{(A 115a)}$$

(A 113) and (A 115a) lead to

$$E(2.1.1) = E(2.1.1.1) + E(2.1.1.2) \quad , \qquad \text{(A 116)}$$

with

$$E(2.1.1.1) = -\iint_W T\cdot x^2\cdot(1 + x^2)^{-3/2}\cdot(\partial x/\partial e')\cdot(1/e')\cdot dw \quad , \qquad \text{(A 117)}$$

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and

$$E(2.1.1.2) = \iint_{w} T \cdot x^2 \cdot (1 + x^2)^{-1/2} \cdot \{1/(2\,e'\,R')\} \cdot dw \quad . \qquad \text{(A 118)}$$

14.6.1.1.1. The formula for $E(2.1.1.1)$

The formula (A 117) undergoes some transformations considering the fact that the integration has to cover only the near surroundings, (A 115). Thus, the spherical surface element dw can be substituted by the plane surface element,

$$dw \longrightarrow e' \cdot de' \cdot dA \quad ,$$

A is the azimuth counted clockwise from the north. (A 117) turns to

$$E(2.1.1.1) = \int_{A=0}^{2\pi} E' \cdot dA \quad , \qquad \text{(A 118a)}$$

with

$$E' = - \int_{e'=0} T \cdot x^2 \cdot (1 + x^2)^{-3/2} \cdot (\partial x / \partial e') \cdot de' \quad .$$

An integration by parts is introduced. It uses the substitutions (considering E')

$$a_1 = - T,$$

$$\partial a_1 / \partial e' = - \partial T / \partial e',$$

$$b_1 = \int_{e'=0}^{e'} x^2 \cdot (1 + x^2)^{-3/2} \cdot (\partial x / \partial e') \cdot de' \quad ;$$

in the above integrand, x is understood that it is a function of e', only,

$$x = x(e') \quad .$$

Thus,

$$\partial b_1 / \partial e' = x^2 \cdot (1 + x^2)^{-3/2} \cdot (\partial x / \partial e') \quad ,$$

$$b_1 = - x \cdot (1 + x^2)^{-1/2} + \operatorname{arsinh} x \quad , \qquad \text{(A 119)}$$

with

$$\partial b_1 / \partial e' = \{(d\, b_1) / (d\, x)\} \cdot (\partial x / \partial e') \quad .$$

The integration by parts gives

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$$E' = \int_{e'=0}^{e'_o} a_1 \cdot (\partial b_1/\partial e') \cdot de' = \Big| a_1 \cdot b_1 \Big|_{e'_u}^{e'_o} - \int_{e'_u}^{e'_o} b_1 \cdot (\partial a_1/\partial e') \cdot de' \ ;$$

the upper and the lower bound have the following transition behavior,

$$e'_u \rightarrow 0 \ ; \quad e'_o \rightarrow 1000 \text{ km} \ .$$

Hence,

$$E' = \Big| (-T) \cdot \Big[(-x) \cdot (1 + x^2)^{-1/2} + \operatorname{arsinh} x \Big] \Big|_{e'_u}^{e'_o} +$$

$$+ \int_{e'_u}^{e'_o} \Big[(-x) \cdot (1 + x^2)^{-1/2} + \operatorname{arsinh} x \Big] (\partial T/\partial e') \cdot de' \ . \qquad \text{(A 120)}$$

According to (A 118a), E' has to be integrated with regard to the azimuth A. Thus, the first term on the right hand side of (A 120) leads to

$$\int_{A=0}^{2\pi} \Big| (-T) \cdot \Big[(-x) \cdot (1 + x^2)^{-1/2} + \operatorname{arsinh} x \Big] \Big|_{e'_u}^{e'_o} \cdot dA \ . \qquad \text{(A 121)}$$

In case, approaching the test point P, $e'_u \rightarrow 0$, the T_Q value tends to its value at the test point P. Thus, if $e'_u \rightarrow 0$, it follows that $T_Q \rightarrow T_P$ = constant. Q is the moving point, Fig. A 6. The slopes of the terrain are considered to be continuous functions, as found in the topographical maps.

Further, if $e'_u \rightarrow 0$, the moving point Q at the surface of the Earth u tends to lie more and more close on the surface element of the tangential plane of the surface u at the test point P. Thus, on this supposition, the x value tends to an expression of the following shape,

$$x = (Z/e') \cong n_1 \cdot \cos A + n_2 \cdot \sin A, \ (e'_u \rightarrow 0) \ . \qquad \text{(A 122)}$$

In (A 122), the coefficients n_1 and n_2 denote the north - south and the east - west component of the slope of the terrain in the test point P. A is in (A 122) the azimuth. The term b_1, (A 119), appearing in the brackets of (A 121), is expressed by an odd function of x, $b_1(x) = - b_1(-x)$. This fact has the following consequence. For small values of e'_u, in the azimuth $A = A_a$, the function $T \cdot b_1$, (see (A 119)), will have an expression of the following shape: $T \cdot b_1 = k_a + k_{a.1} \cdot e'_u + k_{a.2} \cdot (e'_u)^2 + \cdots$. Further, for small values of e'_u, in the azimuth $A = A_a + 180^\circ$, the function $T \cdot b_1$ will have an expression of the ensuing type: $T \cdot b_1 = - k_a + k'_{a.1} \cdot e'_u + k'_{a.2} \cdot (e'_u)^2 + \cdots$. Thus, considering the limit of $T \cdot b_1$ for $e'_u \rightarrow 0$, in the azimuth A_a, we find $T \cdot b_1 \rightarrow k_a$; and, in the azimuth $A_a + 180^\circ$, we will obtain $T \cdot b_1 \rightarrow - k_a$. (For a terrain of continuous slopes).

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Before the background of these deliberations for $e'_u \rightarrow 0$, the two equations (A 119) and (A 122) lead to the fact that the following relation is valid, (A 121),

$$- \int_{A=0}^{2\pi} (T \cdot b_1)_{e'=e'_u} \cdot dA \rightarrow 0 \ , \qquad \text{(A 123)}$$

if

$$e'_u \longrightarrow 0 \ . \qquad \text{(A 123a)}$$

Herewith, the consideration of the relation (A 121) for the lower value of the argument e', (i.e. e'_u), is already settled.

The relation (A 121) for the upper value of the argument e', (i. e. e'_0), is now in the fore. e'_0 is the amount of e' for the periphery of the circle with the radius e'_0 = 1 000 km, and with the test point P as center. The following inequality is valid,

$$|x| \ll 1 \ , \quad \text{if} \quad e'_0 = 1\ 000 \text{ km}. \qquad \text{(A 123b)}$$

In case of (A 123b), the function b_1 has the following convergent series development, (A 119),

$$b_1 = (1/3) \cdot x^3 - + \cdots \ ; \ x^2 < 1 \ . \qquad \text{(A 124)}$$

Consequently, for the upper argument e'_0, the relation (A 121) takes the following shape,

$$- (1/3) \int_{A=0}^{2\pi} (T \cdot x^3)_{e'=e'_0} \cdot dA \ . \qquad \text{(A 124a)}$$

With $x = (2 \text{ km})/(1000 \text{ km}) = 2 \cdot 10^{-3}$, and estimating the height anomalies with $\zeta = T/g' = 100$ m, the term (A 124a) influences the final result of the height anomaly ζ at the test point P by less than (1/1000) millimeter, (see equation (44) of section 4).
For the model potential M (according to the equation (145) of the section 7), the subsequent version of the parameter data is chosen: $\zeta = M/g' = 1000$ m, $x = (10 \text{ km})/(1000 \text{ km}) = 10^{-2}$; thus, by (A 124a), for the effect on the final ζ value, the amount of 0.3 millimeter follows. This latter amount can be taken as the maximal amount of (A 124a).

Finally, (A 118a) (A 119) (A 120),

$$E(2.1.1.1) \cong \int_{A=0}^{2\pi} \int_{e'=0} b_1 \cdot (\partial T / \partial e') \cdot de' \cdot dA \ . \qquad \text{(A 125)}$$

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In order to shorten the writing , the following abbreviating symbolism is now introduced,

$$(A)\ (E) \cdot \Psi \cdot de' \cdot dA = \int_{A=0}^{2\pi} \int_{e'=0} \Psi \; de' \cdot dA \quad . \qquad \text{(A 125a)}$$

Here, the upper bound of the integration over e' is $e' = 1000$ km. If in the relations of the kind of (A 125a) the differentials $d\,e'$ and $d\,A$ appear, in this case, necessarily, the integration area extends to $e' = 1000$ km, only.
The combination of (A 125) and (A 125a) leads to

$$E(2.1.1.1) = (A)\ (E)\ \ b_1 \cdot (\partial T/\partial e') \cdot d\,e' \cdot d\,A \quad . \qquad \text{(A 125b)}$$

The essence of the rearrangements of E(2.1.1.1) by the relations from (A 118a) to (A 125b) is the fact that the derivative of x with regard to e' is replaced by the derivative of T with regard to e'. The latter derivative is much more smoothed than the first one, a great relief for the numerical computations is the consequence. The right hand side of (A 125b) needs no further transformations, it can be introduced in the calculations, directly.

14.6.1.1.2. The formula for E(2.1.1.2)

After E(2.1.1 .1) has a form convenient for numerical computations, (A 125b), the consideration of E(2.1.1.2) is now in the fore, (A 118). The form of the right hand side of (A 118) has already a form convenient for numerical calculations.
Similarly as in the integrand of (A 117), the term x^2 appears in the integrand of (A 118). The amount of x^2 diminishes quickly for growing values of e'. Thus, the integration area on the right hand side of (A 118) can be restricted to the near surroundings of the test point P, of not more than 1 000 km distance from P, as in case of (A 125b). Again, plane polar co-ordinates are introduced. Thus, (A 118) turns to

$$E(2.1.1.2) = (A)\ (E)\ (1/2) \cdot (T/R') \cdot x^2 \cdot (1 + x^2)^{-1/2} \cdot d\,e' \cdot d\,A \quad . \qquad \text{(A 126)}$$

With (A 116), (A 125b), and (A 126), the following form for E(2.1.1) is obtained,

$$E(2.1.1) = (A)\ (E)\ b_1 \cdot (\partial T/\partial e') \cdot d\,e' \cdot d\,A \; +$$

$$+ (A)\ (E)\ (1/2) \cdot (T/R') \cdot x^2 \cdot (1 + x^2)^{-1/2} \cdot d\,e' \cdot d\,A \; . \qquad \text{(A 127)}$$

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14.6.1.1.3. The integrand proportional to x^2 in areas a great distance away from the test point

Considering the integral for E(2.1.1), (A 113), it is obvious that x^2 has very small values if e' is greater than 1000 km. In (A 127), it is intended to integrate only as long as e' is smaller than 1000 km. The following lines intend to verify that the restriction to e'-values smaller than 1000 km is justified. A reliable evidence will be given.

However, in this context, not the simplified form for the sum of the four X_i terms will be applied, (A 104), simplified by superposition with (A 90a). But, it is necessary to base on the precise form for the four X_i terms, (A 83) to (A 86). But, of course, relative errors of the order of (Z/R') can be neglected in the expressions from (A 83) to (A 86), at least in this context discussed in this sub-section. Along these lines, the following formulas are found,

$$X_1 = - \sin p/2 \left[1 + (x'')^2 \right]^{-1/2} , \qquad \text{(A 128)}$$

$$X_2 = - \cos p/2 \left[1 + (x'')^2 \right]^{-1/2} \cdot x'', \qquad \text{(A 129)}$$

$$X_3 = - \cos p/2 \left[1 + (x'')^2 \right]^{-1/2} \cdot \tan n', \qquad \text{(A 130)}$$

$$X_4 = \sin p/2 \left[1 + (x'')^2 \right]^{-1/2} \cdot x'' \cdot (\tan n'); \qquad \text{(A 131)}$$

in the above lines, relative errors of the order of Z/R' are neglected,

$$\left| Z/R' \right| \ll 1 \quad . \qquad \text{(A 132)}$$

Considering (A 132), the relation (A 97) turns to

$$\tan n' = - e' \cdot (\cos p/2) \cdot (\partial x / \partial e') - (\cos p/2) \cdot x \quad . \qquad \text{(A 133)}$$

Consequently, with (A 70a),

$$\tan n' = - e' \cdot (\cos p/2) \cdot (\partial x / \partial e') - x'' \quad . \qquad \text{(A 134)}$$

Thus, if the distance e' is allowed to have values greater than 1 000 km, the equation (A 99) is transformed to

$$X_2 + X_3 = \left[1 + (x'')^2 \right]^{-1/2} \cdot e' \cdot (\cos p/2)^2 \cdot (\partial x / \partial e') \quad . \qquad \text{(A 135)}$$

The relations (A 128), (A 131), (A 135) yield

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$$\left[X_1 + X_2 + X_3 + X_4\right]_{00} = \left[1 + (x'')^2\right]^{-1/2} \cdot \left\{q_1 + q_2 + q_3\right\} , \qquad \text{(A 135a)}$$

with

$$q_1 = - \sin p/2 \quad , \qquad \text{(A 135b)}$$

$$q_2 = - (\sin p/2)\cdot(x'')^2 \quad , \qquad \text{(A 135c)}$$

$$q_3 = \left[(\cos p/2)^2 \cdot e' - (\sin p/2)\cdot(\cos p/2)\cdot x'' \cdot e'\right]\cdot(\partial x/\partial e') \; . \qquad \text{(A 135d)}$$

The relation (A 135a) derives from the universal formulas (A 83), (A 84), (A 85), and (A 86), by neglecting relative errors of the order of Z/R, only. (A 135a) is valid for whole the globe, for all values of p between 0^o and 180^o.

With

$$\sin p/2 = e'/ (2\, R') \quad , \qquad \text{(A 135e)}$$

$$|x| = |Z/e'| << 1 \; , \qquad \text{(A 135f)}$$

$$|x''| = |x \cdot \cos p/2| << 1 \quad , \qquad \text{(A 135 g)}$$

$$x' = 1 + x^2 + Z/R' \cong 1 \; , \qquad \text{(A 135h)}$$

$$e' > 1\,000 \text{ km} \quad , \qquad \text{(A 135i)}$$

the equation (A 135a) turns to

$$\left[X_1 + X_2 + X_3 + X_4\right]_{000} \cong (e'/(2\,R'))\cdot\left[- 1 + (\cos p/2)^2 \cdot 2\cdot R'\cdot(\partial x/\partial e')\right] . \qquad \text{(A 136)}$$

In the construction of (A 136), the following development for the expression in the brackets of (A 135d) is taken into account,

$$(\cos p/2)^2 \cdot e' - (\sin p/2)\cdot(\cos p/2)\cdot x''\cdot e' =$$

$$e'\cdot(\cos p/2)^2 \left[1 - (\sin p/2)\cdot x\right] =$$

$$= e'\cdot(\cos p/2)^2 \left[1 - Z/ (2\, R')\right] \cong$$

$$\cong e'\cdot(\cos p/2)^2 \; .$$

The relation (A136), (the second term in its brackets on the right,only),is put into (A 113). Instead of (A 117), the following expression is obtained for E(2.1.1.1), in order to check the impact exerted by the integration area of (A 135i),

$$E(2.1.1.1) = - \iint T\cdot(\cos p/2)^2 \cdot x^2 \cdot (\partial x/\partial e')\cdot(1/e')\cdot d\, w \; . \qquad \text{(A 137)}$$

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Similarly as in case of (A 117), the integration by parts allows to transform the relation (A 137). The following rearrangements of (A 137) are self - explanatory,

$$d\,w = (R')^2 \sin p \cdot d\,p \cdot d\,A \quad \cong \quad R^2 \cdot (\sin p) \cdot dp \cdot dA \quad ,$$

$$(1/e') \cdot d\,w \cong (R^2 \cdot \sin p \cdot d\,p \cdot d\,A) / (2 \cdot R \cdot \sin p/2),$$

$$(\partial x/\partial e') = (\partial x/\partial p) \cdot (dp/de'),$$

$$(dp/de') \cong 1/(R \cdot \cos p/2) \; ,$$

$$(\partial x/\partial e') \cong (\partial x/\partial p) \cdot \left\{ 1/(R \cdot \cos p/2) \right\} ,$$

$$(\partial x/\partial e') \cdot (1/e') \cdot dw \cong \left[\partial x/\partial (R \cdot p) \right] \; R \cdot dp \cdot dA \;\; ;$$

consequently, (A 137) turns to the subsequent relation, putting $1+(Z/R) \cong 1$,

$$E(2.1.1.1) = - \iint T \cdot (\cos p/2)^2 \cdot x^2 \cdot \left[\partial x/\partial (R \cdot p) \right] \cdot R \cdot dp \cdot dA \quad .$$

A step, analogous as that from (A 117) to (A 125), leads from the above equation to the following one

$$E(2.1.1.1) = (1/3) \iint x^3 \cdot \left[\partial \left\{ T \cdot (\cos p/2)^2 \right\} / \partial (R \cdot p) \right] \cdot R \cdot dp \cdot d\,A \quad ,$$

or,

$$E(2.1.1.1) = (1/3) \iint x^3 \cdot \left[\partial \left\{ T \cdot (\cos p/2)^2 \right\} / \partial (R \cdot p) \right] \cdot \left\{ 1/(R \cdot \sin p) \right\} \cdot dw \, , \quad \text{(A 137a)}$$

Here, the following amounts for the different parameters are introduced, now,

$$(1/3) \cdot x^3 = (1/3) \cdot (2\ \text{km}/\ 10\,000\ \text{km})^3 \cong 3 \cdot 10^{-12} \quad ,$$

$$(1/G) \left[\partial \left\{ T \cdot (\cos p/2)^2 \right\} / \partial (R \cdot p) \right] \cong 10'' \cong (1/2) \cdot 10^{-4} \quad ,$$

(G is the global mean value of the gravity), further, if Δw is the size of the surface compartments,

$$(d\,w) \cdot (1/(R \cdot \sin p)) \cong \Delta w \cdot (1/(R \cdot \sin p)) \cong$$

$$\cong (500\ \text{km} \cdot 500\ \text{km}) /\ 6\,000\ \text{km} \cong 40\ \text{km} \quad .$$

The global total number of the compartments Δw of the constant size of 500 km · 500 km is $2 \cdot 10^3$. Summing over the values of the integrand for the individual compartments by the square root law, the amount of E(2.1.1.1) is estimated as follows, if integrating over distances greater than e' = 1000 km by means of (A 137a),

$$3 \cdot 10^{-12} \cdot (1/2) \cdot 10^{-4} \cdot 40\ \text{km} \cdot (2 \cdot 10^3)^{1/2} = 3 \cdot 10^{-7}\ \text{millimeter.}$$

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In case, the model potential M according to (145) of the section 7 is implied, instead of T,

$$T \rightarrow M \ ,$$

a multiplication with the factor 10 will be necessary. But, also the thus obtained amount of $3 \cdot 10^{-6}$ millimeter is absolutely insignificant in our applications.

Hence, it will be of no use to extend the integration domain of E(2.1.1.1) according to (A 125b) up to a distance e' from the test point P which is beyond of e' = 1 000 km.

Now, the term E(2.1.1.2) is in the fore, (A 118). The share of the integrations covering the domain e' > 1 000 km is to be evaluated. In this context, the first term in the brackets on the right hand side of (A 136) is introduced into (A 113). The global form for E(2.1.1.2) is obtained, instead of (A 118),

$$E(2.1.1.2) = \iint_W T \cdot x^2 \cdot (1/e') \cdot (1/x') \cdot (1/(2 \cdot R')) \cdot dw \ ,$$

and with (A 135 h),

$$E(2.1.1.2) = \iint_W T \cdot x^2 \cdot \left[1/(2 \cdot e' \cdot R') \right] \cdot dw \ . \qquad \text{(A 138)}$$

The integrand of (A 138) has already a shape convenient for numerical evaluations about the impact of the area beyond of e' = 1 000 km. The following parameter values are introduced, globally averaged,

$$T/G \cong 0.05 \text{ km},$$

$$x^2 \cong (2 \text{ km}/ 10\,000 \text{ km})^2 = 4 \cdot 10^{-8} \ ,$$

$$(R \cdot \sin p) / (2 \cdot R' \cdot e') \cong \frac{1}{2 \cdot R'} \cong (1/ 12\,000) \text{km}^{-1} ,$$

$$\Delta w \cdot \left[1/(R \cdot \sin p) \right] \cong 40 \text{ km} \ .$$

Thus, summing in (A 138) over the individual compartments (of total number $2 \cdot 10^3$) by the square root law, (A 138) gives

$$0.05 \text{ km} \cdot 4 \cdot 10^{-8} \cdot (1/12) \cdot 10^{-3} \text{ km}^{-1} \cdot 40 \text{ km} \cdot (2 \cdot 10^{+3})^{1/2} \ ,$$

or,

$$3 \cdot 10^{-10} \text{ km} = 3 \cdot 10^{-4} \text{ millimeter} \ .$$

This amount is absolutely insignificant.

In case, instead of T, the potential M is applied, a multiplication by 10 will be necessary; but the thus found amount of $3 \cdot 10^{-3}$ millimeter is also negligible.

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In (A 118), it is not necessary to extend the integration areas beyond of $e' = 1000$ km. The same is valid in case of the relation (A 117).

14.6.1.2. The formula for E(2.1.2)

The expression for E(2.1.2) is now in the fore, it is given by (A 114). In the brackets of the integrand of (A 114), the very small factor Z/R' turns up, the amount of this factor is of the order of about 10^{-3} or 10^{-4}. In the sum of $X_1 + X_2 + X_3 + X_4$ in the braces of (A 114), it is allowed, consequently, to neglect relative errors of the order of Z/R'. They share to that terms in the integrand of (A 114) which are of the order of $T(Z/R')^2$, an amount not greater than about $T \cdot 10^{-6}$ or $T \cdot 10^{-8}$. A relative error of smaller than 10^{-6} can be neglected in the T potential values in any case, since the impact of it on the height anomalies ζ, being equal to T/g', will be smaller than 0.1 millimeter. A relative error of smaller than 10^{-6} in the amount M/G will be smaller than 1 millimeter, because $|M/G|$ will be smaller than 1000 m, if M is the model potential T − B.

Consequently, the X_i values here to be applied are not the universal expressions by (A 83), (A 84), (A 85), (A 86). Here, for the computation of E(2.1.2), the expression (A 135a) for the sum $[X_1 + X_2 + X_3 + X_4]_{00}$ is recommended. (A 135a) represents the precise values of the X_i terms, globally valid within the interval $0 \leqslant p \leqslant 180^\circ$, but free of terms which cause a relative change by the order of Z/R. Along these lines, the following formula for E(2.1.2) is obtained,

$$E(2.1.2) = -\iint\limits_{w} T\cdot\left[1 + (x'')^2\right]^{-1/2}\cdot(\sin p/2)\cdot a\cdot\left\{Z/(R'\cdot e'^2\cdot x')\right\}\cdot d\,w \quad , \qquad (A\ 139)$$

with

$$a = -1 - (x'')^2 + 2\cdot R'\cdot\left\{(\cos p/2)^2 - (\sin p/2)\cdot(\cos p/2)\cdot x''\right\}\cdot(\partial x/\partial e') \quad . \qquad (A\ 139a)$$

The reader is remembered that

$$x'' = x\cdot(\cos p/2) \quad , \qquad (A\ 140)$$

$$x' = y^2 + Z/R' = 1 + x^2 + Z/R' \quad . \qquad (A\ 141)$$

The expression for the term a is now considered, (A 139a). It contains amounts as -1 and $\{2\cdot R'\cdot(\cos p/2)^2\cdot(\partial x/\partial e')\}$. -1 is constant, and $2\cdot R'\cdot(\cos p/2)^2$ has not an expressed tendency to go to zero for growing values of p. But, the amounts of $-(x'')^2$ and $\{-2\cdot R'\cdot(\sin p/2)\cdot(\cos p/2)\cdot x''\cdot(\partial x/\partial e')\}$ appearing in (A 139a) have a clear tendency to go to zero for growing distances e' from the test point P, on the strength of the fact that these amounts contain x'' and $(x'')^2$, (A 140). Consequently, in (A 139), it will be of use to separate such terms, which diminish rapidly for $p \rightarrow 180^\circ$.

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This division into two parts is described by the following relations, (A 141a) (A 142) (A 143),

$$E(2.1.2) = E(2.1.2.1) + E(2.1.2.2), \qquad \text{(A 141a)}$$

with

$$E(2.1.2.1) = \iint_W T\cdot\left[1 + (x'')^2\right]^{-1/2}\cdot (\sin p/2)\cdot q_4 \cdot\left\{Z/(R'\cdot e'^2\cdot x')\right\}\cdot d\ w\ , \qquad \text{(A 142)}$$

$$q_4 = (x'')^2 + x''\cdot e'\cdot(\cos p/2)\cdot(\partial x/\partial e'); \qquad \text{(A 142a)}$$

$$E(2.1.2.2) = \iint_W T\cdot\left[1 + (x'')^2\right]^{-1/2}\cdot(\sin p/2)\cdot q_5 \cdot\left\{Z/(R'\cdot e'^2\cdot x')\right\}\cdot d\ w\ , \qquad \text{(A 143)}$$

$$q_5 = 1 - 2\cdot R'\cdot(\cos p/2)^2\cdot(\partial x/\partial e')\ . \qquad \text{(A 143a)}$$

14.6.1.2.1. The formula for E(2.1.2.1)

In the near surroundings of the test point P, for $e' < 1\ 000$ km, it is allowed to put $\cos p/2 \cong 1$, $x'' \cong x$, $x' \cong 1 + x^2$. With these simplifications, the relation (A 142) turns to

$$E(2.1.2.1) = \iint_W T\cdot(1 + x^2)^{-1/2}\cdot(\sin p/2)\cdot q_4\cdot\left\{x/(R'\cdot e')\right\}\frac{1}{1 + x^2}\cdot dw\ , \qquad \text{(A 144)}$$

with

$$q_4 \cong x^2 + x\cdot e'\cdot(\partial x/\partial e')\ . \qquad \text{(A 144a)}$$

And further

$$E(2.1.2.1) = \iint_W T\cdot x^2\cdot(1 + x^2)^{-3/2}\cdot\left[x + e'\cdot(\partial x/\partial e')\right]\cdot(1/(2\cdot R^2))\cdot d\ w\ . \qquad \text{(A 145)}$$

The relation (A 145) offers itself to get divided into two parts,

$$E(2.1.2.1) = E(2.1.2.1.1) + E(2.1.2.1.2)\ , \qquad \text{(A 146)}$$

with

$$E(2.1.2.1.1) = \iint_W T\cdot x^3\cdot(1 + x^2)^{-3/2}\cdot(1/(2\cdot R^2))\cdot d\ w, \qquad \text{(A 147)}$$

and

$$E(2.1.2.1.2) = \iint_W T\cdot x^2\cdot(1 + x^2)^{-3/2}\cdot(\partial x/\partial e')\cdot\left\{e'/(2\cdot R^2)\right\}\cdot d\ w, \qquad \text{(A 148)}$$

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for

$$e' < 1\ 000\ \text{km} \quad . \qquad \text{(A 148a)}$$

The form of (A 147) is in close neighborhood to the relation (A 126). The introduction of plane polar co - ordinates and of (A 135e) turns (A 147) to

$$E(2.1.2.1.1) = (A)\ (E)\ T \cdot x^3 \cdot (1 + x^2)^{-3/2} \cdot (\sin p/2) \cdot (1/R) \cdot d\ e' \cdot d\ A \ . \qquad \text{(A 149)}$$

(A 149) has already a form convenient for the calculations.

The reader is already acquainted with the above used abbreviating writing style, (A 125a).

Now, the expression for E(2.1.2.1.2) is considered, (A 148). The representation by plane polar co-ordinates yields, (A 125a),

$$E(2.1.2.1.2) = (A) \cdot E'' \cdot d\ A \quad , \qquad \text{(A 150)}$$

with

$$E'' = (E)\ 2 \cdot T \cdot x^2 \cdot (1 + x^2)^{-3/2} \cdot (\sin p/2)^2 \cdot (\partial x / \partial e') \cdot d\ e' \quad . \qquad \text{(A 151)}$$

In (A 150), this following relation is valid,

$$(A) \cdot \Psi \cdot d\ A = \int_{A=0}^{2\pi} \Psi \cdot d\ A \quad , \qquad \text{(A 151a)}$$

and in (A 151),

$$(E) \cdot \Psi \cdot d\ e' = \int_{e'=0} \Psi \cdot d\ e' \quad ; \qquad \text{(A 151b)}$$

the upper bound of the integration by (A 151b) is $e' = 1\ 000$ km, (see (A 125a)).

The integral of (A 151) is integrated by the method of the integration by parts. The following substitutions are used, (In (A 152b), dp/de' comes from (A 93)) ,

$$a_2 = 2 \cdot T \cdot (\sin p/2)^2 \quad , \qquad \text{(A 152a)}$$

$$\partial a_2 / \partial e' = 2 \cdot (\sin p/2)^2 \cdot (\partial T / \partial e') + 2 \cdot T \cdot (\sin p/2) \cdot (1/R') , \qquad \text{(A 152b)}$$

and, with (A 119),

$$b_2 = b_1 = - x \cdot (1 + x^2)^{-1/2} + \text{arsinh}\ x = (1/3) \cdot x^3 - + \cdots \quad , \qquad \text{(A 152c)}$$

at the end of the above relation, (A 152c), a series development for the function b_2 appears : $(1/3) \cdot x^3 -+ \cdots$. This series development is valid for $x^2 < 1$, only, (see (A 124)) ;

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$$\partial b_2/\partial e' = x^2 \cdot (1 + x^2)^{-3/2} \cdot (\partial x/\partial e') \quad . \tag{A 152d}$$

The integration by parts turns (A 151) into

$$E'' = E''_1 + E''_2 \quad , \tag{A 153}$$

$$E_1'' = \Big| 2 \cdot T \cdot (\sin p/2)^2 \cdot b_2 \Big|_{e'_u}^{e'_0} \quad , \tag{A 153a}$$

$$E''_2 = -2 \int_{e'_u}^{e'_0} b_2 \left[T \cdot (\sin p/2) \cdot (1/R') + (\sin p/2)^2 \cdot (\partial T/\partial e') \right] \cdot d\, e' \quad . \tag{A 153b}$$

In case that e'_u tends to zero, the amount of T and that of b_2 is finite, since T has continuous values, and since a star-shaped Earth is introduced (the slopes of the terrain of it having finite values). Further, if e'_u tends to zero, the amount of $\sin p/2$ tends to zero, simultaneously. Consequently, (A 153a), the following transition behaviour is right,

$$\left[2 \cdot T \cdot (\sin p/2)^2 \cdot b_2 \right]_{e' = e'_u} \longrightarrow 0 \quad , \tag{A 154}$$

if e'_u tends to zero.

As to the upper bound of (A 153a), this bound is defined by $e'_0 = 1000$ km.

Here, the following data are useful,

$$(\sin p/2)^2 = \left[e'_0/(2 R') \right]^2 \cong 1/144 \quad , \tag{A 154a}$$

and, (A 152c),

$$b_2 = (1/3) \cdot x^3 - + \ldots \cong (1/3) \cdot (2 \text{ km}/ 1\,000 \text{ km})^3 = 3 \cdot 10^{-9} \quad ; \tag{A 154b}$$

with $(T/G) = 0.1$ km, the following self - explanatory developments are right, sure, for the upper bound e'_0 appearing in (A 153a), (see (A 154a)(A 154b)),

$$\left[2 \cdot T \cdot (1/G) \cdot (\sin p/2)^2 \cdot b_2 \right]_{e'=e'_0} \cong 10^{-5} \text{ mm} \cong 0 \quad . \tag{A 154c}$$

In case, T is replaced by the model potential $M = T - B$, (see (145) of section 7), the amount of (A 154c) has to be multiplied with a factor of about 10 ; a negligible amount reveals, furthermore .

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Summarizing, the amount of E_1'' can be neglected.

Hence, with (A 150) (A 151) (A 153) (A 153b),

$$E(2.1.2.1.2) = (A)\ (E)\ (-2)\cdot b_2 \cdot q_6 \cdot d\,e' \cdot d\,A \ , \qquad \text{(A 154d)}$$

with

$$q_6 = T\cdot(\sin p/2)\cdot(1/R') + (\sin p/2)^2\cdot(\partial T/\partial e') \ . \qquad \text{(A 154e)}$$

The estimation of the average amount of E(2.1.2.1.2) is now the work which is to be done.
In this context, the following parameter values are introduced in the integrand of (A 154d) (A 154e): $T/G = 0.05$ km; $\sin p/2 = (20\text{ km}/\ 6000\text{ km})$; $1/R' = (1/6000\text{ km})$; $(1/G)\cdot(\partial T/\partial e') = (0.05\text{ km}/\ 1000\text{ km})$; $b_2 = (1/3)\cdot x^3 = (1/3)\cdot(3\text{ km}\ /\ 30\text{ km})^3$; $d\,e' = 100$ km.

These data reveal

$$q_6/G \cong (0.05\text{ km})\cdot(1/300)\cdot(1/6000\text{ km}) + (1/300)^2\cdot 5\cdot 10^{-5} \ ,$$

thus,

$$q_6/G \cong (1/4)\cdot 10^{-7} \ .$$

Consequently, (A 154d),

$$(1/G)\cdot\left|(1/2\pi)\cdot E(2.1.2.1.2)\right| = 2\cdot(1/3)\cdot(1/1000)\cdot(1/4)\cdot 10^{-7}\cdot 100\text{ km},$$

or

$$(1/G)\left|(1/2\pi)\cdot E(2.1.2.1.2)\right| = (1/6)\cdot 10^{-2}\ \text{millimeter}.$$

If T/G is replaced by M/G, again, a multiplication by the factor 10 will bring about this transformation. A value of (1/60) millimeter is now the result, always to be neglected.

In order to avoid misleading deliberations, the nearest surroundings of the test point P, up to a distance of 1 km or 2 km, are now especially considered, for the case of steep cliffs of $|x| > 1$. For $|x| > 1$, the series development for b_1 cannot be applied, (A 152c). The closed expression on the right hand side of (A 152c) is now of use.

For

$$x = \frac{Z}{e'} = \frac{H_Q - H'}{e'} = -1,$$

(A 152c) leads to

$$b_2 = (1/2)^{1/2} + \operatorname{arsinh}(-1),$$

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or,

$$b_2 = 0.707 - 0.881 = -0.174 \quad . \qquad \text{(A 154f)}$$

Integrating in (A 154d) over the interval $0 \leqslant e' \leqslant 2$ km, we choose these data: $b_2 = -0.174$, $T/G = 0.05$ km, $d\,e' = 2$ km, $\sin p/2 = (0.5\ \text{km}/\ 6\ 000\ \text{km}) = (5/6)\cdot 10^{-4}$.
With these data, it follows that the integration over the domain $e' \leqslant 2$ km takes the following share on the amount of $|(1/2\pi)\cdot E(2.1.2.1.2)| \cdot (1/G)$, (see (A 154d)),

$(-2)\cdot(-0.174)\cdot(0.05)\cdot(5/60000)\cdot(1/6000)\cdot 2\ \text{km} = 5\cdot 10^{-4}$ millimeter.

The transition from T/G to M/G leads to $5\cdot 10^{-3}$ millimeter.

Also the very extreme case of $x = -10$ brings no trouble. b_2 is computed by

$$b_2 = 10\cdot(101)^{-1/2} - \operatorname{arsinh} 10,$$

or,

$$b_2 = 0.995 - 2.998 \cong -2. \qquad \text{(A 154g)}$$

arsinh x is an odd function.
A comparison of (A 154f) and (A 154g) shows that now, for $x = -10$, the amount of the integration over the domain $e' \leqslant 2$ km is about ten times greater.
A value of $5\cdot 10^{-3}$ millimeter, resp. $5\cdot 10^{-2}$ millimeter, is now the consequence.
It is always negligible - this amount of $(1/G)\cdot E(2.1.2.1.2)$ -, even in case of very steep cliffs of $x = -10$, too.

Therefore, in the subsequent deductions, it is allowed to put

$$E(2.1.2.1.2) \cong 0. \qquad \text{(A 154h)}$$

Consequently, (A 146) (A 149) (A 154h),

$$E(2.1.2.1) = \text{(A) (E)}\ T\cdot x^3\cdot(1 + x^2)^{-3/2}\cdot(\sin p/2)\cdot(1/R)\cdot de'\cdot dA \quad , \qquad \text{(A 155)}$$

$$e' < 1000\ \text{km}.$$

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14.6.1.2.2. The formula for E(2.1.2.2)

E(2.1.2.1) according to (A 155) is the first term in the expression for E(2.1.2), (A 141a). The second term is E(2.1.2.2), it is defined by the equation (A 143). E(2.1.2.2) is divided into two parts, since q_5 consists of two parts of different kind, (A 143a). Hence, the decomposition is

$$E(2.1.2.2) = E(2.1.2.2.1) + E(2.1.2.2.2), \qquad \text{(A 156)}$$

with the constituents

$$E(2.1.2.2.1) = \iint_W T \cdot \left[1 + (x'')^2\right]^{-1/2} \cdot (\sin p/2) \cdot \left\{ Z/(R' \cdot e'^2 \cdot x') \right\} \cdot dw \quad , \qquad \text{(A 157)}$$

and

$$E(2.1.2.2.2) = - \iint_W T \cdot \left[1+(x'')^2\right]^{-1/2} \cdot (\sin p/2) \cdot (\cos p/2)^2 \cdot (\partial x / \partial e') \cdot \left\{ 2 \cdot Z/(e'^2 \cdot x') \right\} dw. \qquad \text{(A 158)}$$

At first, the consideration of E(2.1.2.2.1) is in the fore, (A 157). In the integrand of (A 157), the height dependence is brought to bear by the expressions Z/R, $(x'')^2$, and by x'. There do not appear any derivatives of height dependent terms, as $\partial x/\partial e'$ for instance. But, to stress the essence of the deliberations about (A 157), it is of great importance for our applications that the form (A 157) can be divided into two parts of different kind. The constituent of the first kind needs only an integration over the near surroundings of the test point P, (A 148a). But, the constituent of the second kind requires an extension of the integration over whole the globe; p covers the interval from 0° to 180°, in the latter kind.

The rearrangements of (A 157) happen along the following self- explanatory lines,

$$e' = 2 \cdot R' \cdot \sin p/2 \ ,$$

$$\sin p/2 = e' / (2 \cdot R') \ ,$$

$$x = Z/ e' \ , \qquad y^2 = 1 + x^2 \geqslant 1 \ ,$$

$$x' = 1 + x^2 + Z/ R' \ , \qquad \text{(A 158a)}$$

$$x'' = x \cdot (\cos p/2)$$

The above 6 equations are rigorously valid. Neglecting a relative error of the order of Z/R', x' follows as

$$x' \cong 1 + x^2 \quad . \qquad \text{(A 159)}$$

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Further on,

$$(x'')^2 = x^2 \cdot (\cos p/2)^2 = x^2 \cdot \left[1 - (\sin p/2)^2\right] ,$$

$$1 + (x'')^2 = 1 + x^2 - x^2 \cdot (\sin p/2)^2 ,$$

$$1 + (x'')^2 = (1 + x^2) \cdot \left[1 - x^2 \cdot (1 + x^2)^{-1} \cdot (\sin p/2)^2\right] ,$$

$$x^2 (\sin p/2)^2 = (Z/e')^2 \cdot (e'/(2 \cdot R'))^2 = \left[Z/(2 \cdot R')\right]^2 ,$$

$$1 + (x'')^2 = (1 + x^2) \cdot \left[1 - (1 + x^2)^{-1} \cdot \{Z/(2 \cdot R')\}^2\right] . \qquad \text{(A 159a)}$$

$$\{Z/(2 \cdot R')\}^2 \cong (2 \text{ km}/(2 \cdot 6000 \text{ km}))^2 \cong 3 \cdot 10^{-8} .$$

Thus, neglecting a relative error of smaller than $\{Z/(2 \cdot R')\}^2 \cong 3 \cdot 10^{-8}$, $1 + (x'')^2$ has the following approximate formula valid over whole the globe

$$1 + (x'')^2 \cong 1 + x^2 \quad ; \quad (p = 0^\circ, \ldots, 180^\circ) . \qquad \text{(A 160)}$$

With (A 159) and (A 160), (A 157) turns to

$$E(2.1.2.2.1) = \iint_W T \cdot \left[\left\{ (1 + x^2)^{-3/2} - 1 \right\} + 1 \right] \cdot (Z/R') \cdot \left[1/(2 \cdot R' \cdot e') \right] \cdot dw . \qquad \text{(A 161)}$$

The integrand of (A 161) is right within relative errors of the order of Z/R', (see the precision of (A 159)).

The expression in the parentheses $\{\}$ of (A 161) diminishes rapidly for growing distances from the test point P. It gives rise to the constituent of the first kind in the integrand of (A 161). Further, it is satisfied with a limitation of the integration domain to the near surroundings of the test point P, only.

The rest of the integrand of (A 161) gives rise to the constituent of the second kind, it requires an extension of the integrations over whole the globe.

The division of (A 161) into these two constituents leads to the following form, (A 162), considering (A 125a) and

$$\text{(A)} \quad \text{(E)} \quad \Psi \cdot dw = \iint_W \Psi \cdot dw . \qquad \text{(A 161a)}$$

Thus, such a form as that on the left hand side of (A 161a), which contains the surface element dw, - even by putting the symbol dw - , this form points out the necessity that it requires the extension of the integrations over whole the globe.

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Hence,

$$E(2.1.2.2.1) = (A)\ (E)\ T\cdot\left[(1+x^2)^{-3/2} - 1\right]\cdot(Z/R')\cdot(1/2\cdot R'^2)\cdot de'\cdot dA +$$

$$+ (A)\ (E)\ T\cdot(Z/R')\cdot\left[1/(2\cdot R'\cdot e')\right]\cdot dw \quad . \qquad (A\ 162)$$

Some short lines will show that the first integral on the right hand side of (A 162) can be neglected, always; since, substantially, it does contain products of x^2 time Z/R'.

Sure, in this first integral on the right hand side of (A 162), the integration is intended for the near surroundings, only, (if $e' > 1000$ km: $(x^2 Z/R') \cong 0$),

$$e' < 1000 \text{ km}. \qquad (A\ 163)$$

In the concerned integrand, the term in the brackets diminishes rapidly for growing values of e'. Now, the amount of this integral is evaluated. At first, the domain (of the first integral on the right of (A 162))

$$0 \leq e' \leq 5 \text{ km} \qquad (A\ 164)$$

is considered. The parameter data are chosen as follows: $T/G = 0.1$ km, $\left|(1+x^2)^{-3/2} - 1\right| \cong 0.5$ (for steep cliffs), $|Z| = 2$ km. Integrating over the area of (A 164), the first integral on the right hand side of (A 162) yields, multiplied with (1/G),

$$(0.1 \text{ km})\cdot 0.5\cdot(2 \text{ km}/6000 \text{ km})\cdot(1/12\,000 \text{ km})\cdot 5 \text{ km} \cong 0.001 \text{ cm} \quad . \qquad (A\ 165)$$

Now, the same integral is evaluated, but for the domain

$$10 \text{ km} \leq e \leq 100 \text{ km} \quad . \qquad (A\ 166)$$

Here, the parameter data are as follows: $T/G = 0.1$ km, $\left|(1+x^2)^{-3/2} - 1\right| \cong (3/2)\cdot x^2 \cong 0.01$ (the here applied series is valid for $x^2 \ll 1$), $|Z| = 2$ km. Integrating over the area (A 166), the first integral on the right hand side of (A 162) contributes, (multiplied with 1/G),

$$(0.1 \text{ km})\cdot 0.01\cdot(2 \text{ km}/6000 \text{ km})\cdot(1/(12\,000 \text{ km}))\cdot 90 \text{ km} \cong 0.0002 \text{ cm} \quad . \qquad (A\ 167)$$

Summarizing (A 164) (A 165) (A 166) (A 167), the first integrand on the right hand side of (A 162) will hardly surmount the value of 0.001 cm. It can be neglected, consequently.

In case, the perturbation potential T is replaced by the model potential M (according to equation (145) of the section 7), we have to multiply with a factor of about 10 in the results of (A 165) and (A 167). The thus obtained results amount to 0.01 cm resp. 0.002 cm; they are negligible, likewise .

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Considering the above lines, (A 165) (A 167), the relation (A 162) turns to

$$E(2.1.2.2.1) = (A)\ (E)\ T\cdot(Z/R)\cdot\left[1/(2\cdot R\cdot e')\right]\cdot dw\ . \qquad (A\ 168)$$

(A 168) has a shape convenient for numerical calculations.

Now, the expression for E(2.1.2.2.2) is in the fore, (A 158). With (A 159) and (A 160), the subsequent relation comes out, neglecting relative errors of the order of Z/R.

$$E(2.1.2.2.2) = -\ (A)\ (E)\ T\cdot(1+x^2)^{-3/2}\cdot(\cos p/2)^2\cdot(Z/R)\cdot(1/e')\cdot(\partial x/\partial e')\cdot dw. \qquad (A\ 169)$$

Further, it follows

$$E(2.1.2.2.2) = -\ (A)\ (E)\ (T/R)\cdot(\cos p/2)^2\cdot x\cdot(1+x^2)^{-3/2}\cdot(\partial x/\partial e')\cdot dw. \qquad (A\ 170)$$

The following lines are self - explanatory,

$$dw = R'^2\cdot(\sin p)\cdot dp\cdot dA, \qquad (A\ 171)$$

$$de' = R'\cdot(\cos p/2)\cdot dp\ ,$$

$$de'/dp = R'\cdot(\cos p/2), \qquad (A\ 172)$$

$$dp/de' = 1/(R'\cdot\cos p/2)\ , \qquad (A\ 173)$$

$$\partial x/\partial e' = (\partial x/\partial p)\cdot(dp/de')\ ,$$

$$\partial x/\partial e' = (\partial x/\partial p)\left[1/(R'\cdot\cos p/2)\right]. \qquad (A\ 174)$$

The relations (A 170), (A 171), and (A 174) are combined to

$$E(2.1.2.2.2) = -\ (A)\ (E)\ T\cdot(\cos p/2)\cdot(\sin p)\cdot(1+x^2)^{-3/2}\cdot x\cdot(\partial x/\partial p)\cdot dp\cdot dA. \qquad (A\ 175)$$

If, in (A 175), only the integration with regard to the parameter p is considered, before the integration over the azimuth A, the following integral is obtained,

$$E''' = -\int_{p=0}^{\pi} T\cdot(\cos p/2)\cdot(\sin p)\cdot(1+x^2)^{-3/2}\cdot x\cdot(\partial x/\partial p)\cdot dp\ . \qquad (A\ 176)$$

The relation (A 176) is transformed by the method of the integration by parts, avoiding forms as $\partial x/\partial p$. Hence,

$$E''' = \Big|a_3\cdot b_3\Big|_{p=0}^{p=\pi} - \int_{p=0}^{\pi} b_3\cdot(\partial a_3/\partial p)\cdot dp; \qquad (A\ 177)$$

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with

$$a_3 = - T\cdot(\cos p/2)\cdot\sin p,$$

$$\partial a_3/\partial p = -(\partial T/\partial p)\cdot(\cos p/2)\cdot\sin p + (1/2)\cdot T\cdot(\sin p/2)\cdot\sin p - T\cdot(\cos p/2)\cos p,$$

$$b_3 = 1 - (1+x^2)^{-1/2}, \qquad \text{(A 178)}$$

$$\partial b_3/\partial p = x\cdot(1+x^2)^{-3/2}\cdot(\partial x/\partial p).$$

Consequently, for (A 177),

$$E''' = \left| - T\cdot(\cos p/2)\cdot(\sin p)\cdot\left[1-(1+x^2)^{-1/2}\right]\right|_{p=0}^{p=\pi} - \int_{p=0}^{\pi} b_3\cdot(\partial a_3/\partial p)\cdot dp. \qquad \text{(A 179)}$$

As to the first term on the right hand side of (A 179), in the two cases $p = 0$ and $p = \pi$, the function $(\sin p)$ is equal to zero. Thus, the first term on the right hand side of (A 179) is equal to zero. The relation (A 179) turns to

$$E''' = \int_{p=0}^{\pi} b_3 \cdot t_1 \cdot(\sin p)\cdot dp , \qquad \text{(A 180)}$$

with

$$t_1 = (\cos p/2)\cdot(\partial T/\partial p) + \left[(\cos p/2)\cdot(\cot p) - (1/2)\cdot(\sin p/2)\right]\cdot T . \qquad \text{(A 180a)}$$

(A 171), (A 175), and (A 180) yield

$$E(2.1.2.2.2) = (A)\ (E)\ b_3\cdot t_1\cdot(1/R'^2)\cdot dw . \qquad \text{(A 181)}$$

This above integral, (A 181), is now evaluated for the more distant area of

$$e' > 1000 \text{ km} . \qquad \text{(A 182)}$$

In the integrand of (A 181), the function $b_3 = b_3(x)$ appears. The averaged value of b_3 according to (A 178), averaged over the exterior domain of (A 182), can be computed by the following self - explanatory line,

$$b_3 \cong (1/2)\cdot x^2 \cong (1/2)\cdot(2 \text{ km}/ 10\,000 \text{ km})^2,$$

$$b_3 \cong 2\cdot 10^{-8} .$$

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Further,

$$(1/G)\cdot(\partial T/\partial p) = R\cdot(1/G)\cdot(\partial T/(R\,\partial p)) \quad ;$$

on the right hand side of the above equation stands a component of the plumb-line deflection, multiplied with the Earth's radius R. For a deflection component of 20'', it follows,

$$(1/G)\cdot(\partial T/\partial p) = R\cdot(20''/\,206\,265'') \cong R\cdot 10^{-4}.$$

In the computation of a rough mean value of t_1, (A 180a), averaged over the domain of (A 182), it is allowed to operate with the subsequent mean values for cos p/2, cot p, and sin p/2,

$$\cos p/2 \cong 1, \quad \cot p \cong 1, \quad \sin p/2 \cong 1 \quad .$$

Thus, for $(T/G) \cong 0.1$ km , the concerned averaged value of $(1/G)\cdot t_1$ is, (A 180a),

$$(1/G)\cdot t_1 \rightarrow R\left[10^{-4} + (0.1\ \text{km}/\ 6000\ \text{km})\right] ,$$

or,

$$(1/G)\cdot t_1 \rightarrow R \cdot 10^{-4} \quad .$$

The concerned averaged value of $(1/G)\cdot b_3 \cdot t_1$ follows by

$$(1/G)\cdot b_3 \cdot t_1 \rightarrow 2\,R\cdot 10^{-12} \quad . \qquad \text{(A 182a)}$$

The integration according to (A 181) is now replaced by a summation over the compartments Δw of a division of the Earth's surface by a net of meshes of 1000 km x 1000 km size. Thus, it is self-explanatory,

$$(1/R'^2)\cdot dw \rightarrow (1/R'^2)\ \Delta w = (1000\ \text{km}/6000\ \text{km})^2 = 1/36 \quad . \qquad \text{(A 182b)}$$

Hence, (A 182a) (A 182b),

$$(1/G)\cdot b_3 \cdot t_1 \cdot (\Delta w/R'^2) = (1/18)\cdot R\cdot 10^{-12}. \qquad \text{(A 183)}$$

A comparison of (A 183) and (A 181) shows that the term of (A 183) is the averaged amount a single compartment of 1000 km x 1000 km size exerts on the value of $(1/G)\cdot E$ (2.1.2.2.2); here, a value of $(1/3)\cdot 10^{-3}$ millimeter is reached. Whole the surface of the Earth has an extension of about 500 millions km^2. Thus, a number of about 500 compartments of 1000 km x 1000 km size come into question. It will be justifiable to introduce the hypothesis the function t_1/G to vary between the individual compartments similarly as a random variate, (A 180a). Thus, the error-effects of the individual 500 compartments propagate to the impact on the sum of these 500 compartments by the square root law, it is plausible. Hence, in (A 183), we have to multiply with $(500)^{1/2} \cong 22$,

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in order to find the average amount of E(2.1.2.2.2), according to (A 181). This amount results to be equal to 0.007 millimeter,

$$\int_{A=0}^{2\pi} \int_{p_1}^{\pi} b_3 \cdot t_1 \cdot (1/R'^2) \cdot dw \rightarrow 0.007 \quad mm \quad , \qquad (A\ 184)$$

$$p_1 = (1000 \text{ km}/ R) \cong (1/6) \cong 10°/\varrho° \qquad (A\ 184a)$$

A transition from the perturbation potential T to the model potential $M = T - B$ in (A 180a) and (A 184) is accompanied by a multiplication with a factor of about 10, since the order of M is about 10 times the order of T, (see equation (145) of the section 7). Thus, this substitution turns the amount according to (A 184) from 0.007 millimeter to 0.07 millimeter, always negligible, too.

Therefore, it is not necessary to integrate in (A 181) over the domain (A 132). Thus, the integration of (A 181) has to cover only the near surroundings of the test point P,

$$e' < 1000 \text{ km} \quad . \qquad (A\ 184b)$$

The integrand of (A 181) has still to be adapted to this speciality, putting

$$\cos p/2 \cong 1,$$

$$\cos p \cong 1,$$

$$\sin p/2 = e' / (2 R'),$$

$$\sin p \cong e'/ R',$$

$$dw \cong e' \cdot de' \cdot dA \quad .$$

With the above lines, t_1 of (A 180a) turns to

$$t_1 \cong \partial T/\partial p + \left[(R'/e') - (1/4) \cdot (e'/R') \right] T \quad . \qquad (A\ 184c)$$

In the brackets of (A 184c), the first term dominates the second one; hence,

$$t_1 \cong \partial T/\partial p + (R'/e') \cdot T \quad . \qquad (A\ 184d)$$

(A 181) and (A 184d) give – **for the constraint (A 184b) given above** –

$$b_3 \cdot t_1 \cdot (1/R'^2) \cdot dw \cong$$

$$b_3 \left[(\partial T/\partial e') \cdot (e'/R') + T/R' \right] \cdot de' \cdot dA \quad .$$

E(2.1.2.2.2) turns to

$$E(2.1.2.2.2) = (A)\ (E)\ b_3 \cdot \left[(\partial T/\partial e') \cdot (e'/R) + T/R \right] de' \cdot dA \quad . \qquad (A\ 185)$$

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With (A 168), we have reached E(2.1.2.2.1). With (A 185), E(2.1.2.2.2) is found. Thus, E(2.1.2.2) is obtained, it is the aim of this sub-section, - (see (A 156)).

It will be of interest to know the order of the amount of E(2.1.2.2.2) according to (A 185). Here, a test point P situated in the high mountains comes into question, only, since the upper bound of the value of E(2.1.2.2.2) should be evaluated. This evaluation of the amount of E(2.1.2.2.2) for steep cliffs happens by the following data: $e' < 2$ km, $x^2 = 1$, $b_3 = 0.3$, $(1/G)\cdot(\partial T/\partial e') = 20''/206\,265''$, $T/G = 0.05$ km, and $(e'/R) = (1\text{ km}/6000\text{ km})$ as an averaged value. (A 185) yields in a self-explanatory way,

$$(1/G)\cdot(1/2\pi)\cdot E(2.1.2.2.2) \cong 0.3\cdot\left[10^{-4}\cdot(1/6000) + (0.05\text{ km}/6000\text{ km})\right]\cdot 2\text{ km}.$$

In the brackets, the second term dominates.

$$(1/G)\cdot(1/2\pi)\cdot E(2.1.2.2.2) \cong 0.3\cdot(5/6)\cdot 10^{-5}\cdot 2\text{ km} \cong 0.5\text{ cm} \quad . \qquad \text{(A 186)}$$

The exchange of the T potential by the model potential M gives here,(A 186), a value of about 5 cm.

In very rugged mountains only, $(1/G)\cdot(1/2\pi)\cdot E(2.1.2.2.2)$ will surmount the value of 1 cm.

In (A 186), the term E(2.1.2.2.2) is considered after the multiplication with the factor $(1/G)\cdot(1/2\pi)$. On the strength of this fact, the amount of 0.5 cm obtained by (A 186) gives directly the full impact which E(2.1.2.2.2) exerts on the height anomaly of the test point P, as can be seen by the equation (44) of the section 4 and by (A 106). $(1/2\pi)\cdot E(2.1.2.2.2)$ is identical with the effect that E(2.1.2.2.2) takes on the T value at the test point P.

The equations (A 156), (A 168), and (A 185) yield

$$E(2.1.2.2) = (A)\ (E)\ b_3\cdot\left[(\partial T/\partial e')\cdot(e'/R) + T/R\right]\cdot de'\cdot dA \; +$$
$$+ (A)\ (E)\ T\cdot(Z/R)\cdot\left[1/(2\cdot R\cdot e')\right]\cdot dw \quad . \qquad \text{(A 187)}$$

14.6.1.2.3. The final expression for E(2.1.2)

The relations (A 141a), (A 155), and (A 187) are combined. They give

$$E(2.1.2) = (A)\ (E)\ (\partial T/\partial e')\cdot b_3\cdot(e'/R)\cdot de'\cdot dA \; +$$
$$+ (A)\ (E)\ (T/R)\cdot\left[b_3 + b_4\right]\cdot de'\cdot dA \; +$$
$$+ (A)\ (E)\ T\cdot(Z/R)\cdot\left[1/(2\cdot R\cdot e')\right]\cdot dw \quad , \qquad \text{(A 188)}$$

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with

$$b_3 = 1 - (1 + x^2)^{-1/2} \quad , \qquad \text{(A 189)}$$

$$b_4 = x^3 \cdot (1 + x^2)^{-3/2} \cdot (\sin p/2) \quad . \qquad \text{(A 190)}$$

14.6.1.3. The final formula for E(2.1)

The expressions (A 112), (A 119), (A 127), and (A 188) lead to the subsequent formula for E(2.1),

$$E(2.1) = (A)\ (E)\ (\partial T/\partial e')\cdot\left[b_1 + b_3\cdot(e'/R)\right]\cdot de'\cdot dA\ +$$

$$+ (A)\ (E)\ (T/R)\cdot\left[b_3 + b_4 + b_5\right]\cdot de'\cdot dA\ +$$

$$+ (A)\ (E)\ \ T\cdot(Z/R)\cdot\left[1/(2\cdot R\cdot e')\right]\cdot dw\ ; \qquad \text{(A 191)}$$

$$b_1 = \text{arsinh}\ x - x\cdot(1 + x^2)^{-1/2}, \qquad \text{(A 192a)}$$

$$b_3 = 1 - (1 + x^2)^{-1/2} \quad , \qquad \text{(A 192b)}$$

$$b_4 = x^3\cdot(1 + x^2)^{-3/2}\cdot(\sin p/2) \quad , \qquad \text{(A 192c)}$$

$$b_5 = (1/2)\cdot x^2\cdot(1 + x^2)^{-1/2} \quad . \qquad \text{(A 192d)}$$

14.6.2. The developments and decompositions of the formula for E(2.2)

14.6.2.1. The decomposition of E(2.2) into expressions in terms of V_1, V_2, V_3

The equations (A 108) and (A 109) deliver the following expression for E(2.2),

$$E(2.2) = \iint_w T\cdot(e')^{-2}\cdot(X_1 + X_2 + X_3 + X_4)\cdot dw \quad . \qquad \text{(A 193)}$$

In case, the integration has to cover whole the globe, the integration element is formed by the surface element dw. Here, the subsequent abbreviating form is used again, (A 161a),

$$\iint_w \Psi\cdot dw = (A)\ (E)\ \ \Psi\cdot dw \quad , \qquad \text{(A 194)}$$

where the arguments cover the domain

$$0 \leqslant p \leqslant \pi , \qquad \text{(A 195a)}$$

$$0 \leqslant A \leqslant 2\pi \quad . \qquad \text{(A 195b)}$$

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But, if the integration extends only over the near environment of the test point P,

$$0 \leqslant p \leqslant (1000 \text{ km}/ 6000 \text{ km}) \quad , \qquad (A\ 196)$$

we have, (A 125a), instead of the writing style on the right hand side of (A 194), the following form,

$$(A)\ (E)\ \Psi \cdot de' \cdot dA \quad . \qquad (A\ 196a)$$

(A 193) and (A 194) yield

$$E(2.2) = (A)\ (E)\ T \cdot (e')^{-2} \cdot (X_1 + X_2 + X_3 + X_4) \cdot dw \quad . \qquad (A\ 197)$$

As to (A 197), the precise expressions for the X_i terms have to be introduced. They are given by (A 83), (A 84), (A 85), and (A 86). It is convenient to introduce a bifurcation of the sum of the X_i terms; the first branch U_1 is free of a horizontal derivation of the x term, but the second branch U_2 involves the slope of the terrain.

$$X_1 + X_2 + X_3 + X_4 = U_1 + U_2 \quad . \qquad (A\ 198)$$

The following developments are self-explanatory,

$$U_1 = -(\sin p/2) \left[\left\{ 1 + (x'')^2 \right\}^{-1/2} \cdot X_{1.1} - 1 \right] - \sin p/2 -$$

$$- (\cos p/2) \cdot x'' \cdot \left\{ 1 + (x'')^2 \right\}^{-1/2} \cdot X_{2.1} \quad . \qquad (A\ 199)$$

$$U_2 = - \left\{ 1 + (x'')^2 \right\}^{-1/2} \cdot q \cdot \tan n' \quad . \qquad (A\ 200)$$

$$q = (\cos p/2) \cdot X_{1.1} - (\sin p/2) \cdot x'' \cdot X_{2.1} \quad , \qquad (A\ 200a)$$

$$X_{1.1} = 1 + (x'')^2 \cdot \left[1 + (x'')^2 \right]^{-1} \cdot \left\{ Z/ (2 \cdot R) \right\} \quad , \qquad (A\ 200b)$$

$$X_{2.1} = 1 - \left[1 + (x'')^2 \right]^{-1} \cdot \left\{ Z/ (2 \cdot R) \right\} \quad . \qquad (A\ 200c)$$

The term tan n', appearing in (A 200), has the following development, (A 97),

$$\tan n' = - e' \cdot (\cos p/2) \cdot \left[1 - Z/R' \right] \cdot (\partial x / \partial e') -$$

$$- (\cos p/2) \cdot \left[1 - Z/R' \right] \cdot x \quad . \qquad (A\ 201)$$

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The relations (A 199), (A 200), and (A 201) imply the following abbreviations, (A 158a),

$$x = Z/e' \quad , \tag{A 202}$$

$$x' = 1 + x^2 + Z/R', \tag{A 203}$$

$$x'' = x \cdot \cos p/2 \quad , \tag{A 204}$$

$$y^2 = 1 + x^2 \quad . \tag{A 205}$$

Neglecting relative errors of the order of Z/R', the term x' changes into these forms,

$$x' = (1 + x^2) \cdot \left[1 + (1 + x^2)^{-1} \cdot \{ Z/R' \} \right] ,$$

$$x' \cong 1 + x^2 = y^2 \quad , \tag{A 206}$$

and with (A 159a),

$$1 + (x'')^2 = (1 + x^2) \cdot \left[1 - (1 + x^2)^{-1} \cdot (Z/(2 \cdot R'))^2 \right] \quad , \tag{A 207}$$

$$1 + (x'')^2 = y^2 \left[1 - y^{-2} \{ Z/(2 \cdot R') \}^2 \right] \quad , \tag{A 208}$$

neglecting relative errors of $(Z/R)^2$,

$$1 + (x'')^2 \cong y^2 \quad , \tag{A 209}$$

$$\left[1 + (x'')^2 \right]^n \cong y^{2n} \quad . \tag{A 209a}$$

The relation (A 201) is introduced into (A 200). The form for U_2 , which is found in this way, is combined with U_1, (A 199). The development for $U_1 + U_2$ found along these lines is brought into a certain order classifying the terms into three types. The first type is free of the topography, V_1. The second type depends on Z, x, and x", but it depends not on the horizontal derivative of x,(2.type: V_2). The third type is labelled by V_3 , V_3 is proportional to $\partial x/\partial e'$. Thus, (A 198), the following relations are found,

$$X_1 + X_2 + X_3 + X_4 = V_1 + V_2 + V_3 \quad . \tag{A 210}$$

$$V_1 = - \sin p/2 \quad , \tag{A 211}$$

$$V_2 = q_1 \cdot (\sin p/2) + q_2 \cdot (\cos p/2) + q_3 \cdot (\cos p/2)^2 + q_4 \cdot (\sin p/2) \cdot (\cos p/2), \tag{A 212}$$

$$V_3 = q_5 \cdot (\cos p/2)^2 + q_6 \cdot (\sin p/2) \cdot (\cos p/2) \quad . \tag{A 213}$$

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The developments for q_1, q_2, q_3, q_4 have the following expressions,

$$q_1 = -\left[1 + (x'')^2\right]^{-1/2} \cdot \left[1 - \{1 + (x'')^2\}^{1/2} + (x'')^2 \cdot \{1 + (x'')^2\}^{-1} \cdot (Z/2R)\right], \quad \text{(A 213a)}$$

$$q_2 = -\left[1 + (x'')^2\right]^{-1/2} \cdot x'' \cdot \left[1 - \{1 + (x'')^2\}^{-1} \cdot (Z/(2 \cdot R))\right], \quad \text{(A 213b)}$$

$$q_3 = \left[1 + (x'')^2\right]^{-1/2} \cdot x \cdot q_{3.1}, \quad \text{(A 213c)}$$

$$q_{3.1} = \left[1 + (x'')^2 \cdot \{1 + (x'')^2\}^{-1} \cdot \{Z/(2 \cdot R)\}\right] \cdot \left[1 - (Z/R)\right], \quad \text{(A 213d)}$$

$$q_4 = -\left[1 + (x'')^2\right]^{-1/2} \cdot x \cdot x'' \cdot q_{4.1}, \quad \text{(A 213e)}$$

$$q_{4.1} = \left[1 - \{1 + (x'')^2\}^{-1} \cdot \{Z/(2 \cdot R)\}\right] \cdot \left[1 - (Z/R)\right], \quad \text{(A 213f)}$$

$$q_5 = \left[1 + (x'')^2\right]^{-1/2} \cdot q_{5.1} \cdot e' \cdot \left[1 - (Z/R)\right] \{\partial x / \partial e'\}, \quad \text{(A 213g)}$$

$$q_{5.1} = 1 + (x'')^2 \cdot \{1 + (x'')^2\}^{-1} \cdot \{Z/(2 \cdot R)\}, \quad \text{(A 213h)}$$

$$q_6 = -\left[1 + (x'')^2\right]^{-1/2} \cdot q_{6.1} \cdot e' \cdot \left[1 - Z/R\right] \cdot x'' \cdot (\partial x / \partial e'), \quad \text{(A 213i)}$$

$$q_{6.1} = 1 - \left[1 + (x'')^2\right]^{-1} \cdot \{Z/(2 \cdot R)\}. \quad \text{(A 213j)}$$

The relation (A 209a) is inserted into the expressions of (213a) to (213j). Hence it follows, neglecting relative errors of the order of $(Z/R)^2$, as in (A 209c),

$$q_1 = -(1/y) \cdot \left[1 - y + (x'')^2 \cdot (1/y)^2 \cdot \{Z/(2 \cdot R)\}\right], \quad \text{(A 214a)}$$

$$q_2 = -(1/y) \cdot x'' \cdot \left[1 - (1/y)^2 \cdot \{Z/(2 \cdot R)\}\right], \quad \text{(A 214b)}$$

$$q_3 = (1/y) \cdot x \cdot \left[1 + (x'')^2 \cdot (1/y)^2 \cdot \{Z/(2 \cdot R)\}\right] \cdot \left[1 - Z/R\right], \quad \text{(A 214c)}$$

$$q_4 = -(1/y) \cdot x \cdot x'' \cdot \left[1 - (1/y)^2 \cdot \{Z/(2 \cdot R)\}\right] \cdot \left[1 - Z/R\right]; \quad \text{(A 214d)}$$

$$q_5 = (1/y) \cdot e' \cdot \left[1 + (x'')^2 \cdot (1/y)^2 \cdot \{Z/(2 \cdot R)\}\right] \cdot \left[1 - Z/R\right] \cdot \{\partial x / \partial e'\}, \quad \text{(A 215a)}$$

$$q_6 = -(1/y) \cdot x'' \cdot e' \cdot \left[1 - (1/y)^2 \cdot \{Z/(2 \cdot R)\}\right] \cdot \left[1 - Z/R\right] \cdot \{\partial x / \partial e'\}. \quad \text{(A 215b)}$$

q_5 turns to, (A 215a),

$$q_5 = (1/y) \cdot e' \cdot \left[1 + \{-2 + (x'')^2 \cdot (1/y)^2 \cdot\} \{Z/(2 \cdot R)\}\right] \{\partial x / \partial e'\}. \quad \text{(A 216)}$$

In the course of the transition from (A 215a) to (A 216), relative errors of the order of $(Z/R)^2$, - being about 10^{-7} - , are neglected. The same is valid for the transition from (A 215b) to (A 217), described subsequently.

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q_6 changes into, (A 215b),

$$q_6 = -(\cos p/2)\cdot(1/y)\cdot x\cdot e'\left[1 - \left\{2 + (1/y)^2\right\}\cdot\left\{Z/(2R)\right\}\right]\cdot\left\{\partial x/\partial e'\right\} . \quad \text{(A 217)}$$

From now, the q_i values of (A 214a) to (A 217) are used instead of the forms from (A 213a) to (A 213j).

Considering the relations (A 213), (A 216), and (A 217), it is possible to distinguish into terms which are free of the factor Z/R or not. Along these lines, V_3 gets the following shape

$$V_3 = (1/y)\cdot q_7\cdot(\partial x/\partial e') + (1/y)\cdot e'\cdot(\cos p/2)^2\cdot q_8\cdot(Z/R)\;(\partial x/\partial e') . \quad \text{(A 218)}$$

$$q_7 = e'\cdot(\cos p/2)^2 - e'\cdot(\sin p/2)\cdot(\cos p/2)^2\cdot x , \quad \text{(A 218a)}$$

$$q_8 = -1 + (1/2)\cdot(x'')^2\cdot(1/y)^2 + (\sin p/2)\cdot x\cdot\left\{1 + (1/2)\cdot(1/y)^2\right\} . \quad \text{(A 218b)}$$

The expression (A 218) for V_3 can be rearranged according to rising powers of x. Hence, it follows, considering

$$Z = x\cdot e' , \quad \text{(A 219)}$$

$$V_3\cdot y = q_9 + q_{10} + q_{11} + q_{12} + q_{13} ; \quad \text{(A 220)}$$

with

$$q_9 = (\cos p/2)^2\cdot e'\cdot(\partial x/\partial e') , \quad \text{(A 220a)}$$

$$q_{10} = -(\sin p/2)\cdot(\cos p/2)^2\cdot e'\cdot x\cdot(\partial x/\partial e') , \quad \text{(A 220b)}$$

$$q_{11} = -(\cos p/2)^2\cdot e'^2\cdot(1/R)\cdot x\cdot(\partial x/\partial e') , \quad \text{(A 220c)}$$

$$q_{12} = (\sin p/2)\cdot(\cos p/2)^2\cdot e'^2\cdot(1/R)\cdot x^2\cdot\left[1 + (1/2)\cdot(1/y)^2\right]\cdot(\partial x/\partial e'), \quad \text{(A 220d)}$$

$$q_{13} = (1/2)\cdot(\cos p/2)^4\cdot e'^2\cdot(1/R)\cdot x^3\cdot(1/y)^2\cdot(\partial x/\partial e') . \quad \text{(A 220e)}$$

As already mentioned,- see also (A 214a) to (A 215b) - , the expression (A 220) for V_3 neglects such terms which cause relative errors of the order of $(Z/R)^2$ in V_3, (A 209),

$$(Z/R)^2 \cong 10^{-7} . \quad \text{(A 221)}$$

The neglection of such terms is justified.

After the expression (A 213) for V_3 is brought into the shape of (A 220), the expression (A 212) for V_2 undergoes a similar rearrangement, too, at which the coefficients q_1, q_2, q_3, q_4 come from (A 214 a, b, c,d). Hence, the rearrangement of V_2 according to rising powers of x leads to (A 222),

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$$V_2 \cdot y = q_{14} + q_{15} + q_{16} + q_{17} + q_{18} + q_{19} + q_{20} + q_{21} \quad . \qquad \text{(A 222)}$$

The terms on the right hand side of (A 222) have the following representations,

$$q_{14} = (\sin p/2)\cdot(y - 1) \quad , \qquad \text{(A 222a)}$$

$$q_{15} = -(\cos p/2)\cdot x'' \quad , \qquad \text{(A 222b)}$$

$$q_{16} = (\cos p/2)^2 \cdot x \quad , \qquad \text{(A 222c)}$$

$$q_{17} = -(\sin p/2)\cdot(\cos p/2)\cdot x\cdot x'' \quad , \qquad \text{(A 222d)}$$

$$q_{18} = -(\sin p/2)\cdot(x'')^2\cdot(1/y)^2\cdot\{Z/(2\cdot R)\} \quad , \qquad \text{(A 222e)}$$

$$q_{19} = (\cos p/2)\cdot x''\cdot(1/y)^2\cdot\{Z/(2\cdot R)\} \quad , \qquad \text{(A 222f)}$$

$$q_{20} = (\cos p/2)^2\cdot x\cdot\{(x'')^2\cdot(1/y)^2 - 2\}\cdot\{Z/(2\cdot R')\} \quad , \qquad \text{(A 222g)}$$

$$q_{21} = (\sin p/2)\cdot(\cos p/2)\cdot x\cdot x''\cdot\{(1/y)^2 + 2\}\cdot\{Z/(2\cdot R)\} \quad . \qquad \text{(A 222h)}$$

From the terms $q_{14} \cdots q_{21}$, it serves the purposes to construct the following four couples, regarding (A 204), also,

$$q_{22} = q_{14} + q_{17} = (\sin p/2)\cdot\{y - 1 - (x'')^2\} \quad , \qquad \text{(A 223)}$$

$$q_{23} = q_{15} + q_{16} = 0 \quad , \qquad \text{(A 224)}$$

$$q_{24} = q_{18} + q_{21} = (\sin p/2)\cdot(\cos p/2)^2\cdot x^2\cdot(Z/R) \quad , \qquad \text{(A 225)}$$

$$q_{25} = q_{19} + q_{20} = (\cos p/2)^2\cdot x\cdot\{(1/2)\cdot(1/y)^2 - 1\}\cdot(Z/R) +$$
$$+ (1/2)\cdot(\cos p/2)^4\cdot x^3\cdot(1/y)^2\cdot(Z/R) \quad . \qquad \text{(225a)}$$

The relations (A 222), (A 223), (A 224), (A 225), (A 225a) can be combined to

$$V_2\cdot y = q_{22} + q_{24} + q_{25} \quad . \qquad \text{(A 225b)}$$

Returning back to the right hand side of (A 210), which gives the sum of $V_1 + V_2 + V_3$: V_1 has the development (A 211), V_2 is represented by (A 225b), (A 223), (A 225), (A 225a), and, finally, V_3 has the expression (A 220), (A 220a,b,c,d,e).

Thus, returning back to (A 197) and (A 210), obviously, E(2.2) can be decomposed into 3 terms,

$$E(2.2) = E(2.2.1) + E(2.2.2) + E(2.2.3) \quad . \qquad \text{(A 226)}$$

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With (A 197), (A 210), (A 211), (A 225b), and (A 220), the individual parts on the right hand side of (A 226) have the following expressions,

$$E(2.2.1) = (A)(E)\ T\cdot(1/e')^2 \cdot V_1 \cdot dw \quad , \qquad \text{(A 227)}$$

$$E(2.2.2) = (A)\ (E)\ T\cdot(1/e')^2 \cdot V_2 \cdot dw \quad , \qquad \text{(A 228)}$$

$$E(2.2.3) = (A)\ (E)\ T\cdot(1/e')^2 \cdot V_3 \cdot dw \quad . \qquad \text{(A 229)}$$

14.6.2.2. The formula for E(2.2.1)

The relations (A 211) and (A 227) yield

$$E(2.2.1) = -\ (A)\ (E)\ T\cdot(1/e')^2\cdot(\sin p/2) \cdot dw \quad . \qquad \text{(A 230)}$$

Obviously, E(2.2.1) is a pure spherical term, it does not imply the topographical heights Z.

14.6.2.3. The formula for E(2.2.2)

The treatment of the procedure that shows the way how to compute E(2.2.2) is a short work only. The consideration of the structure of the expression (A 225b) representing V_2 is in the fore, here, (A 228). V_2 has the essential property that the amount of it diminishes quickly for growing distances e' from the test point. It diminishes as quick as x^2, a fact that will be delivered by the further lines, (A 234), (A 236).

For e' = 1000 km, the amount of x^2 will be of the order of about 10^{-6}. In the expressions (A 223), (A 225), (A 225a), which appear on the right hand side of (A 225b), it is convenient to undertake some transformations. Considering

$$\sin p/2 = e' / (2\cdot R') \quad , \qquad \text{(A 231)}$$

$$dw \cong e' \cdot de' \cdot dA \quad , \qquad \text{(A 232)}$$

$$Z/R' = (x \cdot e')/ R' \quad , \qquad \text{(A 233)}$$

and

$$1 + (x'')^2 \cong y^2 \quad , \qquad \text{(A 233a)}$$

(the latter relation neglects relative errors of the order of $(Z/R)^2$, according to (A 209)), the expression (A 225b) for V_2 in terms of q_{22}, q_{24}, q_{25} turns to the following representation of V_2 in terms of q_{26}, q_{27}, q_{28}, it is self-explanatory,

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$$(1/e')^2 \cdot y \cdot V_2 \cdot dw = (1/R) \cdot (q_{26} + q_{27} + q_{28}) \cdot de' \cdot dA \quad , \qquad \text{(A 234)}$$

here is,

$$q_{26} = (1/2) \cdot (y - y^2) \quad , \qquad \text{(A 234a)}$$

$$q_{27} = x^3 \cdot (\sin p/2) \cdot (\cos p/2)^2 \quad , \qquad \text{(A 234b)}$$

$$q_{28} = x^2 \cdot (\cos p/2)^2 \cdot q_{29} \quad ; \qquad \text{(A 234c)}$$

$$q_{29} = \left\{ (1/2) \cdot (1/y)^2 - 1 \right\} + (1/2) \cdot x^2 \cdot (\cos p/2)^2 \cdot (1/y)^2 \quad . \qquad \text{(A 234d)}$$

The abbreviating symbol b_6 is introduced,

$$b_6 = (1/y) \cdot (q_{26} + q_{27} + q_{28}) \quad ; \qquad \text{(A 235)}$$

(see also (A 343), being a series for b_6 with rising powers of $x : b_6 = -(3/4)x^2 + - \ldots$).

The relations (A 234) and (A 235) are combined with (A 228). Hence it follows

$$E(2.2.2) = (A)\ (F)\ (T/R) \cdot b_6 \cdot de' \cdot dA \quad . \qquad \text{(A 236)}$$

The expression for b_6 diminishes for growing values of e', as the expression x^2, (A 343). Thus, the integral for E(2.2.2) must not be integrated for the area $e' > 1000$ km, (see the integral (A 138) and, at that place, annexed to (A 138), the deliberations about the extension of the integration domain). For the integrations according to (A 236), the coverage of the interval $0 \leqslant e' \leqslant 1000$ km will suffice.

Consequently, the relation (A 236) is the final form of E(2.2.2), convenient for numerical integrations.

14.6.2.4. The formula for E(2.2.3)

The integral for E(2.2.3) is given by (A 229). The integrand contains the term V_3.

14.6.2.4.1. The decomposition of the formula for E(2.2.3)

According to (A 220), V_3 is represented by the sum of 5 terms. (A 220) is introduced into (A 229); with this, the two terms q_{10} and q_{11} are combined. Along these lines, E(2.2.3) gets a form which consists of the sum of 4 terms. Hence it follows

$$E(2.2.3) = E(2.2.3.1) + E(2.2.3.2) + E(2.2.3.3) + E(2.2.3.4) \quad , \qquad \text{(A 237)}$$

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with,

$$E(2.2.3.1) = (A)\ (E)\ T\cdot(1/e')^2\cdot(1/y)\cdot q_9\cdot dw\ ,\qquad (A\ 237a)$$

$$E(2.2.3.2) = (A)\ (E)\ T\cdot(1/e')^2\cdot(1/y)\cdot(q_{10}+q_{11})\cdot dw\ ,\qquad (A\ 237b)$$

$$E(2.2.3.3) = (A)\ (E)\ T\cdot(1/e')^2\cdot(1/y)\cdot q_{12}\cdot dw\ ,\qquad (A\ 237c)$$

$$E(2.2.3.4) = (A)\ (E)\ T\cdot(1/e')^2\cdot(1/y)\cdot q_{13}\cdot dw\ .\qquad (A\ 237d)$$

The relations (A 237) and (A 237a,b,c,d) define the decomposition of E(2.2.3) into 4 parts.

14.6.2.4.2. The formula for E(2.2.3.1)

(A 237a) and (A 220a) give the expression for E(2.2.3.1),

$$E(2.2.3.1) = (A)\ (E)\ T\cdot(1/e')\cdot(\cos p/2)^2\cdot(1/y)\cdot(\partial x/\partial e')\cdot dw\ .\qquad (A\ 238)$$

In the main, the integrand of (A 238) is linear in x. Substantially, (A 238) is not square in x. Thus, we have to take into account a global extension of the integration area. The independent variable e' is replaced by p. In (A 238), by means of (A 174), the derivative $\partial x/\partial e'$ is replaced by $\partial x/\partial p$. A short rearrangement follows. Hence, from (A 238),

$$E(2.2.3.1) = (A)\ (E)\ T\cdot(1/2)\cdot(\cot p/2)\cdot(1/R)^2\cdot(1/y)\cdot(\partial x/\partial p)\cdot dw\ .\qquad (A\ 239)$$

The term $\partial x/\partial p$ variates considerably. Therefore, it is recommended to replace this term by $\partial T/\partial p$, which variates within narrow limits, only. Following up this aim, the integration of (A 239) has to happen by the method of the integration by parts. In this context, dw has to be expressed by the differentials dp and dA. With

$$dw = R'^2\cdot(\sin p)\cdot dp\cdot dA\ ,\qquad (A\ 240)$$

the relation (A 239) turns to

$$E(2.2.3.1) = (A)\ (E)\ T\cdot(\cos p/2)^2\cdot(1/y)\cdot(\partial x/\partial p)\cdot dp\cdot dA\ .\qquad (A\ 240a)$$

The integral on the right hand side of (A 240a) will be treated later on, by the method of the integration by parts with the argument p ranging from 0^o to 180^o. In this context, the two functions a_7 and b_7 are concerned. The product

$$a_7\cdot(\partial b_7/\partial p)\qquad (A\ 240b)$$

is defined to be the integrand of (A 240a). Hence, it follows

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$a_7 = T \ (\cos p/2)^2$, (A 241)

$\partial a_7/\partial p = (\partial T/\partial p)\cdot(\cos p/2)^2 - T\cdot(1/2)\cdot\sin p$, (A 242)

$\partial b_7/\partial p = (1/y)\cdot(\partial x/\partial p)$, (A 243)

$b_7 = \text{arsinh} \ x$; (A 244)

$b_7 = x - (1/6)\cdot x^3 + - \cdots, \quad x^2 < 1$. (A 244a)

The last line corroborates the fact that the integrand on the right hand side of (A 240a) is linear in x, in the main.

14.6.2.4.3. The formula for E(2.2.3.2)

E(2.2.3.2) is defined by (A 237b).
Here is, (A 220b) (A 220c),

$(1/y)\cdot(q_{10} + q_{11}) = -(3/2)\cdot(\cos p/2)^2\cdot(e')^2\cdot(1/R)\cdot(1/y)\cdot x\cdot(\partial x/\partial e')$. (A 244b)

The above expression (A 244b) is square in the height Z, since the product

$x\cdot(\partial x/\partial e')$

appears. Thus, in the integration, the argument e' ranges from 0 to 1000 km, only. In this area, a plane co-ordinate system is an adequate approximation. Consequently,

$dw \cong e' \cdot de' \cdot dA$, (A 244c)

$(1/e')^2\cdot dw \cong (1/e')^2\cdot 2\cdot R'\cdot(\sin p/2)\cdot de'\cdot dA$. (A 244d)

The combination of (A 244b) and (A 244d) yields

$(1/e')^2 \cdot dw\cdot(1/y)\cdot(q_{10} + q_{11}) =$

$= -3\cdot(\sin p/2)\cdot(\cos p/2)^2\cdot(1/y)\cdot x\cdot(\partial x/\partial e')\cdot de'\cdot dA$. (A 244e)

Hence,

$E(2.2.3.2) = (A)\ (E)\ (-3)\cdot T\cdot(\sin p/2)\cdot(\cos p/2)^2\cdot(1/y)\cdot x\cdot(\partial x/\partial e')\cdot de'\cdot dA$. (A 245)

Here, the integration by parts has the following substitutions (regarding the relation (A 173) for dp/ de') ,

$a_8 = -3\cdot T\cdot(\sin p/2)\cdot(\cos p/2)^2$, (A 246)

$\partial a_8/\partial e' = -3\cdot(\partial T/\partial e')\cdot(\sin p/2)\cdot(\cos p/2)^2 -$

$- 3\cdot T\cdot\left\{\partial\left[(\sin p/2)\cdot(\cos p/2)^2\right]/\partial p\right\}\cdot(dp/de')$. (A 246a).

For the term in the parentheses {} of the above equation, the following rearrangement is self-explanatory,

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$$\partial\left\{(\sin p/2)\cdot(\cos p/2)^2\right\}/\partial p =$$

$$= (1/2)\cdot(\cos p/2)\cdot(\cos p/2)^2 - (\sin p/2)\cdot 2\cdot(\cos p/2)\cdot(\sin p/2)\cdot(1/2) =$$

$$= (1/2)\cdot(\cos p/2)^3 - (\sin p/2)^2\cdot(\cos p/2) =$$

$$= (\cos p/2)\cdot\left\{(1/2)\cdot(\cos p/2)^2 - (\sin p/2)^2\right\} =$$

$$= (\cos p/2)\cdot(1/2)\cdot\left\{1 - 3\cdot(\sin p/2)^2\right\} .$$

With (A 173), the second term on the right hand side of (A 246a) turns to

$$- (3/2)\cdot(T/R)\cdot\left\{1 - 3\cdot(\sin p/2)^2\right\} . \qquad \text{(A 246b)}$$

(A 246b) is introduced into (A 246a),
hence it follows

$$\partial a_8/\partial e' = - 3\cdot(\partial T/\partial e')\cdot(\sin p/2)\cdot(\cos p/2)^2 - (3/2)\cdot(T/R)\cdot\left\{1 - 3\cdot(\sin p/2)^2\right\} . \text{(A 247)}$$

Further, regarding (A 245),

$$\partial b_8/\partial e' = (1/y)\cdot x\cdot(\partial x/\partial e') , \qquad \text{(A 248)}$$

$$b_8 = y - 1 ; \qquad \text{(A 249)}$$

the series development for b_8 is

$$b_8 = (1/2)\cdot x^2 - (1/8)\cdot x^4 + - \cdots , \quad x^2 < 1 . \qquad \text{(A 249a)}$$

(A 249a) corroborates that the term b_8 diminishes as quick as x^2 ,
for rising e' values.

14.6.2.4.4. The formula for E(2.2.3.3)

E(2.2.3.3) has the expression of (A 237c). The term q_{12} comes from
(A 220d). (A 220d) and (A 244d) are combined to

$$(1/e')^2\cdot(1/y)\cdot q_{12}\cdot dw =$$

$$= 2\cdot(\sin p/2)^2\cdot(\cos p/2)^2\cdot x^2\cdot(1/y)\cdot\left\{1 + (1/2)\cdot(1/y)^2\right\}\cdot(\partial x/\partial e')\cdot de'\cdot dA . \qquad \text{(A 249b)}$$

Since (A 249b) implies the term x^2, the integration must not range further
than to e' = 1000 km. (A 237c) and (A 249b) lead to

$$E(2.2.3.3) = (A)\ (E)\ T\cdot(1/2)\cdot(\sin p)^2\cdot(1/y)\cdot\left\{1 + (1/2)\cdot(1/y)^2\right\}\cdot x^2\cdot(\partial x/\partial e')\cdot de'\cdot dA.$$

(A 250)

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Here, the integration by parts makes use of the following substitutions:

$$a_9 = (1/2)\cdot T\cdot(\sin p)^2 \quad ; \qquad \text{(A 251)}$$

in the derivation of a_9 with regard to e', (A 251), the following expression appears, obviously, (see (A 173)),

$$\left\{\partial(\sin p)^2/\partial p\right\}\cdot(dp\,/\,de') =$$
$$= 2\cdot(\sin p)\cdot(\cos p)\cdot\left\{1/\ (R\cdot\cos p/2)\right\} = (4/R)\cdot(\sin p/2)\cdot\cos p.$$

Hence it follows,

$$\partial a_9/\partial e' = (\partial T/\partial e')\cdot(1/2)\cdot(\sin p)^2 + (T/R)\cdot 2\cdot(\sin p/2)\cdot\cos p\ . \qquad \text{(A 252)}$$

The rest of the integrand of E(2.2.3.3), (A 250), left over by the term a_9, is

$$\partial b_9/\partial e' = \left\{(1/y)\cdot x^2 + (1/2)\cdot(1/y)^3\cdot x^2\right\}\cdot(\partial x/\partial e')\ , \qquad \text{(A 253)}$$

The integration gives

$$b_9 = (1/2)\cdot x^3\cdot(1/y)\ , \qquad \text{(A 254)}$$

it has the series development

$$b_9 = (1/2)\cdot x^3 + - \cdots, \quad x^2 < 1\ . \qquad \text{(A 254a)}$$

The amount of b_9 diminishes very quickly for growing values of e'. Thus, the limitation of the integration to the cap of $e' < 1000$ km is justified, (A 250).

14.6.2.4.5. The formula for E(2.2.3.4)

The term E(2.2.3.4) is represented by (A 237d). The term q_{13} appearing in (A 237d) has the expression (A 220e).
Consequently, (A 244d),

$$(1/e')^2\cdot(1/y)\cdot q_{13}\cdot dw =$$
$$(\sin p/2)\cdot(\cos p/2)^4\cdot(1/y)^3\cdot x^3\cdot(\partial x/\partial e')\cdot de'\cdot dA\ . \qquad \text{(A 254b)}$$

Hence it follows,

$$E(2.2.3.4) = (A)\ (E)\ T\cdot(\sin p/2)\cdot(\cos p/2)^4\cdot(1/y)^3\cdot x^3\cdot(\partial x/\partial e')\cdot de'\cdot dA\ . \qquad \text{(A 255)}$$

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Here, the integration by parts comes about by the following substitutions

$$a_{10} = T\cdot(\sin p/2)\cdot(\cos p/2)^4 \quad , \qquad \text{(A 256)}$$

In the derivation of a_{10} with regard to e', the following expression is needed, (A 173),

$$\left[\partial\{(\sin p/2)\cdot(\cos p/2)^4\} / \partial p\right]\cdot(dp/de') = q_{30}\{1/(R\cdot\cos p/2)\} ,$$

with

$$q_{30} = (1/2)\cdot(\cos p/2)\cdot(\cos p/2)^4 + (\sin p/2)\cdot 4\cdot(\cos p/2)^3\cdot(-\sin p/2)\cdot(1/2).$$

Thus,

$$q_{30}\{1/(R\cdot\cos p/2)\} = (1/R)\cdot\{(1/2)\cdot(\cos p/2)^4 - 2(\sin p/2)^2\cdot(\cos p/2)^2\} =$$

$$= (1/R)\cdot(1/2)\cdot\{(\cos p/2)^4 - (\sin p)^2\} .$$

Hence, it follows by the derivation of (A 256)

$$\partial a_{10}/\partial e' = (\partial T/\partial e')\cdot(\sin p/2)\cdot(\cos p/2)^4 + \{T/(2\cdot R)\}\cdot\{(\cos p/2)^4 - (\sin p)^2\} . \qquad \text{(A 257)}$$

The rest of the integrand of E(2.2.3.4), left over by the term a_{10}, has the following shape,

$$\partial b_{10}/\partial e' = (1/y)^3\cdot x^3\cdot(\partial x/\partial e') \quad . \qquad \text{(A 258)}$$

The integration of (A 258) gives

$$b_{10} = y + (1/y) - 2 \quad , \qquad \text{(A 259)}$$

it has the series development

$$b_{10} = (1/4)\cdot x^4 - + \ldots , \quad x^2 < 1 \quad . \qquad \text{(A 259a)}$$

b_{10} implies the term x^4. Thus, the integration range does not need to surpass an upper bound of $e' = 1000$ km.

14.6.2.4.6. The integration by parts

Now, the integration by parts of the integrals for E(2.2.3.1), E(2.2.3.2), E(2.2.3.3), and E(2.2.3.4) is discussed, (A 239) (A 245) (A 250) (A 255). If the integration ranges from $p = 0^\circ$ to $p = 180^\circ$, the spherical distance p serves as the independent variable argument. If the integration procedure covers only the cap around the test point P of 1000 km radius, the length e' of the chord is the independent variable argument.

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In the course of these **different examples** of an integration by parts, now to be developed, in the first step, the integration over the values of the azimuth A is not considered. This integration is considered in the succeeding second step, later on. During the first step, it is split off.

Considering (A 194) and (A 196a), the symbolic relation (A 260) is introduced,

$$\int_A^{2\pi} dA = (A) \cdot dA \text{ ; or, } \int_A^{2\pi} \Gamma \cdot dA = (A) \; \Gamma \cdot dA. \quad \text{(A 260)}$$

The four expressions E(2.2.3.i), (with i = 1,2,3,4), are represented by four integrals. If the integration over the azimuth A is split off, the remaining integrals W(i), (i = 1,2,3,4), have the integration with regard to p or e', only. Hence, the expressions for E(2.2.3.i) can be written in the following shape, (A 260), (A 240a)(A 245)(A 250)(A 255),

$$E(2.2.3.1) = (A) \;\; W(1) \cdot dA \quad , \qquad \text{(A 261)}$$

$$E(2.2.3.2) = (A) \;\; W(2) \cdot dA \quad , \qquad \text{(A 262)}$$

$$E(2.2.3.3) = (A) \;\; W(3) \cdot dA \quad , \qquad \text{(A 263)}$$

$$E(2.2.3.4) = (A) \;\; W(4) \cdot dA \quad . \qquad \text{(A 264)}$$

The integrations in the global domain $0 \leqslant p \leqslant \pi$, or , alternately , in the domain of the cap $0 \leqslant e' \leqslant 1000$ km, are denoted symbolically by

$$(E) \; dp = \int_{p=0}^{\pi} dp; \text{ resp.,} \quad (E) \; de' = \int_{e'=0} de' \quad . \qquad \text{(A 265)}$$

Thus, the 4 functions W(i) can be brought into the following shape, (A 261) to (A 264), (A 241) (A 243), (A 246) (A 248), (A 251) (A 253), (A 256) (A 258),

$$W(1) = (E) \;\; a_7 \cdot (\partial b_7 / \partial p) \cdot dp, \qquad \text{(A 266)}$$

$$W(2) = (E) \; a_8 \cdot (\partial b_8 / \partial e') \cdot de' \quad , \qquad \text{(A 267)}$$

$$W(3) = (E) \; a_9 \cdot (\partial b_9 / \partial e') \cdot de' \quad , \qquad \text{(A 268)}$$

$$W(4) = (E) \;\; a_{10} \cdot (\partial b_{10} / \partial e') \cdot de'. \qquad \text{(A 269)}$$

The procedure of the integration by parts is governed by the following relation, it is well-known from the text-books,

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$$\int u \cdot v' \cdot dx = uv - \int v \cdot u' \cdot dx \quad . \qquad \text{(A 270)}$$

(A 266) and (A 270) give

$$W(1) = W(1.1) + W(1.2) \quad , \qquad \text{(A 271)}$$

$$W(1.1) = \left| a_7 \cdot b_7 \right|_{p=0}^{\pi} \quad , \qquad \text{(A 271a)}$$

$$W(1.2) = -(E)\ b_7 \cdot (\partial a_7 / \partial p) \cdot dp \; ; \qquad \text{(A 271b)}$$

$$W(2) = W(2.1) + W(2.2) \quad , \qquad \text{(A 272)}$$

$$W(2.1) = \left| a_8 \cdot b_8 \right|_{e'=0}^{2R'} \quad , \qquad \text{(A 272a)}$$

$$W(2.2) = -(E)\ b_8 \cdot (\partial a_8 / \partial e') \cdot de' \; ; \qquad \text{(A 272b)}$$

$$W(3) = W(3.1) + W(3.2) \quad , \qquad \text{(A 273)}$$

$$W(3.1) = \left| a_9 \cdot b_9 \right|_{e'=0}^{2R'} \quad , \qquad \text{(A 273a)}$$

$$W(3.2) = -(E)\ b_9 \cdot (\partial a_9 / \partial e') \cdot de' \; ; \qquad \text{(A 273b)}$$

$$W(4) = W(4.1) + W(4.2) \quad , \qquad \text{(A 274)}$$

$$W(4.1) = \left| a_{10} \cdot b_{10} \right|_{e'=0}^{2R'} \quad , \qquad \text{(A 274a)}$$

$$W(4.2) = -(E)\ b_{10} \cdot (\partial a_{10} / \partial e') \cdot de' \quad . \qquad \text{(A 274b)}$$

At first, W(1) is considered, (A 271).

The formula for W(1.1) contains the term $(a_7 \cdot b_7)$ for the upper bound $p = 180^o$. The cosine function (cos p/2) is equal to zero for $p = 180^o$. Thus, a_7 is equal to zero at the upper bound, also. Consequently, $(a_7 \cdot b_7)$ is equal to zero for $p = 180^o$. Hence, it follows

$$W(1.1) = -\left\{ T \cdot (\cos p/2)^2 \cdot \operatorname{arsinh} x \right\}_{p=0} \quad . \qquad \text{(A 275)}$$

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The term on the right hand side of (A 275) necessitates special deliberations, similar as (A 121) and (A 122). These deliberations are governed by three facts. At first, $\{T \cdot (\cos p/2)^2\}$ tends to the constant value $(T)_P$, the value of T at the test point P, if p tends to zero. Secondly, the function arsinh x is an odd function,

$$\text{arsinh} \quad x \quad = \quad - \quad \text{arsinh} \quad (-x) \quad . \qquad \text{(A 276)}$$

Thirdly, the expression x tends to the value of the slope of the terrain in the azimuth A, at the place of the test point P, if p tends to zero.

Thus, (A 275),

$$W(1.1) = - (T)_P \cdot \{\text{arsinh} \quad x\}_{p \to 0} \quad . \qquad \text{(A 277)}$$

And, considering (A 122),

$$W(1.1) = - (T)_P \cdot \text{arsinh} \quad (n_1 \cdot \cos A + n_2 \cdot \sin A) \quad . \qquad \text{(A 278)}$$

n_1 and n_2 are constant values. Before the background of (A 276), the following equations are important,

$$\cos (A + 180^o) = - \cos A \quad , \qquad \text{(A 278a)}$$

$$\sin (A + 180^o) = - \sin A \quad . \qquad \text{(A 278b)}$$

Consequently, regarding (A 276),

$$\text{arsinh} \ (n_1 \cdot \cos A + n_2 \cdot \sin A) =$$

$$= - \text{arsinh} \left(n_1 \cdot \cos (A + 180^o) + n_2 \cdot \sin (A + 180^o)\right) \quad . \qquad \text{(A 279)}$$

Thus, (A 278) (A 279), if c is the value of $W(1.1)$ for the azimuth A, then, $- c$ is the value of $W(1.1)$ for the azimuth $A + 180^o$. Consequently, it is obvious that the integration of the expression (A 278) over the full range of the azimuth A, ($0 \leq A \leq 360^o$), will lead to the following relation, (A 261),

$$\text{(A)} \quad W(1.1) \cdot dA \quad = 0 \quad . \qquad \text{(A 280)}$$

(A 280) is right, because the $W(1.1)$ value for the azimuth A and for the azimuth $A + 180^o$ will cancel each other.

Hence, the expression for $E(2.2.3.1)$ given by (A 261) turns to (A 281), regarding (A 271) (A 280) (A 271b) (A 244) (A 242),

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$$(A)\ W(1)\cdot dA = (A)\ W(1)_0 \cdot dA + (A)\ W(1)_{00} \cdot dA \quad , \qquad \text{(A 281)}$$

with the following two equations (A 281a)(A 281b), integrating over whole the globe,

$$W(1)_0 = -\ (E)\ (\partial T/\partial p)\cdot(\cos p/2)^2\cdot(\text{arsinh}\ \ x)\cdot dp \quad , \qquad \text{(A 281a)}$$

$$W(1)_{00} = (E)\ (1/2)\cdot T\cdot(\sin p)\cdot(\text{arsinh}\ \ x)\cdot dp \quad . \qquad \text{(A 281b)}$$

Now, W(2) is considered, (A 272).

The formula (A 272a) for W(2.1) contains the product $(a_8 \cdot b_8)$ for the argument $e' = 0$, (i. e. $p = 0$). For $p = 0$, b_8 is finite, (A 249); (star-shaped Earth). For $p = 0$, a_8 is equal to zero, since sin p/2 is equal to zero in this case, (A 246). Thus, the product $(a_8 \cdot b_8)$ is equal to zero, for $e' = 0$. For $e' = 2\cdot R'$ or for $p = 180^\circ$, b_8 is finite, (A 249). Further, for $p = 180^\circ$, a_8 is equal to zero, since cos p/2 is equal to zero in this case, (A 246). Thus, the product $(a_8 \cdot b_8)$ is equal to zero also for the upper bound $e' = 2\cdot R'$.

Consequently,

$$\Big[a_8 \cdot b_8\Big]_{e'=0} = \Big[a_8 \cdot b_8\Big]_{e'=2R'} = 0 \quad . \qquad \text{(A 282)}$$

Thus,

$$W(2.1) = 0 \quad . \qquad \text{(A 283)}$$

Hence, the relations (A 272) and (A 272b) lead to

$$(A)\ W(2)\cdot dA = (A)\ W(2)_0 \cdot dA + (A)\ W(2)_{00} \cdot dA \quad , \qquad \text{(A 284)}$$

with, (A 247) (A 249),

$$W(2)_0 = (E)\ 3\cdot(\partial T/\partial e')\cdot(\sin p/2)\cdot(\cos p/2)^2\cdot(y-1)\cdot de' \quad , \qquad \text{(A 284a)}$$

$$W(2)_{00} = (E)\ (3/2)\cdot(T/R)\cdot\left\{1 - 3\cdot(\sin p/2)^2\right\}\cdot(y-1)\cdot de' \quad . \qquad \text{(A 284b)}$$

The integration described by (A 284) covers the spherical cap defined by $e' < 1000$ km, only.

The next step is the consideration of W(3), (A 273). According to (A 273a), the expression for W(3.1) is governed by the product $a_9 \cdot b_9$. b_9 has always finite amounts, (A 254). At the lower bound, at $e' = 0$ or $p = 0$, a_9 is equal to zero; it is evidenced from (A 251), since we have the fact: $\sin p = 0$ if $p = 0$. At the upper bound, at $e' = 2\cdot R'$ or $p = 180^\circ$, the same property is found for a_9: namely $a_9 = 0$. Thus, for a star-shaped Earth, being an Earth of finite slopes of the terrain,

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$$\left[a_9 \cdot b_9\right]_{e'=0} = \left[a_9 \cdot b_9\right]_{e' = 2R'} = 0 \quad . \qquad \text{(A 285)}$$

Hence it follows, (A 273a),

$$W(3.1) = 0 \quad . \qquad \text{(A 286)}$$

Finally, the relations (A 273) and (A 273b) yield

$$\text{(A) } W(3) \cdot dA = \text{(A) } W(3)_0 \cdot dA + \text{(A) } W(3)_{00} \cdot dA, \qquad \text{(A 287)}$$

with, (A 252) (A 254),

$$W(3)_0 = - \text{(E) } (\partial T/\partial e') \cdot (1/4) \cdot (\sin p)^2 \cdot x^3 \cdot (1/y) \cdot de' \quad , \qquad \text{(A 287a)}$$

$$W(3)_{00} = - \text{(E) } (T/R) \cdot (\sin p/2) \cdot (\cos p) \cdot x^3 \cdot (1/y) \cdot de' \quad . \qquad \text{(A 287b)}$$

The terms x^3 in the expressions for $W(3)_0$ and $W(3)_{00}$ diminish rapidly for growing values of e'. For e' = 1000 km and Z = 2 km, x has the amount $2 \cdot 10^{-3}$. Thus, x^3 is not more than $8 \cdot 10^{-9}$. Consequently, it is out-of-place here to think on an integration over distances e' of more than 1000 km, in the relation (A 287).

As the last one of the W(i) values, for i = 4, the term W(4) has to be developed into a shape convenient for routine calculations, substituting the horizontal derivatives of x by the derivatives of the two-dimensional surface values T of the perturbation potential. The meaning of W(4) is explained by (A 274), (A 274a), and (A 274b). The first part in the expression for W(4) is W(4.1), (A 274). This term is defined by the product $a_{10} \cdot b_{10}$. The amount of b_{10} is always finite, for a star-shaped Earth, (A 259). At the lower bound of (A 274a), at e' = 0 or at p = 0, the amount of a_{10} is equal to zero; it is evidenced from (A 256), since:(sin p/2) = 0 if p = 0. At the upper bound, for e' = 2·R' or for p = 180°, the amount of cos p/2 is equal to zero. Hence, the relation (A 256) leads to the fact that a_{10} is equal to zero at the upper bound, also. Consequently,

$$\left[a_{10} \cdot b_{10}\right]_{e'=0} = \left[a_{10} \cdot b_{10}\right]_{e' = 2R'} = 0 \quad . \qquad \text{(A 288)}$$

The equations (A 288), (A 274a), and (A 274) yield

$$W(4.1) = 0 \quad , \qquad \text{(A 289)}$$

and

$$W(4) = W(4.2) \quad . \qquad \text{(A 289a)}$$

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Hence it follows

$$(A)\ W(4)\cdot dA = (A)\ W(4)_0 \cdot dA + (A)\ W(4)_{00} \cdot dA, \qquad (A\ 290)$$

with, (A 274b) , (A 289a), (A 259), (A 257),

$$W(4)_0 = -\ (E)\ (\partial T/\partial e')\cdot(\sin p/2)\cdot(\cos p/2)^4\cdot\left\{y + (1/y) - 2\right\}\cdot de' , \qquad (A\ 290a)$$

$$W(4)_{00} = -\ (E)\ (1/2)\cdot(T/R)\cdot\left\{(\cos p/2)^4 - (\sin p)^2\right\}\cdot\left\{y + (1/y) - 2\right\}\cdot de' . \qquad (A\ 290b)$$

14.6.2.4.7. The final formula for the calculation of E(2.2.3)

E(2.2.3) has the expression of a sum of 4 constituents, (A 237). The detailed formulas for the calculation of these individual 4 constituents can be taken from the above derivations. They are obtained in the following way.

E(2.2.3.1): By (A 261), (A 281) (A 281a) (A 281b).

E(2.2.3.2): By (A 262), (A 284) (A 284a) (A 284b).

E(2.2.3.3): By (A 263), (A 287) (A 287a) (A 287b).

E(2.2.3.4): By (A 264), (A 290) (A 290a) (A 290b).

From the above sources, the comprehensive expression for the numericsl calculation of the amount of E(2.2.3) is found. It gives this amount in terms of $\partial T/\partial p$, $\partial T/\partial e'$, and T. The topography of the Earth comes from the 4 terms b_7, b_8, b_9, b_{10}; (A 244) (A 249) (A 254) (A 259).
Hence it follows,

$$E(2.2.3) =$$

$$= (A)\ (E)\ (-\ \partial T/\partial p)\cdot(\cos p/2)^2 \cdot b_7 \cdot dp \cdot dA\ +$$

$$+\ (A)\ (E)\ (1/2)\cdot T\cdot(\sin p)\cdot b_7 \cdot dp \cdot dA\ +$$

$$+\ (A)\ (E)\ 3\cdot(\partial T/\partial e')\cdot(\sin p/2)\cdot(\cos p/2)^2\cdot b_8 \cdot de' \cdot dA\ +$$

$$+\ (A)\ (E)\ (3/2)\cdot(T/R)\cdot\left\{1 - 3\cdot(\sin p/2)^2\right\}\cdot b_8 \cdot de' \cdot dA\ +$$

$$+\ (A)\ (E)\ (-\ 1/2)\cdot(\partial T/\partial e')\cdot(\sin p)^2\cdot b_9 \cdot de' \cdot dA\ +$$

$$+\ (A)\ (E)\ (-2)\cdot(T/R)\cdot(\sin p/2)\cdot(\cos p)\cdot b_9 \cdot de' \cdot dA\ +$$

$$+\ (A)\ (E)\ (-\ \partial T/\partial e')\cdot(\sin p/2)\cdot(\cos p/2)^4\cdot b_{10} \cdot de' \cdot dA\ +$$

$$+\ (A)\ (E)\ (-1/2)\cdot(T/R)\cdot\left\{(\cos p/2)^4 - (\sin p)^2\right\}\cdot b_{10}\cdot de' \cdot dA\ . \qquad (A\ 291)$$

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As to the integrations on the right hand side of (A 291), in the first and second term, the integration has to cover whole the globe. But, from the 3. to the 8. term, the integrations can be limited to the interval $0 \leq e' \leq 1000$ km.

14.6.2.5. The final shape of the formula for the computation of E(2.2)

The relation (A 226) represents the amount of E(2.2) as the sum of three constituents. E(2.2.1) comes from (A 230). E(2.2.2) is obtained from (A 236). E(2.2.3) has the expression (A 291). It is

$$E(2.2.1) = (A)\ (E)\ (-T)\cdot(1/e')^2\cdot(\sin p/2)\cdot dw \quad , \qquad \text{(A 292)}$$

and

$$E(2.2.2) = (A)\ (E)\ (T/R)\cdot b_6\cdot de'\cdot dA. \qquad \text{(A 293)}$$

Hence, (A 226),

$$\begin{aligned} E(2.2) &= \\ &= (A)\ (E)\ (-T)\cdot(1/e')^2\cdot(\sin p/2)\cdot dw\ + \\ &+ (A)\ (E)\left\{-\partial T/\partial(Rp)\right\}\cdot(1/R)\cdot(\cos p/2)^2\cdot(1/\sin p)\cdot b_7\cdot dw\ + \\ &+ (A)\ (E)\ (T/R)\cdot\left\{1/(2\ R)\right\}\cdot b_7\cdot dw\ + \\ &+ (A)\ (E)\ (\partial T/\partial e')\cdot u_1\cdot de'\cdot dA\ + \\ &+ (A)\ (E)\ (T/R)\cdot u_2\cdot de'\cdot dA\ . \end{aligned} \qquad \text{(A 294)}$$

The abbreviations u_1 and u_2 of (A 294) have the following meaning

$$\begin{aligned} u_1 = &\ 3\cdot(\sin p/2)\cdot(\cos p/2)^2\cdot b_8\ - \\ &- (1/2)\cdot(\sin p)^2\cdot b_9\ - \\ &- (\sin p/2)\cdot(\cos p/2)^4\cdot b_{10}\ , \end{aligned} \qquad \text{(A 295)}$$

$$\begin{aligned} u_2 = &\ b_6 + (3/2)\cdot\left\{1 - 3\cdot(\sin p/2)^2\right\}\cdot b_8\ - \\ &- 2\cdot(\sin p/2)\cdot(\cos p)\cdot b_9\ - \\ &- (1/2)\cdot\left\{(\cos p/2)^4 - (\sin p)^2\right\}\cdot b_{10}\ . \end{aligned} \qquad \text{(A 296)}$$

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According to (A 294), the terms u_1 and u_2 appear in the integrations over the cap of the near surroundings of the test point P, only. Therefore, in (A 295) and (A 296), (cos p/2) and (cos p) can be replaced by the unity.

14.6.3. The formula for E(2.3)

The chapter 14.6.1. gives the expression for E(2.1), it has the shape of (A 191). The chapter 14.6.2. gives the expression for E(2.2), by (A 294). Now, in this chapter 14.6.3., the expression for E(2.3) is to be developed; the developments start from (A 110). This relation gives

$$E(2.3) = -(A)\ (E)\ T\cdot\left\{\partial(1/e')/\partial r\right\}\cdot dw \quad . \qquad \text{(A 297)}$$

Hence,

$$E(2.3) = (A)\ (E)\ T\cdot(1/e')^2\cdot(\partial e'/\partial r)\cdot dw \quad . \qquad \text{(A 298)}$$

The equation (19) of the chapter 3, (The spherical solution), yields, (Fig. 2,3, A 2, A 5),

$$\partial e'/\partial r \;=\; \sin p/2 \;=\; e'/(2\,R') \quad . \qquad \text{(A 299)}$$

(A 298) and (A 299) are combined to

$$E(2.3) = (A)\ (E)\ T\cdot(1/e')^2\cdot(\sin p/2)\cdot dw \quad . \qquad \text{(A 300)}$$

Obviously, E(2.3) is a pure spherical term, free of any impact caused by the topography, which, for instance, could be brought to bear here by the term x. Here, it is certainly true, the relation (A 300) is free of x.

14.7. The formula for E(2)

14.7.1. The expression for the computation of E(2)

E(2) is a sum of three terms, (A 108),

$$E(2) = E(2.1) + E(2.2) + E(2.3) \quad . \qquad \text{(A 301)}$$

The relations (A 191), (A 294), and (A 300) give, with (A 301),

$$E(2) =$$

$$= (A)\ (E)\left\{-\partial T/\partial(Rp)\right\}\cdot(1/R)\cdot(\cos p/2)^2\cdot(1/\sin p)\cdot b_7\cdot dw\ +$$

$$+ (A)\ (E)\ (T/R)\cdot\left\{1/(2R)\right\}\cdot(b_7 + x)\cdot dw\ +$$

$$+ (A)\ (E)\ (\partial T/\partial e')\left\{u_1 + b_1 + (e'/R)\cdot b_3\right\} de'\cdot dA\ +$$

$$+ (A)\ (E)\ (T/R)\cdot(u_2 + b_3 + b_4 + b_5)\cdot de'\cdot dA \quad . \qquad \text{(A 302)}$$

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For abbreviation, the symbols v_1, v_2 and v_3 are introduced, now. In the 2., 3., and 4. term on the right hand side of (A 302), the topography is implied by these expressions: v_1, v_2, v_3 . Hence,

$$v_1 = (1/2)\cdot(b_7 + x) \quad , \qquad (A\ 303)$$

$$v_2 = u_1 + b_1 + (e'/R)\cdot b_3 \ , \qquad (A\ 304)$$

$$v_3 = u_2 + b_3 + b_4 + b_5 \ . \qquad (A\ 305)$$

These relations are introduced in (A 302).
The final form for E(2) is found,

$$E(2) =$$

$$= (A)\ (E) \left\{ - \partial T/\partial(Rp) \right\} (1/R)\cdot(\cos p/2)^2\cdot(1/\sin p)\cdot b_7 \cdot dw \ +$$

$$+ (A)\ (E)\ (T/R)\cdot(1/R)\cdot v_1 \cdot dw \ +$$

$$+ (A)\ (E)\ (\partial T/\partial e')\cdot v_2 \cdot de' \cdot dA \ +$$

$$+ (A)\ (E)\ (T/R)\cdot v_3 \cdot de' \cdot dA. \qquad (A\ 306)$$

In the 1. und 2. term on the right hand side of (A 306), the integration has global coverage; the 3. and 4. term covers the surroundings of $e' < 1000$ km, only, in the course of the integration.

14.7.2. The terms b_1, b_2, ..., b_{10}

The individual functions b_1, b_2, ..., b_{10} ,which appear in the relations (A 295) (A 296), and from (A 303) to (A 306), have the following representations,

b_1: By (A 119), (A 124),

$$b_1 = - x\cdot(1/y) + \text{arsinh}\ x \ , \qquad (A\ 307)$$

$$b_1 = (1/3)\cdot x^3 - + \cdots \ , \quad x^2 < 1 \ . \qquad (A\ 308)$$

b_2: By (A 152c),

$$b_2 = b_1 \quad , \qquad (A\ 309)$$

$$b_2 = (1/3)\cdot x^3 - + \cdots \ , \quad x^2 < 1 \ . \qquad (A\ 310)$$

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b_3: By (A 178),

$$b_3 = 1 - (1/y) \ , \qquad \text{(A 311)}$$

$$b_3 = (1/2)\cdot x^2 \ - + \dots \ , \quad x^2 < 1 \ . \qquad \text{(A 312)}$$

b_4: By (A 190),

$$b_4 = x^3\cdot(1/y)^3\cdot \sin p/2 \ , \qquad \text{(A 313)}$$

$$b_4 = (\sin p/2)\cdot\left\{x^3 - (3/2)\cdot x^5 + - \dots\right\}, \quad x^2 < 1 \ . \qquad \text{(A 314)}$$

b_5: By (A 192d) ,

$$b_5 = (1/2)\cdot x^2\cdot(1/y) \ , \qquad \text{(A 315)}$$

$$b_5 = (1/2)\cdot x^2 - (1/4)\cdot x^4 + - \dots \ , \ x^2 < 1 \ . \qquad \text{(A 316)}$$

b_6: By (A 235),

$$b_6 = (1/y)\cdot(1/2)\cdot(y-y^2) + (1/y)\cdot x^2\cdot(\cos p/2)^2\cdot b_{6.1}, \qquad \text{(A 317)}$$

$$b_{6.1} = x\cdot\sin p/2 + (1/2)\cdot(1/y)^2 - 1 + (1/2)\cdot x^2\cdot(\cos p/2)^2\cdot(1/y)^2 \ . \qquad \text{(A 318)}$$

b_7: By (A 244) (A 244a),

$$b_7 = \operatorname{arsinh} \ x, \qquad \text{(A 319)}$$

$$b_7 = x - (1/6)\cdot x^3 + - \dots \ , \ x^2 < 1 \ . \qquad \text{(A 320)}$$

b_8: By (A 249) (A 249a),

$$b_8 = y - 1 \ , \qquad \text{(A 321)}$$

$$b_8 = (1/2)\cdot x^2 - (1/8)\cdot x^4 + - \dots \ , \ x^2 < 1 \ . \qquad \text{(A 322)}$$

b_9: By (A 254) (A 254a),

$$b_9 = (1/2)\cdot x^3\cdot(1/y) \ , \qquad \text{(A 323)}$$

$$b_9 = (1/2)\cdot x^3 + - \dots \ , \ x^2 < 1 \ . \qquad \text{(A 324)}$$

b_{10}: By (A 259) (A 259a),

$$b_{10} = y + 1/y - 2 \ , \qquad \text{(A 325)}$$

$$b_{10} = (1/4)\cdot x^4 - + \dots \ , \ x^2 < 1 \ . \qquad \text{(A 326)}$$

The term y has the relation

$$y^2 = 1 + x^2 \ . \qquad \text{(A 326a)}$$

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14.7.3. The term v_1

The expression for v_1 appears in (A 306), in an integral of global extension. By (A 303), the complete expression is, (A 319),

$$v_1 = (1/2)\cdot(x + \operatorname{arsinh} x) \quad , \qquad \text{(A 327)}$$

it has the series development

$$v_1 = x - (1/12)\cdot x^3 + - \cdots , \quad x^2 < 1 . \qquad \text{(A 327a)}$$

14.7.4. The term v_2

The full expression for v_2 is explained by (A 304). But, in (A 306), v_2 appears only in an integral which covers the cap of the near surroundings, (e' < 1000 km), of the test point P, solely. Thus, it is allowed to put here

$$(\cos p/2) \cong \cos p \cong 1, \; e' < 1000 \text{ km}, \qquad \text{(A 328)}$$

and

$$(\sin p/2)^2 \cong (\sin p)^2 \cong 0, \; e' < 1000 \text{ km}. \qquad \text{(A 329)}$$

Regarding (A 328) and (A 329), the form (A 295) for u_1 turns to, $(2(\sin p)^2 b_9 \cong x(Z/R)^2 \cong 0)$,

$$u_1 = 3\cdot(\sin p/2)\cdot b_8 - (\sin p/2)\cdot b_{10} \quad . \qquad \text{(A 330)}$$

(A 330) and (A 304) yield

$$v_2 = b_1 + (\sin p/2)\cdot\left\{2\cdot b_3 + 3b_8 - b_{10}\right\} , \quad e' < 1000 \text{ km} \quad . \qquad \text{(A 331)}$$

This is the value of v_2 which is to be applied in the near surroundings of the test point P, universally, for all amounts of x, even for steep cliffs in the near vicinity of the point P. In (A 331), x is allowed to be greater than the unity. The extensive expression for (A 331) has the following shape, (A 307) (A 311) (A 321) (A 325),

$$v_2 = - x\cdot(1/y) + \operatorname{arsinh} x + (\sin p/2)\cdot\left\{1 - (3/y) + 2\cdot y\right\} , \qquad \text{(A 332)}$$

valid for

$$e' < 1000 \text{ km} , \qquad \text{(A 332a)}$$

and for a star-shaped Earth,

$$- \infty < x < + \infty \quad . \qquad \text{(A 332b)}$$

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Now, in the consideration of v_2, (A 332), the inequality (A 332b) is ignored, but (A 332a) is still valid. The reason is the intention to specialize (A 332) for the case that the absolute amount of x has relative small values. Thus, (A 332b) is replaced by the inequality

$$x^2 \ll 1 \quad . \tag{A 333}$$

Along these lines, the series developments for b_1, b_3, b_8, and b_{10} are introduced in (A 331) and (A 332). The relation (A 332) turns to, (neglecting x^4, x^5, ...),

$$v_2 = (1/3) \cdot x^3 + (\sin p/2)\cdot(5/2)\cdot x^2 + - \cdots , \tag{A 334}$$

valid for

$$e' < 1000 \text{ km} , \qquad \text{(A 334a)},$$

and for

$$x^2 \ll 1 \quad . \tag{A 334b}$$

14.7.5. The term v_3

The term v_3 undergoes a similar treatment as v_2 . The underlying constituents are shown by (A 305). According to (A 306), and similarly as v_2, the v_3 values are needed for the argument domain $e' < 1000$ km, only. Thus, it is allowed to take over the approximations (A 328) and (A 329). These approximations are introduced into (A 296). Hence,

$$u_2 = b_6 + (3/2)\cdot b_8 - 2\cdot(\sin p/2)\cdot b_9 - (1/2)\cdot b_{10} \quad . \tag{A 335}$$

And, with (A 305),

$$v_3 = b_3 + b_4 + b_5 + b_6 + (3/2)\cdot b_8 - 2\cdot(\sin p/2)\cdot b_9 - (1/2)\cdot b_{10} \quad . \tag{A 336}$$

(A 336) is valid for

$$e' < 1000 \text{ km} \quad . \tag{A 336a}$$

With b_3 from (A 311), b_4 from (A 313), b_5 from (A 315), b_6 from (A 317), b_8 from (A 321), b_9 from (A 323), and b_{10} from (A 325), the equation (A 336) turns to, (with (A 328), (A 329)),

$$\begin{aligned} v_3 = {} & 1 - (1/y) + x^3\cdot(1/y)^3\cdot(\sin p/2) + \\ & + (1/2)\cdot x^2\cdot(1/y) + (1/y)\cdot(1/2)\cdot(y - y^2) + \\ & + (1/y)\cdot x^2\cdot\left\{x\cdot(\sin p/2) + (1/2)\cdot(1/y)^2 - 1 + (1/2)\cdot x^2\cdot(1/y)^2\right\} + \\ & + (3/2)\cdot(y - 1) - 2\cdot(\sin p/2)\cdot(1/2)\cdot x^3\cdot(1/y) - \\ & - (1/2)\cdot\left\{y + (1/y) - 2\right\} \quad . \end{aligned} \tag{A 337}$$

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Some self-explanatory rearrangements of (A 337) lead to, (for $e' < 1000$ km),

$$v_3 = (1/2) - (3/2)\cdot(1/y) + y + x^2\cdot\left\{-(1/2)\cdot(1/y) + (1/2)\cdot(1/y)^3\right\} +$$
$$+ x^3\cdot\left\{(1/y)^3\cdot(\sin p/2)\right\} + x^4\cdot\left\{(1/2)\cdot(1/y)^3\right\} + (1/2) - (1/2)\cdot y \quad . \qquad \text{(A 338)}$$

A short step leads from (A 338) to (A 339), it is the final complete shape of v_3,

$$v_3 = 1 + (1/2)\cdot y - (3/2)\cdot(1/y) + x^2\cdot(1/2)\cdot\left\{-(1/y) + (1/y)^3\right\} +$$
$$+ x^3\cdot(1/y)^3\cdot(\sin p/2) + x^4\cdot(1/2)\cdot(1/y)^3 \quad . \qquad \text{(A 339)}$$

(A 339) is valid for

$$e' < 1000 \text{ km} \qquad \text{(A 339a)}$$

and

$$-\infty < x < +\infty \, . \qquad \text{(A 339b)}$$

(A 339) is the full expression for v_3, valid for all values x of a star-shaped Earth, (A 339b). (A 339) has only one sole restriction, that is (A 339a). (A 339) is valid, however great the steepness of the cliffs in the vicinity of the surface test point P may be.

At many places of the area described by (A 339a), the absolute amount of x will be considerably smaller than the unity. This fact leads to a relief for the computations of v_3. Thus, in (A 339), the condition (A 339b) is abandoned, it is replaced by the inequality (A 333). But (A 339a) is still valid.

Obviously, along these lines, v_3 is expressed as a series development with rising powers of x. Starting from (A 336), the power series developments for b_3, b_4, b_5, b_8, b_9, and b_{10} lead to the following form for v_3, (A 312) (A 314) (A 316) (A 322) (A 324) (A 326),

$$v_3 = (1/2)\cdot x^2 + (\sin p/2)\cdot x^3 + (1/2)\cdot x^2 + b_6 +$$
$$+ (3/2)\cdot(1/2)\cdot x^2 - 2\cdot(\sin p/2)\cdot(1/2)\cdot x^3 + - \cdots \quad , \; x^2 < 1 \, . \qquad \text{(A 340)}$$

The higher powers x^4, x^5, ... are neglected in (A 340).

A simple rearrangement of (A 340) leads to

$$v_3 = b_6 + (7/4)\cdot x^2 + - \cdots \, , \; x^2 < 1 \, . \qquad \text{(A 341)}$$

For (A 328) and (A 329), the relation for b_6 given by (A 317) (A 318) turns to

$$b_6 = (1/2) - (1/2)\cdot y + (1/y)\cdot x^2\cdot\left\{x\cdot(\sin p/2) + (1/2)\cdot(1/y)^2 - 1 + (1/2)\cdot x^2\cdot(1/y)^2\right\} . \qquad \text{(A 342)}$$

Neglecting higher powers of x (as x^4, x^5, ...), (A 342) changes to (A 343), for $x^2 < 1$,

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$$b_6 = (1/2) - (1/2)y + (\sin p/2)\cdot x^3 - (1/2)\cdot x^2 + - \dots , \quad x^2 < 1 . \qquad (A\ 343)$$

(A 341) and (A 343) are combined to

$$v_3 = (1/2) - (1/2)y + (\sin p/2)\cdot x^3 + (5/4)\cdot x^2 + - \dots , \quad x^2 < 1 ; \qquad (A\ 344)$$

and with

$$y \cong 1 + (1/2)\cdot x^2 , \quad (x^2 \ll 1) ,$$

$$v_3 = (\sin p/2)\cdot x^3 + \left\{ - (1/4) + (5/4) \right\} x^2 + - \dots , \quad x^2 < 1 ,$$

and, introducing

$$\sin p/2 = e'/ (2\cdot R') \cong e'/ (2\cdot R),$$

and, regarding

$$x = Z/ e' ,$$

v_3 gets the following shape,

$$v_3 = \left\{ e'/ (2\cdot R) \right\} \left\{ Z/e' \right\} \cdot x^2 + x^2 + - \dots , \quad x^2 < 1 ;$$

or

$$v_3 = x^2 \cdot \left\{ 1 + Z/(2\cdot R) \right\} + - \dots , \quad x^2 < 1 .$$

The neglection of relative errors of the order of Z/R can be tolerated. Thus, finally,

$$v_3 = x^2 + - \dots , \quad x^2 \ll 1 ; \qquad (A\ 345)$$

(A 345) is valid for the following constraints,

$$e' < 1000 \text{ km}, \quad x^2 \ll 1 . \qquad (A\ 346)$$

14.8. The formula for E(3)

The formula for E(1) is given by (A 50) (A 51) (A 52). The formula for E(2) has the shape of (A 306). Now, the second term on the right hand side of the representation of D(2.1), given by the equation (45) of the ealier section 4, has to be transformed. It is to be brought into a shape suitable for routine computations. It is denoted by E(3), (see (45c), section 4),

$$E(3) = - (A)\ (E)\ (\partial T/\partial r)\cdot(1/e')\cdot D(1.4) . \qquad (A\ 347)$$

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The equation (39) of the earlier treated section 4 leads to the following relation,

$$du \cdot \cos(g', n) = dw + D(1.4); \qquad (A\ 348)$$

du is the surface element of the oblique surface of the Earth u,
dw is the surface element of the sphere w which does pass through the test point P, (see Fig. A 7). (g', n) is the angle of the slope of the terrain. Hence, (A 348),

$$D(1.4) = du \cdot \cos(g', n) - dw \quad . \qquad (A\ 349)$$

Further, the following denotation is introduced,

$$H' = H_P \quad , \qquad (A\ 349a)$$

H_P is the height in which the test point P does lie, above the geocentric sphere v having the radius R, (see Fig. A 2). Thus,

$$dw = (R + H')^2 \cdot (\cos\varphi) \cdot d\varphi \cdot d\lambda \ ; \qquad (A\ 350)$$

and with

$$R' = R + H' \quad , \qquad (A\ 350a)$$

follows

$$dw = (R')^2 \cdot (\cos\varphi) \cdot d\varphi \cdot d\lambda \quad . \qquad (A\ 351)$$

The formula (A 349) is transformed, now. It is rearranged in order to find such an expression for D(1.4) that develops in terms of the following three expressions:
dw, the height difference Z taken with regard to the test point P, and the radius R' of the sphere w.

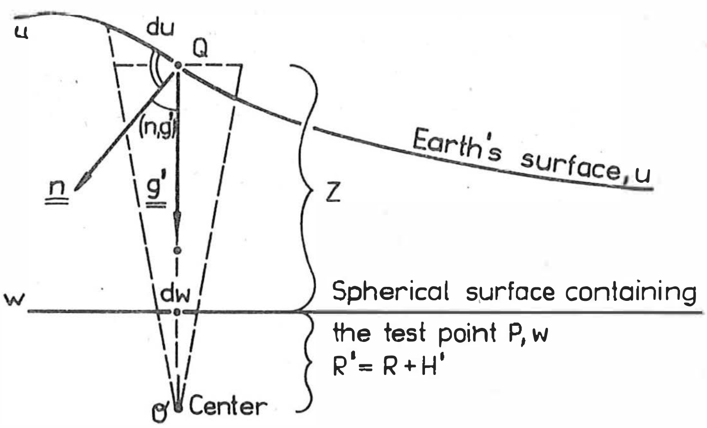

Fig. A 7.

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The interdependence between the surface element du and the surface element dw of the globe w is visualized by Fig. A 7. This interdependence is constructed by the slope of the terrain described by the inclination angle $\sphericalangle(n, g')$, further, by the geocentric radius of the sphere w, (being $R + H' = R'$), and, finally, by the radius of the point Q on the surface u, (being $R + H' + Z$). An infinitesimal cone is introduced. The vertex of this cone is identical with the gravity center σ of the Earth. This cone is introduced on the understanding that the vertex angle of it has an infinitesimal small amount, or, to be more precise, that the cone cuts out an infinitesimal small area out of the concentric unit sphere, Fig. A 7. Out of the sphere w passing through the test point P, this cone cuts out the horizontal surface element dw, (A 350) (A 351). Out of the oblique surface of the Earth u, even the same cone cuts out the surface element du, situated at the point Q. The oblique surface element du is projected into the horizontal plane which passes through the surface point Q. Out of this horizontal plane, the considered cone cuts out the surface element of the following amount,

$$\cos(g', n) \cdot (du). \tag{A 352}$$

It is learnt from Fig. A 7, the following relation connects the amount described by (A 352) and the surface element dw, it is self-explanatory,

$$\cos(g', n) \cdot (du) = dw \cdot (R + H' + Z)^2 / (R + H')^2 \ . \tag{A 353}$$

Regarding (A 350a), the relation (A 353) turns to

$$\cos(g', n) \cdot (du) = dw \cdot \left\{ (R' + Z)^2 / (R')^2 \right\} \ . \tag{A 354}$$

Hence it follows

$$(du) \cdot \cos(g', n) = (1 + Z/R')^2 \cdot dw = \left\{ 1 + 2 \cdot Z/R' + (Z/R')^2 \right\} \cdot dw \ . \tag{A 355}$$

(A 349) and (A 355) yield

$$D(1.4) = \left\{ 2 \cdot Z/R' + (Z/R')^2 \right\} \cdot dw \ . \tag{A 356}$$

For the points Q situated at the surface of the Earth u, the fundamental equation of the physical geodesy has the following shape, (see equation (A 2)),

$$\Delta g_T = -(\partial T / \partial r) - 2 \cdot T/r \ , \tag{A 357}$$

hence, regarding Fig. A 7,

$$\Delta g_T = -(\partial T / \partial r) - 2 \cdot T / (R + H' + Z) \ . \tag{A 358}$$

As to (A 358), we have, by (A 350a), the following series development

$$(1/R') \left[1 / \left\{ 1 + (Z/R') \right\} \right] = (1/R') \left\{ 1 - (Z/R') +- \cdots \right\} = (1/R') - \left\{ Z/(R')^2 \right\} +- \cdots \ . \tag{A 358a}$$

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Considering (A 350a), the second term on the right hand side of (A 358) turns to the term described by (A 359), accounting for the relation (A 358a),

$$- 2\cdot T/(R + H' + Z) = - 2\cdot T/R' + 2\cdot Z\cdot T/(R')^2 \quad . \qquad \text{(A 359)}$$

The two relations (A 358) and (A 359) are combined giving

$$- (\partial T/\partial r) = \Delta g_T + 2\cdot T/R' - 2\cdot Z\cdot T/(R')^2 \quad . \qquad \text{(A 360)}$$

The relation (A 356) for D(1.4) and the expression (A 360) for the radial derivative of the perturbation potential T are now utilized for a transformation of the expression (A 347) representing E(3). Thus,

$$E(3) = (A)\ (E) \left\{\Delta g_T + 2\cdot T/R' - 2\cdot Z\cdot T/(R')^2\right\}\cdot(1/e')\cdot\left\{2\cdot Z/R' + (Z/R')^2\right\}\cdot dw \quad . \qquad \text{(A 361)}$$

Some simple rearrangements of (A 361) lead to (A 362), neglecting powers of $(Z/R')^3 \ldots$,

$$E(3) = (A)\ (E)\ \Delta g_T\cdot(1/e')\cdot\left\{2\cdot Z/R' + (Z/R')^2\right\}\cdot dw +$$

$$+ (A)\ (E)\ (T/R')\cdot(1/e')\cdot\left\{4\cdot Z/R' - 2\cdot(Z/R')^2\right\}\cdot dw. \qquad \text{(A 362)}$$

e' is equal to

$$e' = 2\cdot(R + H')\cdot \sin p/2 = 2\cdot R'\cdot \sin p/2 \quad . \qquad \text{(A 363)}$$

(A 362) is a form of E(3) convenient for routine calculations.

14.9. The formula for E(4)

In the chapter 4, the equation (45d) represents the term E(4). It appears also as the third term on the right hand side of the relation (45) of that chapter. This term E(4) is now in the fore. The cited relations give

$$E(4) = (A)\ (E)\ T\cdot\left\{\partial(1/e')/\partial r\right\}\cdot D(1.4). \qquad \text{(A 364)}$$

The expression for D(1.4) is taken from (A 356). Further, as to the term in the braces{} of (A 364), the radial derivative of e' is considered, now. In this context, the point Q^{**} is introduced, (Fig. A 2, A 8). This point lies perpendicular below the moving surface point Q, and, moreover, on the spherical surface w. Now, the reader is asked to imagine that this point Q^{**} does move upwards, in vertical direction, by an enlargement of the radius of it from the amount R' up to the amount R' + dr.

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The impact this upwards movement exerts on the length of e' is now described by the radial derivative of e', i. e. $\partial e'/\partial r$; Fig. A 8.

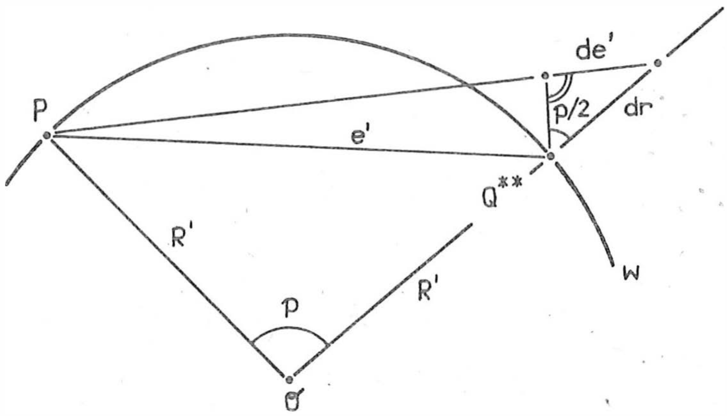

Fig. A 8.

Fig. A 8 shows how the derivation of e' with regard to r is constructed. This derivation is taken at the point Q^{**} which is situated on the sphere w.

The following lines are self-explanatory, Fig. A 8,

$$\partial(1/e')/\partial r = -(1/e')^2 \cdot (\partial e'/\partial r), \quad \text{(A 365)}$$

$$\partial e'/\partial r = \sin p/2 = e'/(2 \cdot R') , \quad \text{(A 366)}$$

hence,

$$\partial(1/e')/\partial r = -1/(2 \cdot e' \cdot R') . \quad \text{(A 367)}$$

Regarding (A 356) and (A 367), the expression (A 364) for E(4) turns to

$$E(4) = -(A)\ (E)\ T \cdot \left\{ 2 \cdot Z/R' + (Z/R')^2 \right\} \cdot \left\{ 1/(2 \cdot e' \cdot R') \right\} \cdot dw . \quad \text{(A 368)}$$

A short transformation gives finally

$$E(4) = -(A)\ (E)\ (T/R') \cdot (1/e') \cdot \left\{ Z/R' + (1/2) \cdot (Z/R)^2 \right\} \cdot dw . \quad \text{(A 369)}$$

This expression is good for numerical routine calculations.

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14.10. The formula for E(5)

Finally, considering the relation (45) representing the term D(2.1), (see chapter 4), the 5. expression on the right hand side of this equation is to be brought into a shape which suits to calculation purposes. This term is denominated by E(5), as shown by the relation (45e) of chapter 4.

Hence,

$$E(5) = (A)\ (E)\ D(1.1)\cdot D(1.2)\cdot dw \quad . \qquad (A\ 370)$$

At this occasion, a principle remark may be given. In the computation of the height anomalies ζ in terms of the gravity anomalies, the integration of the traditional Stokes integral contributes the main share, Here, the Faye-anomalies are inserted; they are defined to be the free-air anomalies supplemented by the plane topographical correction C: cf. equation (2) and (3) of the chapter 1, being the introduction into this publication. This integration calculation ranks at the first place. The formulas for E(3) and E(4) rank at the second place. They are given by (A 362) and (A 369), and they necessitate a global integration. Further, the effect the expression $C_1(M)$ exerts on the height anomaly ζ constructs a term which does rank at the second place: cf. the equation (3) of the chapter 1. These terms of the second rank will contribute to the ζ values by an amount being smaller than 1 meter, in general.

But, as to the term E(5) treated now, (A 370), it will be of the third rank. This term will have an amount which is generally much more small than the amount of E(3) and E(4), and the effect of $C_1(M)$. The reason why, in the following lines, the term E(5) is transformed into a shape convenient for numerical calculations lies also in the intention to follow up another aim, this is the intention to show that the formulas (2) and (3) of chapter 1 can be completed by very small and tiny terms: The theoretical error of the solution of the geodetic boundary value problem according to (2) and (3) can be depressed down to arbitrary small amounts. The intention to depress this theoretical error down to any arbitrary small amount has no principle limitation, it is a procedure free of any fundamental difficulty. The demonstration of this fact is one of the aims followed up by the deliberations of this chapter, (theoretical error = neglected residuum).

The definition of the term D(1.1) of (A 370) comes from the relation (36) of the 4. chapter,

$$D(1.1) = (\partial T/\partial n)\cdot(1/\cos\ (g',\ n)) + \partial T/\partial r, \qquad (A\ 371)$$

whereas the term D(1.2) is found with (37),

$$D(1.2) = 1/e \ - \ 1/e' \quad . \qquad (A\ 372)$$

e is the oblique distance between the two surface points P and Q, Fig. A 2. e' is the length of the cord between the point P and the point Q^{**} on the sphere w, Fig. A 2, A 8. Some rearrangements of (A 371) result the following relation, (A 21),

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$$D(1.1) = D_T(1.1) = \Theta \cdot g \cdot \tan(g', n) \cdot \cos(A'' - A') \quad . \tag{A 373}$$

Θ is here the full amount of the deflection of the vertical in the field of the potential T, the Θ values refer to the surface of the Earth u.

Also, the term D(1.2) given by (A 372) is transformed; the relation (A 37) yields,

$$D(1.2) = -(e')^{-3} \cdot \left\{ Z^2 + e'^2 (Z/R') \right\} \cdot \left\{ x' + (x')^{1/2} \right\}^{-1} \quad , \tag{A 374}$$

with, (see (A 39),(A 40), and (A 41)) ,

$$x' = 1 + x^2 + Z/R' \quad . \tag{A 375}$$

In (A 373), A" is the azimuth of the deflection of the vertical Θ , Fig. A 1. A' is the azimuth of the inclination of the terrain. tan (g', n) is the amount of this inclination.

Now, the plumb-line deflection Θ is decomposed into its north-south and its east-west component, i.e. t_1 and t_2,

$$t_1 = \Theta \cdot \cos A'' \quad , \tag{A 376}$$

$$t_2 = \Theta \cdot \sin A'' . \tag{A 377}$$

The text-books on physical geodesy show that t_1 derives from T by

$$t_1 = -\left[(1/g') \cdot (\partial T / \partial \bar{x}) \right]_u \quad , \tag{A 378}$$

and t_2 by

$$t_2 = -\left[(1/g') \cdot (\partial T / \partial \bar{y}) \right]_u \quad ; \tag{A 379}$$

the symbol u denotes here that the values of t_1 and t_2 are to be computed for points situated on the Earth's surface u. The horizontal arc elements $d\bar{x}$, and $d\bar{y}$ of (A 378) and (A 379) are here understood that they are plotted at the points of the Earth's surface u; hence it follows for the horizontal differentials $d\bar{x}$ and $d\bar{y}$, at the moving surface point Q on u,

$$d\bar{x} = (R' + Z) \cdot d\varphi \quad , \tag{A 379a}$$

$$d\bar{y} = (R' + Z) \cdot (\cos \varphi) \cdot d\lambda \quad . \tag{A 379b}$$

In a similar way, the component of the plumb-line deflection in the radial direction (that is the direction of a constant azimuth A plotted at the test point P) has the following relation, which is given by (A 380).

(As to the horizontal arc elements $d\bar{x}$ and $d\bar{y}$, they are found in the following way : Through the point Q at the oblique surface of the Earth u, the geocentric sphere , having the radius of $R + H_P + Z = R' + Z$, is constructed. Along this sphere , the two arc elements $d\bar{x}$ and $d\bar{y}$ are plotted even in our special point Q. Thus, in the point Q, $d\bar{x}$ and $d\bar{y}$ lie also on the tangential plane . In order to avoid misunderstandings, it may be stated : $d\bar{x}$ and $d\bar{y}$ lie not on the oblique surface u of the Earth, unless u is horizontal in the point Q !).

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$$t_p = - \left\{ (1/g') \cdot (\partial T/(R' + Z) \cdot \partial p) \right\}_u \quad . \qquad \text{(A 380)}$$

Thus, t_p is the component of the plumb-line deflection at the surface point Q taken for the direction in which only the p values do grow. p is the spherical distance between the fixed test point P and the point Q (which is moving during the integrations), Fig. A 2. Consequently, $d\bar{x}$ and $d\bar{y}$, and $(R' + Z) \cdot dp$ are horizontal arc elements plotted at the point Q situated at the surface u of the Earth. $d\bar{x}$ is heading to the north, $d\bar{y}$ points to the east, and $(R' + Z) \cdot dp$ is directed into the direction in which the p values grow (This is the direction of the tangent of the great circle through P and Q, taken at Q).

By means of t_1 and t_2, (A 378) (A 379), it is possible to construct a vector $\underline{\underline{t}}$. In this context, t_1 and t_2 are two-parametric surface functions along the surface u, $t_1 = t_1(\varphi, \lambda)$ and $t_2 = t_2(\varphi, \lambda)$. The t_1 value at the point Q is mapped into the point Q^{**}, by an identical mapping. Q^{**} lies vertical below the point Q on the surface w, Fig. A 2. Thus, after this mapping, the t_1 value of the point Q is now attached to the point Q^{**}.
The amount of t_2 undergoes a similar mapping from the point Q down to the point Q^{**}. Furthermore, on the sphere w which has the radius R', two unit vectors $\underline{\underline{e}}_1$ and $\underline{\underline{e}}_2$ are introduced. They are horizontal vectors. Consequently, they are tangential vectors with regard to the sphere w. They are plotted at the point Q^{**}. $\underline{\underline{e}}_1$ is heading to the north, $\underline{\underline{e}}_2$ is heading to the east. By means of the values t_1 and t_2 at the point Q^{**}, it is possible to construct a vector $\underline{\underline{t}}$ which is situated on the sphere w, as a tangential vector. Hence it follows,

$$\underline{\underline{t}} = t_1 \cdot \underline{\underline{e}}_1 + t_2 \cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 381)}$$

Here is,

$$\underline{\underline{e}}_1^{\,2} = 1, \qquad \underline{\underline{e}}_2^{\,2} = 1 \quad . \qquad \text{(A 382)}$$

Considering (A 376) (A 377) (A 381), and introducing , by the symbol t, the length of the vector $\underline{\underline{t}}$, the following relation is obtained,

$$\underline{\underline{t}}^2 = t^2 = \Theta^2 = t_1^{\,2} + t_2^{\,2} \quad . \qquad \text{(A 383)}$$

Here, the expressions for t, t_1, t_2, and t_p are functions of φ and λ. They can be understood as functions distributed along the sphere w. Hence,

$$t = t(\varphi, \lambda) \quad , \qquad \text{(A 383a)}$$

$$t_1 = t_1(\varphi, \lambda) \quad , \qquad \text{(A 383b)}$$

$$t_2 = t_2(\varphi, \lambda) \quad , \qquad \text{(A 383c)}$$

$$t_p = t_p(\varphi, \lambda) \quad . \qquad \text{(A 383d)}$$

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In a similar way as the vector $\underline{\underline{t}}$ can be decomposed into a north-south and an east-west component, (A 376) (A 377), the slope of the terrain tan (g', n) can be decomposed also into a north-south and an east-west component, s_1 and s_2, Fig. A 1. The following relations can be constructed, regarding the fact that the angle A' is the azimuth of the slope of the terrain in the point Q,

$$s_1 = \tan (g',n) \cdot \cos A' \quad , \qquad \text{(A 384)}$$

$$s_2 = \tan (g', n) \cdot \sin A' . \qquad \text{(A 385)}$$

Or, describing s_1 and s_2 by the horizontal derivatives of the height difference Z, ($Z = H_Q - H_P$; in these derivations, H_P is constant and H_Q is variable),

$$s_1 = - (1/(R' + Z)) \cdot (\partial Z/\partial \varphi) \quad , \qquad \text{(A 386)}$$

$$s_2 = - (1/(R' + Z)) \cdot (\partial Z/(\cos \varphi) \partial \lambda) \quad . \qquad \text{(A 387)}$$

For the derivatives of Z, given by (A 386) and (A 387), the following relations are valid,

$$(\partial Z/\partial \varphi) = \partial(H_Q - H_P) /\partial \varphi = \partial H_Q/ \partial \varphi \quad . \qquad \text{(A 387a)}$$

$$(\partial Z/\partial \lambda) = \partial(H_Q - H_P)/\partial \lambda = \partial H_Q/\partial \lambda \quad . \qquad \text{(A 387b)}$$

In most cases, in the relations (A 386) and (A 387), a relative error of the order of Z/R' can be tolerated in the amounts of s_1 and s_2 . The question is here a factor of about 1/1000 or 1/10 000. With these simplifications, (A 386) and (A 387) change to

$$s_1 = - (./R') \cdot (\partial Z/\partial \varphi) \quad , \qquad \text{(A 387c)}$$

$$s_2 = - (1/(R' \cdot \cos \varphi)) \cdot (\partial Z/\partial \lambda) \quad . \qquad \text{(A 387d)}$$

In a similar way, as the functions t_1 and t_2 did lead to the vector $\underline{\underline{t}}$, (A 381), it is possible to construct a vector $\underline{\underline{s}}$, by means of the functions s_1 and s_2 , (A 384) (A 385).
Hence,

$$\underline{\underline{s}} = s_1 \cdot \underline{\underline{e}}_1 + s_2 \cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 388)}$$

The operator of the gradient of a scalar field distributed along the sphere w is now introduced,

$$\nabla = \text{grad} = + (1/R') \cdot \left\{\frac{\partial}{\partial \varphi}\right\} \cdot \underline{\underline{e}}_1 + (1/(R' \cdot \cos \varphi)) \cdot \left\{\frac{\partial}{\partial \lambda}\right\} \cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 388a)}$$

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This gradient operator is applied to the scalar field of the H_Q values. In this context, the H_Q values are understood that they are distributed along the sphere w, having the radius R'. Consequently, the operator of (A 388a) leads to

$$\text{grad } H_Q = +(1/R')\cdot(\partial H_Q/\partial\varphi)\cdot \underline{\underline{e}}_1 + (1/(R'\cdot\cos\varphi))\cdot(\partial H_Q/\partial\lambda)\cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 388b)}$$

In this context, H_P is a constant value . Thus, considering the scalar function $Z = H_Q - H_P$, the derivatives of Z with regard to the latitude and longitude are eqaul to the derivatives of H_Q with regard to these arguments, consequently. Along these lines, the relation (A 388b) can be transformed into the following shape,

$$\text{grad } Z = +(1/R')\cdot(\partial Z/\partial\varphi)\cdot \underline{\underline{e}}_1 + (1/(R'\cdot\cos\varphi))\cdot(\partial Z/\partial\lambda)\cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 388c)}$$

A comparison of (A 388c) with (A 387c), (A387d), and with (A 388) shows that the vector $\underline{\underline{s}}$ can be represented by the gradient of the Z field,

$$\underline{\underline{s}} = -\text{ grad } Z \quad . \qquad \text{(A 389)}$$

s is the length of the vector $\underline{\underline{s}}$,

$$\underline{\underline{s}}^2 = s^2 \quad . \qquad \text{(A 389a)}$$

Regarding (A 384) (A 385) (A 388) (A 389a), the following equation is found

$$\underline{\underline{s}}^2 = s^2 = \{\tan(g', n)\}^2 = s_1^{\,2} + s_2^{\,2} \quad . \qquad \text{(A 390)}$$

Before the background of the above vector developments, the expression (A 373) for $D_T(1.1)$ can be brought into the form of a scalar product or of an inner product of two vectors.
In this context, (A 373) is rearranged, as follows

$$D_T(1.1) = \Theta\cdot g\cdot\tan(g',n)\{\cos A'' \cos A' + \sin A'' \sin A'\} \quad . \qquad \text{(A 390a)}$$

Regarding (A 376) (A 377), and in view of (A 384) (A 385), the above expression for $D_T(1.1)$ turns to

$$D_T(1.1) = g\,(t_1\cdot s_1 + t_2\cdot s_2) \quad . \qquad \text{(A 391)}$$

g is here the real gravity intensity for the real potential W, taken at the surface u of the Earth. The braces on the right hand side of (A 391) contain the scalar product of the two vectors $\underline{\underline{t}}$ and $\underline{\underline{s}}$, (A 381) (A 388). Hence it follows

$$D_T(1.1) = g\cdot\underline{\underline{t}}\cdot\underline{\underline{s}} \quad . \qquad \text{(A 392)}$$

Now, after the rearrangement of $D_T(1.1)$, the expression (A 374) for D(1.2) is transformed; this transformation happens by the introduction of the quotient

$$x = Z/e' \qquad \text{(A 393)}$$

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which was already of service before now. (A 393) and (A 374) are combined to

$$D(1.2) = -(1/e')\cdot(x^2 + e'\cdot x/R')\cdot\left[x' + (x')^{1/2}\right]^{-1} . \qquad (A\ 394)$$

For the product of the term in the second braces of (A 394), on the one hand, and of the term in the brackets of (A 394), on the other hand, a sign of abbreviation is introduced, now,

$$x^{*}(P,Q) = (x^2 + e'\cdot x/R')\cdot\left[x' + (x')^{1/2}\right]^{-1} , \qquad (A\ 395)$$

or,

$$x^{*}(P,Q) = (x^2 + Z/R')\cdot\left[x' + (x')^{1/2}\right]^{-1} . \qquad (A\ 395a)$$

Consequently, (A 394) changes to

$$D(1.2) = -(1/e')\cdot x^{*}(P,Q). \qquad (A\ 396)$$

This is the final expression for D(1.2).

In view of (A 392) and (A 396), the development (A 370) for E(5) transforms into

$$E(5) = -g\ (A)\ (E)\ \underline{\underline{t}}\cdot\underline{\underline{s}}\cdot(1/e')\cdot x^{*}(P, Q)\cdot dw . \qquad (A\ 397)$$

$\underline{\underline{t}}$ and $\underline{\underline{s}}$ are the above defined vectors, (A 381) (A 388). $x^{*}(P, Q)$ is a scalar function, it is evidenced from (A 395a); in our applications, this function is understood that it varies with the moving point Q, only, in the course of one integration. Within such an integration, the test point P is fixed. The vector $\underline{\underline{t}}$ and the function $x^{*}(P, Q)$ are combined yielding the vector $\underline{\underline{k}}$,

$$\underline{\underline{k}} = x^{*}(P, Q)\cdot\underline{\underline{t}} . \qquad (A\ 398)$$

The equations (A 397) and (A 398) lead to

$$E(5) = -g\ (A)\ (E)\ \underline{\underline{k}}\cdot\underline{\underline{s}}\cdot(1/e')\cdot dw . \qquad (A\ 399)$$

Regarding (A 389), E(5) takes the following shape

$$E(5) = g\ (A)\ (E)\ \underline{\underline{k}}\cdot(\text{grad } Z)\cdot(1/e')\cdot dw . \qquad (A\ 399a)$$

This above expression for E(5) offers the possibility for essential rearrangements. They have the aim to avoid the horizontal derivatives of the topographical heights which are implied in the term (grad Z). In the course of these rearrangements, (grad Z) comes to be replaced by Z, and, further, instead of $\underline{\underline{k}}$, the horizontal derivatives of the components of the vector $\underline{\underline{k}}$ appear. The horizontal derivatives of $\underline{\underline{k}}$ are much more smoothed than the corresponding amounts of Z. Even this fact is the essential reason for the coming rearrangements of E(5).

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Following up this aim of these rearrangements, a new vector $\underline{\underline{a}}$ is introduced by

$$\underline{\underline{a}} = (Z/e') \cdot \underline{\underline{k}} \quad . \qquad \text{(A 400)}$$

As to the 3 symbols on the right hand side of the equation above, the scalar functions Z, e', and the two components of the vector $\underline{\underline{k}}$ have values which are understood (in the now discussed rearrangements of E(5)) that they are distributed along the surface w of the sphere with the radius R'. They are functions of the two variable co-ordinates of the surface point Q, at least in the here discussed problem. The co-ordinates of the point P are constant. Z has finite values, as so as the components of the vector $\underline{\underline{k}}$. In (A 398), the components of $\underline{\underline{t}}$ are always finite, since the components of the plumb-line deflection are finite, always; and $x^*(P,Q)$ is also always finite, (A 395a), it tends to the unity if x^2 tends to infinity, a property easily verified before the background of (A 203) (A 206) for x', (see also (A 414) and (A 415)) ,

Now, a short excursion into the field of vector analysis is to be undertaken. Along the sphere w, a general continuous scalar function q, having continuous first derivatives, is introduced,

$$q = q(\varphi, \lambda) \quad . \qquad \text{(A 400a)}$$

φ and λ are the geocentric latitude and longitude. The gradient of the function q has the following shape, (A 388a),

$$\text{grad } q = (1/R') \cdot (\partial q/\partial \varphi) \cdot \underline{\underline{e}}_1 + (1/(R' \cdot \cos\varphi)) \cdot (\partial q/\partial \lambda) \cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 401)}$$

Along the sphere w, it is possible to introduce the two arc elements $d\overline{\overline{x}}$ and $d\overline{\overline{y}}$, being defined by

$$d\overline{\overline{x}} = R' \cdot d\varphi \quad , \qquad d\overline{\overline{y}} = (R' \cdot \cos\varphi) \cdot d\lambda \quad . \qquad \text{(A 401a)}$$

With (A 401a), the expression of (A 401) turns to

$$\text{grad } q = (\partial q/\partial \overline{\overline{x}}) \cdot \underline{\underline{e}}_1 + (\partial q/\partial \overline{\overline{y}}) \cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 402)}$$

The meaning of $\underline{\underline{e}}_1$ and $\underline{\underline{e}}_2$ was already explained, some lines before the equations (A 381), (A 382). Furthermore, besides of the function q, a tangential vector of the sphere w is introduced. It is denoted by $\underline{\underline{q}}$,

$$\underline{\underline{q}} = q_1 \cdot \underline{\underline{e}}_1 + q_2 \cdot \underline{\underline{e}}_2 \quad ; \qquad \text{(A 402a)}$$

q_1 and q_2 are continuous functions of φ and λ , they have continuous first derivatives,

$$q_1 = q_1(\varphi, \lambda) \quad , \qquad \text{(A 402b)}$$

$$q_2 = q_2(\varphi, \lambda) \quad . \qquad \text{(A 402c)}$$

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The scalar product of the gradient vector, (according to (A 388a)), with the vector $\underline{\underline{q}}$ gives the divergence of the vector field $\underline{\underline{q}}$,

$$\text{div } \underline{\underline{q}} = \nabla \cdot \underline{\underline{q}} = \text{grad} \cdot \underline{\underline{q}} \quad . \qquad \text{(A 402d)}$$

The divergence of a vector field is a scalar function. Thus,

$$\text{div } \underline{\underline{q}} = \frac{1}{R'} \cdot \frac{\partial q_1}{\partial \varphi} + \frac{1}{R' \cdot \cos\varphi} \cdot \frac{\partial q_2}{\partial \lambda} - \frac{\tan\varphi}{R'} \cdot q_1 \quad . \qquad \text{(A 403)}$$

After this excursion into the field of the vector analysis, demonstrated with the help of the function q and the vector $\underline{\underline{q}}$, we return now back to the vector field $\underline{\underline{a}}$, (A 400). The divergence of the vector field $\underline{\underline{a}}$ is obtained by (A 400) and (A 403), hence

$$\text{div } \underline{\underline{a}} = \text{div} \left[(Z/e') \cdot \underline{\underline{k}} \right] \quad , \qquad \text{(A 403a)}$$

and further,

$$\text{div } \underline{\underline{a}} = \nabla \cdot \left[(Z/e') \cdot \underline{\underline{k}} \right] = \nabla \cdot \underline{\underline{a}} =$$

$$= (\nabla \cdot Z) \cdot (1/e') \cdot \underline{\underline{k}} + Z \left[\nabla \cdot (1/e') \right] \underline{\underline{k}} + (Z/e') \cdot \left[\nabla \cdot \underline{\underline{k}} \right] \quad . \qquad \text{(A 404)}$$

Now, the singularity of the function 1/e' has to be considered. In case, the length e' tends to zero, the function 1/e' tends to infinity. But, in (A 404), the function (1/e') can be tolerated only as long as it is a continuous function . In order to avoid this discrepancy, the function div $\underline{\underline{a}}$ is not treated for whole the surface w of the sphere with the radius R', (A 404). Around the test point P, an area w" which does surround this point P is separated from the surface w ; w has global extension. The remaining part of w is w'. Thus,

$$w = w' + w'' \quad . \qquad \text{(A 404a)}$$

As long as div $\underline{\underline{a}}$ according to (A 404) is discussed for the partial area w' only, any singularity of the function 1/e' does not exist, since the distance between the point P and the margin of the area w" has never to be equal to zero, - this is a necessary constraint.

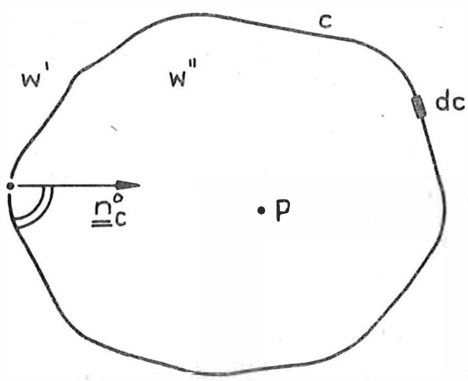

Fig. A 9.

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From Fig. A 9, the reader learns that the boundary-line between w' and w" is denominated by c, dc is the arc element. $\underline{\underline{n}}_c^o$ is the unit normal vector of the line c, $\underline{\underline{n}}_c^o$ is simultaneously a tangential vector along the sphere w. $\underline{\underline{n}}_c^o$ is heading into the exterior of the domain w', and, thus, into the interior of w".

Obviously, it is allowed to apply the integral theorem of Gauss to the vector field $\underline{\underline{a}}$. Here, this theorem is specialized on the area w' and its boundary c. Hence it follows

$$\iint\limits_{w'} (\text{div}\ \underline{\underline{a}}) \cdot dw = \int\limits_{c} (\underline{\underline{a}} \cdot \underline{\underline{n}}_c^o) \cdot dc \quad . \qquad \text{(A 405)}$$

Here, w' is a part of the surface w, and c is the boundary-line of w'.

Usually, in the text-books, the Gaussian theorem is described for a three-dimensional space and its boundary-surface. The transition from the three-dimenional case to the two-dimensional case of (A 405) is easily done by considering the fact that the vector $\underline{\underline{a}}$ has two horizontal components, only, further, that $\underline{\underline{a}}$ does not depend on the distance r to the center of the Earth, and, finally, that $\underline{\underline{a}}$ has no component in the radial direction. These special properties transform the problem from the three-dimensional case to the two-dimensional one, (A 405).

The validity of the integral theorem of Gauss for the two -dimensional vector field $\underline{\underline{a}}$, (see (A 405)), is easily proved along the following lines. Just to take an example, one arbitrary infinitesimal mesh is singled out from the co-ordinate grid covering the area w'. This mesh is constructed by lines of Gauss' co-ordinates φ = const. and λ = const., spread out over the area w'. Thus, the boundary-lines of this mesh are lines of constant latitude, on the one hand, and lines of constant longitude, on the other hand. The situation is shown by Fig. A 10. The area of this mesh is equal to dw; the side lengthes of it are equal to $R' \cdot d\varphi$, $R' \cdot (\cos\varphi)_3 \cdot d\lambda$, and $R' \cdot (\cos\varphi)_4 \cdot d\lambda$. If (A 405) is applied to this infinitesimal mesh (instead of the domain w') and to the vector field $\underline{\underline{q}}$, described by (A 402a) (A 402b) (A 402c), (instead of the vector field $\underline{\underline{a}}$), the relation (A 405) turns to

$$(\text{div}\ \underline{\underline{q}}) \cdot dw = \sum_{i=1}^{4} \left[(\underline{\underline{q}} \cdot \underline{\underline{n}}_c^o) \cdot dc \right]_i \quad . \qquad \text{(A 405a)}$$

Here, in equation (A 405a), dw denotes again the surface element of the spherical surface w. And, dc is again the arc element of the boundary-line c which separates the two partial areas w' and w" of the spherical surface w.

The smaller the amount of dw, the better valid the equation (A 405a). In (A 405a), the summation over the suffix i, (i = 1,2,3,4), means the summation over the four sides of the infinitesimal trapezoid represented by Fig. A 10. For these 4 sides, the concerned values of $\underline{\underline{q}}$, $\underline{\underline{n}}_c^o$, and dc have to be quoted. Thus, these 4 values are as follows,

$$(\underline{\underline{q}})_i \ , \ (\underline{\underline{n}}_c^o)_i \ , \ (dc)_i \quad ; \quad (i = 1,2,3,4) \quad .$$

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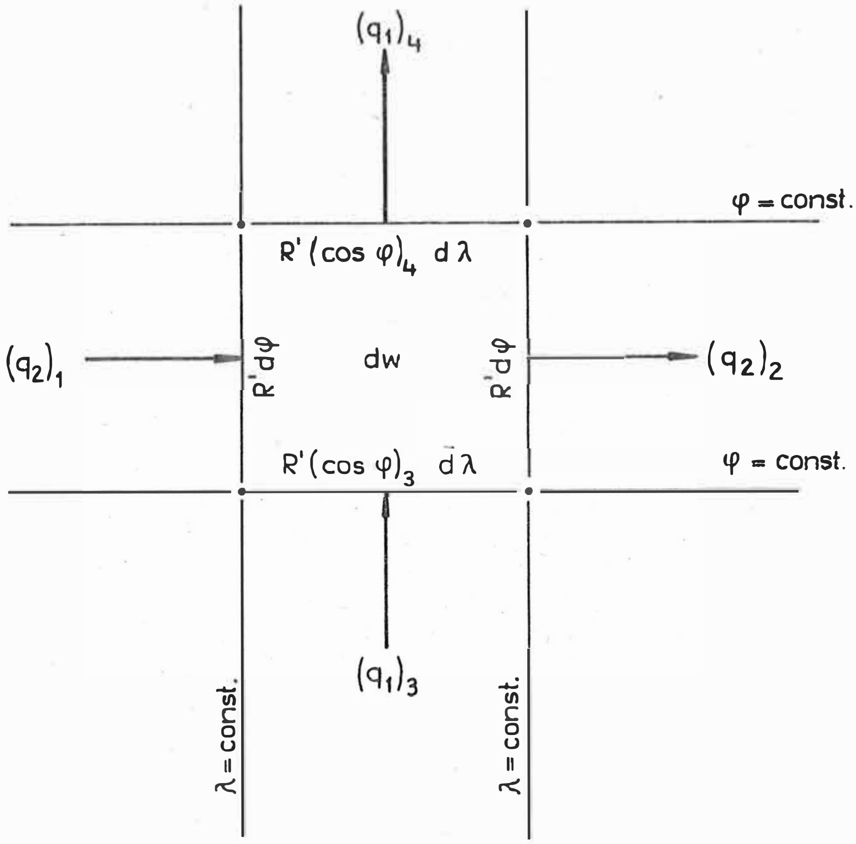

Fig. A 10.

q_1 is the component of the vector $\underline{\underline{q}}$ in the north-south direction,
q_2 is the component in the east-west direction.

Now, the validity of (A 405a) is easily proved by the developments of (A 405b).

The summation on the right hand side of (A 405a) refers to the four sides of the mesh, represented by Fig. A 10. The sum on the right hand side of (A 405a) develops in the following way, it follows from a look on Fig. A 10.

$$\sum_{i=1}^{4} \left[\left(\underline{\underline{q}} \cdot \underline{\underline{n}}_c^o \right) \cdot dc \right]_i =$$

$$= - (q_2)_1 \cdot R' \cdot d\varphi + (q_2)_2 \cdot R' \cdot d\varphi - (q_1)_3 \cdot R' \cdot (\cos\varphi)_3 \cdot d\lambda +$$

$$+ (q_1)_4 \cdot R' \cdot (\cos\varphi)_4 \cdot d\lambda =$$

$$= \left[(q_2)_2 - (q_2)_1 \right] R' \cdot d\varphi + \left[(q_1)_4 - (q_1)_3 \right] R' \cdot (\cos\varphi)_3 \cdot d\lambda +$$

$$+ (q_1)_4 \cdot R' \cdot d\lambda \cdot \left[(\cos\varphi)_4 - (\cos\varphi)_3 \right] =$$

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$= (\partial q_2/\partial\lambda)\cdot d\lambda \cdot R' \cdot d\varphi + (\partial q_1/\partial\varphi)\cdot d\varphi \cdot R' \cdot (\cos\varphi)\cdot d\lambda +$

$+ (q_1)_4 \cdot R' \cdot d\lambda \cdot (-\sin\varphi)\cdot d\varphi =$

$= \Big[(1/R')\cdot(\partial q_1/\partial\varphi) + (1/(R'\cdot\cos\varphi))\cdot(\partial q_2/\partial\lambda) -$

$- (\tan\varphi)\cdot(1/R')\cdot q_1\Big]\cdot R'^2 \cdot (\cos\varphi)\cdot d\varphi \cdot d\lambda =$

$$= (\mathrm{div}\ \underline{\underline{q}})\cdot dw \quad . \qquad \text{(A 405b)}$$

The developments given by the above lines are self-explanatory. They prove ,by (A 403), the validity of (A 405a). The integration over the whole of the infinitesimal meshes of the domain w' leads from (A 405a) to (A 405). Thus, the validity of (A 405) is corroborated.

Now, we return back to the relations (A 404) and (A 405), and to the specialities connected with the division of the surface w into two parts, w' and w", Fig. A 9. For the subsequent mathematical deliberations, the close surroundings w" around the test point P get the form of a small spherical cap with the spherical radius $R'\cdot\vartheta$. This cap is concentric to the test point P, and it is situated on the sphere w. Thus, the figure A 9 changes to the figure A 11.

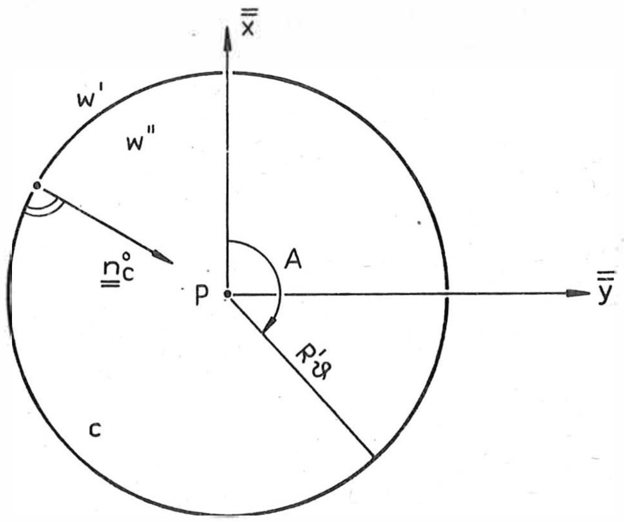

Fig. A 11.

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In Fig. A 11, the symbol A is again the azimuth measured clockwise from the north. The line $\bar{\bar{x}}$ leads to the north, the line $\bar{\bar{y}}$ to the east. The vertex of the azimuth A is the center point P of the cap w".

Consequently, if (A 404a) is considered, the relation (A 406) follows,

$$\iint_{w'} (\text{div}\ \underline{\underline{a}}) \cdot dw = \iint_{w - w''} (\text{div}\ \underline{\underline{a}}) \cdot dw \quad . \qquad \text{(A 406)}$$

With (A 406), (A 405) turns to

$$\iint_{w-w''} (\text{div}\ \underline{\underline{a}}) \cdot dw = \int_{c} \left(\underline{\underline{a}} \cdot \underline{\underline{n}}_c^{o}\right) \cdot dc \quad . \qquad \text{(A 407)}$$

In (A 407), we refer to the special situation shown by Fig. A 11.

In case, the radius ϑ of the cap w" tends to zero, the area of w" tends to zero simultaneously. Here, the radius was measured by the geocentric angle ϑ which belongs to w", Fig. A 11. Now, the specialities are to be considered which set in during the transition to an infinitesimal small area for w". This transition procedure comes about if $R' \cdot \vartheta$ tends to zero,

$$R' \cdot \vartheta \rightarrow 0 \quad . \qquad \text{(A 408)}$$

The integral on the left hand side of (A 407) covers the area $w' = w - w''$. The coverage of the area w" needs a special consideration, since the integrand contains the inverse of e'. In case of (A 408), this inverse does tend to infinity. Hence, it is necessary to show that the integral

$$K = \iint_{w''} (\text{div}\ \underline{\underline{a}}) \cdot dw \qquad \text{(A 409)}$$

tends to zero, if the transition (A 408) takes place. For a sufficient small value of $R' \cdot \vartheta$, the surface element dw has the approximative plane-geometry-representation (the precise shape of dw is: $(R')^2 \cdot (\sin p) \cdot dp \cdot dA$) by (A 410)

$$dw = e' \cdot de' \cdot dA \quad + \quad l_1\ (e') \quad , \qquad \text{(A 410)}$$

$l_1(e')$ symbolizes a relative error of the order or $(e'/R')^2$ in the value of dw. $l_1(e')$ is a function depending on e'.($\sin p = (e'/R') - (1/8) \cdot (e'/R')^3 + - \ldots$).

(A 404) is introduced into (A 409). In doing so, K divides into three constituents,

$$K = K_1 + K_2 + K_3 \quad . \qquad \text{(A 410a)}$$

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They have the following expressions, regarding (A 410) (A 404), (neglecting the term $l_1(e')$, i.e. relative errors of the order of $(e'/R')^2$ in the integrands),

$$K_1 = \iint_{w''} (\nabla \cdot Z) \cdot \underline{\underline{k}} \cdot de' \cdot dA \quad , \qquad \text{(A 411)}$$

$$K_2 = \iint_{w''} Z \cdot \left[\nabla \cdot (1/e') \right] \underline{\underline{k}} \cdot e' \cdot de' \cdot dA \quad , \qquad \text{(A 412)}$$

$$K_3 = \iint_{w''} Z \cdot \left[\nabla \cdot \underline{\underline{k}} \right] \cdot de' \cdot dA \quad . \qquad \text{(A 413)}$$

The surface of the Earth was presupposed to be that of a star-shaped Earth, the slopes of the terrain have never infinite amounts. Thus, Z, x, and $\nabla \cdot Z$ have always finite amounts. If (A 408) is applied, Z tends to zero.

The length of $\underline{\underline{k}}$ is viewed by (A 398): The length of the vector $\underline{\underline{t}}$ is always finite, because the plumb-line deflection θ has finite amounts, always, (A 383), (A 376) (A 377), and, because, moreover, $x^*(P,Q)$ is a function of finite values, too. The latter fact is evidenced by (A 395a). Regarding (A 206), the relation (A 395a) yields

$$x^*(P,Q) \cong (x^2 + Z/R') \cdot \left[1 + x^2 + (1 + x^2)^{1/2} \right]^{-1} \quad . \qquad \text{(A 414)}$$

In case, the topographical heights tend to zero, the x values tend to zero simultaneously (for finite values of e'). Consequently, (A 414) tends to zero, in this case. And, furthermore, in the adverse case, if the x^2 values tend to infinity, the amount of (A 414) tends to the unity. Thus, obviously,

$$0 \leqslant |x^*| < 1 \quad . \qquad \text{(A 415)}$$

Hence, the length of the vector $\underline{\underline{k}}$ is finite.

Furthermore, the amount of the scalar value $\nabla \cdot \underline{\underline{k}}$, being equal to div $\underline{\underline{k}}$, has to be discussed, since this amount appears in (A 413). In this context, the question is in the fore whether div $\underline{\underline{k}}$ has finite values. Regarding the relations (A 398) (A 378) (A 379) (A 414), and substituting the vector $\underline{\underline{q}}$ in (A 403) by the vector $\underline{\underline{k}}$, the following relation is obtained,

$$\text{div } \underline{\underline{k}} = \nabla \cdot \underline{\underline{k}} =$$
$$= (1/R') \left\{ \partial(x^* \cdot t_1)/\partial\varphi \right\} + (1/(R' \cdot \cos\varphi)) \left\{ \partial(x^* \cdot t_2)/\partial\lambda \right\} - (x^* \cdot t_1) \cdot (\tan\varphi) \cdot (1/R') \quad . \qquad \text{(A 416)}$$

As it is evidenced by (A 414), the function $x^*(P,Q)$ is a continuous function with continuous first derivatives, since x is a continuous function of Z, and since Z is a continuous function with continuous first derivatives, depending on the latitude and longitude.

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t_1 and t_2 are the components of the plumb-line deflection. It is well-known that these functions are continuous with continuous first and higher derivatives. Thus, the values of $x^* \cdot t_1$, the values of the derivative of $(x^* \cdot t_1)$ with regard to the latitude, and the values of the derivative of $(x^* \cdot t_2)$ with regard to the longitude, (which appear in (A 416)), all these three values have finite amounts. Consequently, it can be taken for granted that the amount of (div $\underline{\underline{k}}$) in (A 413) has finite amounts. In case of $\varphi = 90^o$, the right hand side of (A 416) has, (tan $\varphi \rightarrow \infty$), a removable singularity. It can be removed by the choice of another convenient pole for co-ordinate system. The operator (div $\underline{\underline{k}}$) depends not on the choice of the co-ordinate system.

As to the here discussed properties of the integrands appearing in (A 411), (A 412), and (A 413), finally, the amount of the scalar

$$\left[\text{grad } (1/e')\right] \cdot \underline{\underline{k}} \cdot e' \qquad \text{(A 417)}$$

appearing in (A 412) is to be considered, and that in case of the transition described by (A 408).
Obviously, the gradient vector of $1/e'$ has the following shape, (A 402),

$$\text{grad } (1/e') = \left[\partial(1/e')/\partial\bar{\bar{x}}\right] \cdot \underline{\underline{e}}_1 + \left[\partial(1/e')/\partial\bar{\bar{y}}\right] \cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 418)}$$

Here is

$$\left[\partial(1/e')/\partial\bar{\bar{x}}\right] \cdot \underline{\underline{e}}_1 = -(1/e')^2 \cdot (\partial e'/\partial\bar{\bar{x}}) \cdot \underline{\underline{e}}_1 \quad . \qquad \text{(A 419)}$$

The expression (A 419) is understood that it is taken for a point in the near surroundings of the test point P. The values of (A 419) cover the area of w", Fig. A 11. The differential quotient $\partial e'/\partial\bar{\bar{x}}$ can be interpreted as the cosinus of the angle α between the directions of de' and $d\bar{\bar{x}}$. Thus, for $e' \leq R' \cdot \vartheta$,

$$\partial e'/\partial\bar{\bar{x}} = \cos\alpha \quad , \qquad \text{(A 419a)}$$

$$\partial e'/\partial\bar{\bar{y}} = \sin\alpha \quad . \qquad \text{(A 419b)}$$

Hence, (A 418) turns to

$$\text{grad } (1/e') = -(1/e')^2 \cdot \left[(\cos\alpha) \cdot \underline{\underline{e}}_1 + (\sin\alpha) \cdot \underline{\underline{e}}_2\right] \quad . \qquad \text{(A 420)}$$

In case of approaching the point P, the value of $\cos\alpha$ tends to $\cos A$, and $\sin\alpha$ tends to $\sin A$. A is here the azimuth along which the approach to P happens, (See Fig. A 11) .

Returning back to (A 417), the vector $\underline{\underline{k}} = \overset{*}{x} \cdot \underline{\underline{t}}$ has to be considered also, (A 398). The following relation is obtained referring to (A 381) and (A 398) ,

$$\underline{\underline{k}} = \overset{*}{x} \cdot t_1 \cdot \underline{\underline{e}}_1 + \overset{*}{x} \cdot t_2 \cdot \underline{\underline{e}}_2 \quad . \qquad \text{(A 421)}$$

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Regarding (A 417), the product of (A 420) and (A 421) needs to be considered, now. This product is multiplied with Z and with the length e'. Hence it follows, for the values within the area w",

$$Z \cdot \left[\text{grad } (1/e') \right] \cdot \underline{\underline{k}} \;\; e' = - \overset{*}{x} \cdot x \cdot \left[t_1 \cdot \cos \alpha + t_2 \cdot \sin \alpha \right] . \qquad \text{(A 422)}$$

After these investigations about the integrands of K_1, K_2, K_3, conducted from (A 414) to (A 422), it is possible to estimate the amount of (A 411), (A 412), and (A 413), for the special case that the area of w" tends to zero, or, that the transition (A 408) is carried out.

At first, the integral for K_1 is considered. Because the two vectors (grad Z) and $\underline{\underline{k}}$ have limited lengthes, as proved in the lines above, the scalar or inner product of these two vectors has a limited scalar amount, too. This fact follows from the Schwarz inequality, which has the following form in the here discussed problem,

$$\left| (\nabla Z) \cdot \underline{\underline{k}} \right| \leqslant \left| \nabla Z \right| \cdot \left| \underline{\underline{k}} \right| . \qquad \text{(A 423)}$$

Since the two factors on the right hand side of (A 423) have finite amounts, the left hand side of this inequality yields a finite amount , also. If k_1 is the upper bound of the amount of $\left| (\text{grad } Z) \cdot \underline{\underline{k}} \right|$, obtained within the area w", the relation (A 411) gives for the absolute amount of K_1

$$\left| K_1 \right| \leqslant 2 \cdot \pi \cdot k_1 \cdot R' \cdot \vartheta \;; \qquad \text{(A 423a)}$$

$$k_1 = \text{fin sup} \left| ((\text{grad } Z) \cdot \underline{\underline{k}}) \right| . \qquad \text{(A 423b)}$$

The smaller the value of ϑ, the more precise the relation (A 423a). If ϑ tends to zero, (A 408), $\left| K_1 \right|$ tends to zero, too, because $2 \cdot \pi \cdot k_1 \cdot R'$ has an upper bound. Thus,

$$K_1 \to 0, \text{ if (A 408) is valid.} \qquad \text{(A 423c)}$$

At the second place, the integral for K_2 comes into the fore, (A 412). The relations (A 412) and (A 422) yield

$$K_2 = - \iint\limits_{w''} \overset{*}{x} \cdot x \cdot \left[t_1 \cdot \cos \alpha + t_2 \cdot \sin \alpha \right] \cdot de' \cdot dA , \qquad \text{(A 424)}$$

As it was found above, the terms $\overset{*}{x}$, x, t_1, t_2, cos α , and sin α which appear in the integrand of K_2 have finite amounts. Consequently, the absolute amount of the integrand of (A 424) has an upper bound, k_2. Hence it follows

$$k_2 = \text{fin sup} \left| \overset{*}{x} \cdot x \cdot \left[t_1 \cdot \cos \alpha + t_2 \cdot \sin \alpha \right] \right| . \qquad \text{(A 424a)}$$

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The relation (A 424a) is inserted into (A 424); the transition behaviour described by (A 408) is regarded. The inequality (A 424b) is the consequence

$$\left| K_2 \right| \leqslant 2 \cdot \pi \cdot k_2 \cdot R' \cdot \vartheta \quad \text{(A 424b)}$$

If ϑ tends to zero, the absolute amount of K_2 tends to zero, too. This behaviour follows from (A 424a) and (A 424b). Thus, the following relation is obtained,

$$K_2 \rightarrow 0 \text{ , if (A 408) is valid.} \quad \text{(A 424c)}$$

At the third place, the integral for K_3 is evaluated, for an area w'' which tends to zero, (A 413). Within the area w'', Z was proved to be a finite value. In case of the transition procedure (A 408), the amount of Z tends to zero. Further, in the lines which follow the relation (A 416), it was shown that div $\underline{\underline{k}}$ has always finite amounts. Thus, if k_3 is the upper bound of the absolute amount of the integrand of (A 413),

$$k_3 = \text{fin sup} \left| Z \cdot (\text{div } \underline{\underline{k}}) \right| , \quad \text{(A 424d)}$$

the relation (A 413) leads to

$$\left| K_3 \right| \leqslant 2 \cdot \pi \cdot k_3 \cdot R' \cdot \vartheta . \quad \text{(A 424e)}$$

Hence it follows

$$\left| K_3 \right| \rightarrow 0, \text{ if (A 408) is valid.} \quad \text{(A 424f)}$$

Regarding (A 423c), (A 424c), and (A 424f), the relation (A 410a) gives

$$K \rightarrow 0 \text{ , if (A 408) is valid.} \quad \text{(A 425)}$$

Hence, with (A 407) and (A 409),

$$\iint\limits_{w-w''} (\text{div } \underline{\underline{a}}) \cdot dw \rightarrow \iint\limits_{w} (\text{div } \underline{\underline{a}}) \cdot dw \quad , \quad \text{(A 425a)}$$

if (A 408) is valid.

Returning back to the equation (A 407), the relation (A 425a) describes the transition behaviour of the left hand side of (A 407), for a vanishing area of w''.
Now, the transition behaviour of the right hand side of (A 407) is in the fore,

$$K' = \int\limits_{c} (\underline{\underline{a}} \cdot \underline{\underline{n}}_c^{\circ}) \cdot dc. \quad \text{(A 425b)}$$

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The vector $\underline{\underline{a}}$ in the above integrand comes from (A 400) and (A 398),

$$\underline{\underline{a}} = x \cdot \underline{\underline{k}} = x \cdot x^{*} \cdot \underline{\underline{t}} \quad . \qquad \text{(A 425c)}$$

The above investigations, (A 414) (A 415), did show that the 4 terms x, x^{*}, and $\underline{\underline{t}}$, and $\underline{\underline{k}}$ have finite amounts. Thus, the length of the vector $\underline{\underline{a}}$, along the spherical circle c, has always finite amounts. The Schwarz inequality gives

$$\left|(\underline{\underline{a}} \cdot \underline{\underline{n}}_c^o)\right| \leqslant \left|\underline{\underline{a}}\right| \cdot \left|\underline{\underline{n}}_c^o\right| = \left|\underline{\underline{a}}\right| \,, \qquad \text{(A 425d)}$$

the vector $\underline{\underline{n}}_c^o$ was introduced as a unit vector. The inequality (A 425d) shows that the absolute amount of $(\underline{\underline{a}} \cdot \underline{\underline{n}}_c^o)$ has an upper bound, because $|\underline{\underline{a}}|$ has an upper bound, since we consider a star-shaped Earth with finite amounts of the slopes. k' denotes this upper bound,

$$k' = \text{fin sup} \left| \underline{\underline{a}} \cdot \underline{\underline{n}}_c^o \right| \quad . \qquad \text{(A 425c)}$$

(A 425e) is introduced into (A 425b). Hence, it follows

$$|K'| \leqslant \int_c k' \cdot dc = k' \int_c dc \quad . \qquad \text{(A 425f)}$$

The relation (A 425f) yields

$$|K'| \leqslant k' \int_{A=0}^{2\pi} (R' \vartheta) \cdot dA = 2 \cdot \pi \cdot k' \cdot R' \cdot \vartheta \,. \qquad \text{(A 425g)}$$

Thus,

$$K' \rightarrow 0, \text{ if (A 408) is valid.} \qquad \text{(A 425h)}$$

Finally, regarding (A 425) and (A 425h), the Gauss' integral relation (A 407) turns to

$$\iint_w (\text{div } \underline{\underline{a}}) \cdot dw = 0, \qquad \text{(A 425i)}$$

if the radius of the area w" tends to zero. In (A 425i), (A 425a) was used, also.

Now, we return back to (A 404). The relation (A 404) develops the expression (div $\underline{\underline{a}}$) into 3 terms. Thus, the introduction of (A 404) into (A 425i) yields

$$0 = (A)\,(E)\,(\text{div } \underline{\underline{a}}) \cdot dw = B_1 + B_2 + B_3 \quad . \qquad \text{(A 426)}$$

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For B_1 follows

$$B_1 = (A)\ (E)\ (\text{grad } Z) \cdot (1/e') \cdot \underline{\underline{k}} \cdot dw \quad ; \tag{A 427}$$

accounting for (A 389) and (A 399a), the relation (A 427) turns to

$$E(5) = g \cdot B_1 \quad . \tag{A 428}$$

E(5) is the term for which an expression convenient for routine calculations is to be found.

B_2 has the following expression ,

$$B_2 = (A)\ (E)\ Z \cdot x^* \cdot (\text{grad } (1/e')) \cdot \underline{\underline{t}} \cdot dw \quad . \tag{A 429}$$

In (A 429), the vector grad (1/e') is a tangential vector of even those great circles of the sphere w which are plotted through the point P. grad (1/e') is here the gradient vector of the field of the (1/e') values taken along the sphere w, (A 401). If e_3 is this unit tangential vector heading into the direction of growing p values, it follows

$$\text{grad } (1/e') = -(1/e')^2 \cdot (1/R') \cdot (\partial e'/\partial p) \cdot \underline{\underline{e}}_3 \quad . \tag{A 430}$$

The component of the vector $\underline{\underline{t}}$ in the direction of the above defined great circles is, (A 380),

$$t_p \cdot \underline{\underline{e}}_3 \quad . \tag{A 431}$$

With

$$e' = 2 \cdot R' \cdot \sin p/2 \quad , \tag{A 432}$$

the following derivative is obtained

$$(1/R') \cdot (\partial e'/\partial p) = \cos\ p/2 \quad . \tag{A 433}$$

Regarding (A 380) (A 430) (A 433), the scalar product in the expression (A 429) takes the following shape

$$\left[\text{grad } (1/e')\right] \cdot \underline{\underline{t}} = (1/R) \cdot (1/g') \cdot (1/e')^2 \cdot (\cos\ p/2) \cdot (\partial T/\partial p) \quad . \tag{A 434}$$

Inserting (A 380) in (A 434), relative errors of the order of Z/R are neglected, here. A comparison of (A 429) and (A 434) gives

$$B_2 = (A)\ (E)\ Z \cdot x^* \cdot (\cos\ p/2) \cdot (1/R) \cdot (1/g') \cdot (1/e')^2 \cdot (\partial T/\partial p) \cdot dw \quad . \tag{A 435}$$

In (A 435), a simple rearrangement is now undertaken. Considering

$$x = Z/e' \quad , \tag{A 436}$$

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and accounting for (A 432), we find

$$Z\cdot(1/e')^2 = x\cdot(1/(2\cdot R'))\cdot(1/(\sin p/2)) \quad ; \qquad (A\ 437)$$

hence it follows

$$Z\cdot x^{*}\cdot(1/e')^2 = x\cdot x^{*}\cdot(1/(2\cdot R'))\cdot(1/(\sin p/2)) \quad . \qquad (A\ 438)$$

The symbol b_{11} serves as an abbreviation , (A 395),

$$b_{11} = x\cdot x^{*} = x(x^2 + Z/R')\cdot\left[x' + (x')^{1/2}\right]^{-1} \quad . \qquad (A\ 439)$$

Thus, B_2 has the following final shape convenient for routine calculations, (with $R \cong R'$),

$$B_2 = (A)\ (E)\ \left[\partial T/(R\cdot\partial p)\right]\cdot(1/g')\cdot(1/(2R))\cdot(\cos p/2)\cdot\left\{1/(\sin p/2)\right\}\cdot b_{11}\cdot dw \quad . \qquad (A\ 440)$$

The term B_3 of (A 426) has the following expression, introducing the third term on the right hand side of (A 404) in (A 426),

$$B_3 = (A)\ (E)\ (Z/e')\cdot(\mathrm{div}\ \underline{\underline{k}})\cdot dw \quad . \qquad (A\ 441)$$

The relation (A 398) gives

$$\mathrm{div}\ \underline{\underline{k}} = \mathrm{div}\ (x^{*}\cdot\underline{\underline{t}}) \quad , \qquad (A\ 442)$$

where the vector $\underline{\underline{k}}$ is divided into 2 components, k_1 and k_2,

$$\underline{\underline{k}} = \begin{Bmatrix} x^{*}\cdot t_1 \\ \\ x^{*}\cdot t_2 \end{Bmatrix} = \begin{Bmatrix} k_1 \\ \\ k_2 \end{Bmatrix} \quad . \qquad (A\ 443)$$

In the numerical calculations, the vector $\underline{\underline{k}}$ appears in form of its components k_1 and k_2, the numerical values of which can be treated, if wanted, in the computations. Consequently, in the following investigations, the divergence operator for the vector $\underline{\underline{k}}$ is now replaced by an operator for the components k_1 and k_2, adapting the symbolic expression of the divergence to the specialities of the numerical applications. Hence, regarding (A 403), and with $R \cong R'$,

$$\mathrm{div}\ \underline{\underline{k}} = \nabla\cdot\underline{\underline{k}} = \Phi(k_1, k_2) =$$

$$= (1/R)\cdot(\partial k_1/\partial\varphi) + (1/(R\cdot\cos\varphi))\cdot(\partial k_2/\partial\lambda) - (1/R)\cdot(\tan\varphi)\cdot k_1 \quad . \qquad (A\ 444)$$

A comparison of (A 441) and (A 444) leads to the following expression for B_3

$$B_3 = (A)\ (E)\ Z\cdot\Phi(k_1, k_2)\cdot(1/e')\cdot dw \quad . \qquad (A\ 445)$$

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It is the aim of this chapter to find an expression for E(5) which is convenient for routine calculations, (A 370). This aim is reached by (A 426) (A 428) (A 440) and (A 445). Hence, regarding, in addition to (A 446), the formulas for B_2 and B_3,

$$E(5) = - g \cdot B_2 - g \cdot B_3 \quad . \tag{A 446}$$

$$E(5) = - (A)\ (E)\ g \cdot Z \cdot \Phi(k_1, k_2) \cdot (1/e') \cdot dw -$$

$$- (A)\ (E) \left[\partial T/(R\, \partial p) \right] \cdot (1/2R) \cdot (\cos p/2) \cdot \left[1/(\sin p/2) \right] \cdot b_{11} \cdot dw \quad . \tag{A 447}$$

With (A 443), we find (A 448)

$$\Phi(k_1, k_2) = \Phi(x^* \cdot t_1, \quad x^* \cdot t_2) \quad . \tag{A 448}$$

Usually, in the geodetic text-books, t_1 is denominated by ξ, and t_2 by η; thus, putting

$$t_1 = \xi \quad , \qquad t_2 = \eta \quad , \tag{A 449}$$

the relation (A 450) follows

$$\Phi(k_1, k_2) = \Phi(x^* \cdot \xi, \quad x^* \cdot \eta) =$$

$$= (1/R) \left[\partial(x^* \cdot \xi)/\partial\varphi \right] + (1/(R \cdot \cos\varphi)) \cdot \left[\partial(x^* \cdot \eta)/\partial\lambda \right] - (1/R) \cdot (\tan\varphi) \cdot x^* \cdot \xi \quad . \tag{A 450}$$

The amount of x^* diminishes rapidly for growing distances from the test point P, since x^* is quadratic in x. Consequently, the amount of the operator

$$\Phi(x^* \cdot \xi, \quad x^* \cdot \eta)$$

diminishes also rapidly if the distances from the point P are growing. Therefore, in the first term on the right hand side of (A 447), the integration can be limited to the near surroundings of the test point P. Further, in the second term on the right hand side of (A 447), the following rearrangements can be carried out, (A 439),

$$b_{11} \cdot dw = x \cdot x^* \cdot dw = Z \cdot x^* \cdot (1/e') \cdot dw \quad . \tag{A 451}$$

Here, in (A 451), the factor x^* appears, also. Thus, it can be taken for granted, that the integrand of the second term on right hand side of (A 447) diminishes rapidly, too. Consequently, the integration of this term can be limited to the near surroundings of the test point P, too. Therefore, in (A 451), the plane surface element $e' \cdot de' \cdot dA$ can be substituted for dw; hence

$$b_{11} \cdot dw \cong Z \cdot x^* \cdot de' \cdot dA \quad , \text{ for } e' < 1000 \text{ km} \quad . \tag{A 452}$$

Along these lines, accounting for (A 452), (A 447) can be transformed into

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$$E(5) = -\ (A)\ (E)\ g\cdot Z\cdot \Phi(x^{*}\cdot\xi,\ \ x^{*}\cdot\eta)\cdot de'\cdot dA\ -$$

$$-\ (A)\ (E)\ (\partial T/\partial e')\cdot(1/2)\cdot(Z/R)\cdot(\cos p/2)\cdot\left\{1/(\sin p/2)\right\} x^{*}\cdot de'\cdot dA\ . \qquad (A\ 453)$$

Here is, putting approximately $R' \cong R$,

$$x^{*} = (x^{2}\ + Z/R)\cdot\left[\,x'\ + (x')^{1/2}\right]^{-1}\ . \qquad (A\ 454)$$

14.11. The formulae for D(2.1)

14.11.1. The universal formula for D(2.1)

Now, we return back to the expression for D(2.1), which is described by the equations (45) and (45f) of the section 4. Hence it follows, (45f), on page 23,

$$D(2.1) = E(1) + E(2) + E(3) + E(4) + E(5)\ . \qquad (A\ 455)$$

The detailed developments for the 5 terms on the right hand side of (A 455) can be found at the following places of this publication;

E(1): (A 50) (A 51) (A 52),

E(2): (A 306),

E(3): (A 362),

E(4): (A 369),

E(5): (A 453).

In order to have these formulae easy to survey, they are here put together.

$$E(1) = (A)\ (E)\ \Delta g\cdot(-x^{2})\cdot(y + y^{2})^{-1}\cdot de'\cdot dA\ +$$

$$+\ (A)\ (E)\ (T/R)\cdot(-2\cdot x^{2})\cdot(y + y^{2})^{-1}\cdot de'\cdot dA\ +$$

$$+\ (A)\ (E)\ \Delta g\cdot(-\ Z/R)\cdot(y + y^{2})^{-1}\cdot (e')^{-1}\cdot dw\ +$$

$$+\ (A)\ (E)\ (T/R)\cdot(-\ 2\cdot Z/R)\cdot(y + y^{2})^{-1}\cdot (e')^{-1}\cdot dw\ . \qquad (A\ 456)$$

$$E(2) = (A)\ (E)\ (T/R)\cdot v_{3}\cdot de'\cdot dA\ +$$

$$+\ (A)\ (E)\ (\partial T/\partial e')\cdot v_{2}\cdot de'\cdot dA\ +$$

$$+\ (A)\ (E)\ (T/R)\cdot(1/R)\cdot v_{1}\cdot dw\ +$$

$$+\ (A)\ (E)\ (\partial T/\partial p)\cdot(-\ 1/R^{2})\cdot(\cos p/2)^{2}\cdot(1/\sin p)\cdot b_{7}\cdot dw\ . \qquad (A\ 457)$$

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$$E(3) = (A)\ (E)\ \Delta g \cdot (2 \cdot Z/R) \cdot (1/e') \cdot dw\ +$$

$$+\ (A)\ (E)\ (T/R) \cdot (4 \cdot Z/R) \cdot (1/e') \cdot dw\ . \qquad (A\ 458)$$

$$E(4) = (A)\ (E)\ (T/R) \cdot (-Z/R) \cdot (1/e') \cdot dw\ . \qquad (A\ 459)$$

$$E(5) = (A)\ (E)\ (\partial T/\partial e') \cdot (-1/2R) \cdot (\cos p/2) \cdot (1/(\sin p/2)) \cdot b_{11} \cdot e' \cdot de' \cdot dA\ +$$

$$+\ (A)\ (E)\ (-g \cdot Z) \cdot \Phi(x^{*} \cdot \xi\ ,\ x^{*} \cdot \eta) \cdot de' \cdot dA\ . \qquad (A\ 460)$$

In the above formulae for E(1), E(2), E(3), E(4), E(5) , relative errors of the order of Z/R are neglected.

In view of the numerical applications, a regrouping of the right hand side of (A 455) is now carried out. It is recommended to group the development for the term D(2.1), (A 455), according to certain aspects which originate from the facts appearing in the numerical calculations. Following up this aim, terms with similar integrands are assigned into the same new group. Making this new classification on the right hand side of (A 455), the following new 7 groups E(a), E(b), E(c), E(d), E(e), E(f), and E(g) appear in the expression for the term D(2.1),

$$D(2.1) = E(a) + E(b) + E(c) + E(d) + E(e) + E(f) + E(g). \qquad (A\ 461)$$

These new groups have the following shape,

$$E(a) = (A)\ (E)\ \Delta g \cdot (-Z/R) \cdot (y + y^{2})^{-1} \cdot (e')^{-1} \cdot dw + (A)\ (E)\ \Delta g \cdot (2Z/R) \cdot (1/e') \cdot dw\ . \qquad (A\ 462)$$

$$E(b) = (A)\ (E)\ (T/R) \cdot (-2Z/R) \cdot (y + y^{2})^{-1} \cdot (1/e') \cdot dw + (A)(E)\ (T/R) \cdot (1/R) \cdot v_{1} \cdot dw\ +$$

$$+\ (A)\ (E)\ (T/R) \cdot (4 \cdot Z/R) \cdot (1/e') \cdot dw + (A)\ (E)\ (T/R) \cdot (-Z/R) \cdot (1/e') \cdot dw\ . \qquad (A\ 463)$$

$$E(c) = (A)\ (E)\ (\partial T/\partial p) \cdot (-1/R^{2}) \cdot (\cos p/2)^{2} \cdot (1/\sin p) \cdot b_{7} \cdot dw\ . \qquad (A\ 464)$$

$$E(d) = (A)\ (E)\ \Delta g \cdot (-x^{2}) \cdot (y + y^{2})^{-1} \cdot de' \cdot dA\ . \qquad (A\ 465)$$

$$E(e) = (A)\ (E)\ (T/R) \cdot (-2x^{2}) \cdot (y + y^{2})^{-1} \cdot de' \cdot dA + (A)\ (E)\ (T/R) \cdot v_{3} \cdot de' \cdot dA\ . \qquad (A\ 466)$$

$$E(f) = (A)\ (E)\ (\partial T/\partial e') \cdot v_{2} \cdot de' \cdot dA + (A)\ (E)\ (\partial T/\partial e') \cdot (-b_{11}) \cdot de' \cdot dA\ . \qquad (A\ 467)$$

$$E(g) = (A)\ (E)\ (-g \cdot Z) \cdot \Phi(x^{*} \cdot \xi,\ x^{*} \cdot \eta) \cdot de' \cdot dA\ . \qquad (A\ 468)$$

In the second term on the right hand side of (A 467), the approximately valid relations

$$\cos p/2 \cong 1\ ,\ \text{for}\ e' < 1000\ \text{km}\ , \qquad (A\ 469)$$

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and

$$e' = 2\cdot R' \cdot \sin p/2 \quad \text{, or, approximately,} \quad e' \cong 2 \cdot R \cdot (\sin p/2), \qquad \text{(A 470)}$$

are made use of. In (A 470), we have used the approximation $R \cong R'$.

The following rearrangements of the relations from (A 462) up to (A 468) are self-explanatory.

$$E(a) = (A)\ (E)\ \Delta g \cdot (Z/R)\cdot\left[2 - (y + y^2)^{-1}\right]\cdot(1/e')\cdot dw \quad . \qquad \text{(A 471)}$$

$$E(b) = (A)\ (E)\ (T/R)\cdot(Z/R)\cdot\left[3 - 2\cdot(y + y^2)^{-1}\right]\cdot(1/e')\cdot dw \ +$$

$$+ (A)\ (E)\ (T/R)\cdot(1/R)\cdot v_1 \cdot dw \quad . \qquad \text{(A 472)}$$

E(c) : (see (A 464)). (A 473)

E(d) : (see (A 465)). (A 474)

$$E(e) = (A)\ (E)\ (T/R)\cdot\left[v_3 - 2x^2\cdot(y + y^2)^{-1}\right]\cdot de' \cdot dA \quad . \qquad \text{(A 475)}$$

$$E(f) = (A)\ (E)\ (\partial T/\partial e')\cdot\left[v_2 - b_{11}\right]\cdot de' \cdot dA \quad . \qquad \text{(A 476)}$$

E(g) :(see (A 468)). (A 477)

The meaning of the here appearing terms v_1, v_2, v_3 can be found by (A 327), (A 332), (A 339). The meaning of the term b_7, appearing in E(c), is found by (A 319). The meaning of b_{11} is as follows, (A 439),

$$b_{11} = x\cdot x^{*} = x\cdot(x^2 + Z/R')\cdot\left[x' + (x')^{1/2}\right]^{-1}. \qquad \text{(A 478)}$$

The meaning of $\Phi(x^{*}\cdot\xi\ ,\ x^{*}\cdot\eta\ )$ is explained by (A 450). The meaning of x, y, x', x", e' is as follows, (A 27), (A 39), (A 31), (A 40), (A 70a),

$$x = Z/e' \quad , \qquad \text{(A 479)}$$

$$y^2 = 1 + x^2 \quad , \qquad \text{(A 480)}$$

$$x' = y^2 + Z/R' \quad , \qquad \text{(A 481)}$$

$$x'' = x \cdot \cos p/2 \quad , \qquad \text{(A 482)}$$

$$e' = 2\cdot R' \cdot \sin p/2 \quad . \qquad \text{(A 483)}$$

The integrals for E(a), E(b), and E(c) have the surface element dw at the integrand. These integrations have to cover whole the globe. But, the integrals for E(d), E(e), E(f), and E(g) have the product of the two differentials $de' \cdot dA$ under the integration symbol. Thus, these latter 4 integrations cover only the cap of $e' < 1000$ km around the test point P.

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In the development (A 461) for D(2.1), it is recommendable to draw a clear distinction between the integrals of global and those of regional coverage. Therefore, the relation (A 461) is written in the following form,

$$D(2.1) = F_1 + F_2 \quad , \qquad \text{(A 484)}$$

with

$$F_1 = E(a) + E(b) + E(c), \qquad \text{(A 485)}$$

and

$$F_2 = E(d) + E(e) + E(f) + E(g) \ . \qquad \text{(A 486)}$$

The term F_1 comprises the integrals of global integration; the term F_2 encloses the terms of regional coverage, only, (i. e. for $e' < 1000$ km).

Finally, it is to be stated that the relation (A 484) is the universally valid representation of D(2.1); may the test point P be situated in the lowlands or in the high mountains, the relation (A 484) meets all requirements. In (A 484), F_1 comes from (A 485) and F_2 from (A 486). In (A 485): E(a), E(b), and E(c) come from (A 471), (A 472), and (A 473). In (A 486): E(d), E(e), E(f), and E(g) come from (A 474), (A 475), (A 476), and (A 477).

Consequently, (A 484) is the fundamental form representing D(2.1). It is of universal efficiency.

14.11.2. The lowland formula for D(2.1)

Sure, mostly, in the different cases of the geodetic applications, the universal formula (A 484) for D(2.1) is not fully exhausted, by far not. The potentiality of the expression (A 484) is fully exploited only, if the test point P is situated in high mountains, and if, simultaneously, the height anomalies to be determined have to have centimeter precision, - a very rare case. In most cases, the test points P, for which the height anomalies ζ are to be determined, are situated in the lowlands, or in hilly areas with small terrain inclination, or on the oceans. In these special situations now in the fore for the place of the test point P, the amount of the term x^2 is very small. Consequently, in this case, $x^2 << 1$, many parts of the formulae from (A 471) up to (A 477) are so small that it is allowed to neglect them, in the lowlands.

Hence, in order to save work, the universal expression, (A 484), is now simplified for the case the test point P is situated in the lowlands, exclusing high mountain test points.

The mathematical formulation of the constraint that the test point is situated in the lowlands is given by the inequality

$$x^2 << 1 \quad . \qquad \text{(A 487)}$$

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(A 487) is the definition of the condition that a lowland test point is under consideration, to speak with other words.

Since the terms of F_2 are quadratic in the argument x, (A 486), (see E(d), E(e), E(f), E(g)), the amount of F_2 will always be very small, if the inequality (A 487) is right. Thus,

$$F_2 \cong 0 \quad , \text{ if } \quad (A\ 487) \text{ is right.} \qquad (A\ 488)$$

Furthermore, considering the three expressions E(a), E(b), and E(c) on the right hand side of (A 485), the developments (A 471), (A 472), and (A 473) for these three expressions will simplify enormously applying (A 487).

At first, the term in the brackets of (A 471) is simplified by the application of (A 487). In case, the amount of x^2 is very small, (A 487), the relation (A 480) leads to the following approximately valid equations

$$y = |y| \cong 1, \quad y^2 \cong 1; \text{ if (A 487) is valid.} \qquad (A\ 488a)$$

Thus,

$$2 - (y + y^2)^{-1} \cong 3/2 \quad ; \quad \text{if} \quad x^2 \ll 1 \, . \qquad (A\ 489)$$

Further on, the relations (A 488a) turn the brackets of (A 472) to

$$3 - 2\cdot(y + y^2)^{-1} \cong 2 \quad ; \quad \text{if} \quad x^2 \ll 1 \, . \qquad (A\ 490)$$

In the second term on the right hand side of (A 472), the expression v_1 is implied. For small values of x, the relation (A 327a) leads to

$$v_1 \cong x = Z/e', \text{ if (A 487) is valid.} \qquad (A\ 491)$$

The relations (A 490) and (A 491) are introduced into (A 472); hence it follows

$$(Z/R)\cdot\left[3 - 2(y + y^2)^{-1}\right]\cdot(1/e') + (1/R)\cdot v_1 \cong (Z/R)\cdot(3/e') \quad ; \text{if } x^2 \ll 1. \qquad (A\ 492)$$

Finally, the function E(c) given by (A 473) is adapted to (A 487). For small values of x^2, the term b_7 gets the following shape, (A 319) (A 320),

$$b_7 \cong x = Z/e', \text{ if (A 487) is valid.} \qquad (A\ 493)$$

Returning back to the relation (A 485) representing F_1, and following up the adaptation of it to the lowland conditions, the relations (A 489), (A 492), and (A 493) are introduced into the equations (A 471), (A 472), and (A 473) for E(a), E(b), and E(c). The sum of these three values

$$E(a) + E(b) + E(c) \qquad (A\ 494)$$

computed observing the lowland condition (A 487) is denominated by $F_{1.1}$ or by $[\,]_1$.

$$F_{1.1} = \left[E(a) + E(b) + E(c)\right]_1 \quad . \qquad (A\ 495)$$

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Along these lines, the combination of (A 489), (A 492), and (A 493) with the expressions on the right hand side of (A 485) leads to the following shape of $F_{1.1}$, (A 497); - here the relation (A 496) was made use of,

$$(-1/R)\cdot(\cos p/2)^2\cdot(1/\sin p)\cdot(1/e') = (-1/4R^2)\cdot(\cos p/2)\cdot(1/(\sin p/2))^2. \quad \text{(A 496)}$$

$$F_{1.1} = (A)\ (E)\ \Delta g\cdot(Z/R)\cdot(3/2)\cdot(1/e')\cdot dw + (A)\ (E)\ (T/R)\cdot(Z/R)\cdot(3/e')\cdot dw +$$

$$+ (A)\ (E)\ (\partial T/\partial p)\cdot(Z/R)\cdot\left(-1/(4R^2)\right)\cdot(\cos p/2)\cdot(1/(\sin p/2))^2\cdot dw \quad . \quad \text{(A 497)}$$

This simplified formula (A 497) for $F_{1.1}$ is **right** if the lowland condition (A 487) is valid. This simple formula (A 497) representing $F_{1.1}$ is a convenient substitute for the extensive formula for F_1 as long as our geodetic applications do without test points situated in the high mountains.

Returning back to the expression for D(2.1), the universal formula (A 484) gets the simplified shape if the lowland condition (A 487) is taken into regard. Thus, accounting for (A 488), and with the transition behaviour of (A 498)

$$F_1 \rightarrow F_{1.1} \text{ , (if (A 487) is valid),} \quad \text{(A 498)}$$

the universal case (A 484) turns to the lowland version (A 499),

$$D(2.1) \cong F_{1.1} \text{ ,(if (A 487) is valid).} \quad \text{(A 499)}$$

Finally, summarizing the considerations about the computation of D(2.1), the simple formula (A 499) will be of prominent importance, it will governe most cases of our applications. (A 499) can be handled easily in the numerical calculations. The field of application of (A 499) will be much more broad than that of (A 484). The application of the universal formula (A 484) will be restricted to the seldom cases of high mountain test points P, only.

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