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# Geodetic Boundary Value Problems II

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# A. A test for the Marussi condition.

## Contents

## Summary

A test is stated for the oase that the Marussi oondition in regard of a gravitation (gravity) potential is not fulfilled; this means that the Marussi tensor is singular. It relates the main ourvatures of the equipotential surface to the oomponents of the curvature of the plumbline in the directions of the main curvatures. A generally valid necessary condition follows. The equipotential surface in the test point must be hyperbolioally curved, i. e. a saddle point must exist. In the geodetio practice, the Marussi-condition will be fulfilled.



## Zusammenfassung

Es wird ein Kriterium für die Erfüllung der Marussi-Bedingung in bezug auf ein Gravitationspotential (Schwerepotential) angegeben. Es wird eine Beziehung zwischen den Hauptkrümmungsradien der Niveaufläche und den Komponenten der Krümmung der Lotlinie in den Hauptkrümmungsrichtungen aufgestellt für den Fall, daß der Marussi-Tensor singulär ist. Der Tensor ist singulär, wenn die Äquipotentialfläche im Aufpunkt hyperbolisch gekrümmt ist, so daß ein Sattelpunkt vorliegt. Dieser Fall tritt sehr selten ein. Die Marussi-Bedingung dürfte daher praktisch immer erfüllt sein.



## Аннотация

Указывается критерий выполнения условия Марусси относительно гравитационного потенциала (потенциал силы тяжести).
Устанавливается соотношение между основными радиусами искревленной поверхности и компонентами искривления к линии лота в направлении основного искривления для того случая, если Марусси-тензор сингулярен. Тензор сингулярен в том случае, если эквипотенциальная поверхности гиперболически изогнута в точке восхождения, так что имеется сетловая точка. Этот случай наблюдается очень редко. Поэтому, условия Марусси должны практически постоянно выполняться.


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## 1. The Marussi condition.

The Marussi tensor has the following form

$$\underline{\underline{M}} = \begin{pmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{yx} & W_{yy} & W_{yz} \\ W_{zx} & W_{zy} & W_{zz} \end{pmatrix} , \qquad (1)$$

x, y, z are rectangular Cartesian co-ordinates in space and W is the gravitation potential of the non-rotating Earth, [1] [2] [3]. The x, y, z-system is non-rotating and fixed in the space; the Earth is fixed in the x, y, z system, in the considerations of section 1 and 2. In equation (1), the following abbreviations were set for the second derivatives of the potential function.

$$\frac{\partial^2 W}{\partial x^2} = W_{xx}, \quad \frac{\partial^2 W}{\partial x \partial y} = W_{xy}, \; \ldots ; \qquad (2)$$

similar relations hold for the other elements of the Marussi-tensor. The x, y-plane is the plane of the equator of the Earth; the z-axis is directed to the north, and perpendicular to the x, y-plane. The well-known relations of the differential calculus yield

$$W_{xy} = W_{yx} , \qquad (3)$$

$$W_{xz} = W_{zx} , \qquad (4)$$

$$W_{yz} = W_{zy} , \qquad (5)$$

The Marussi tensor (1) is symmetric, as follows from the equations (3), (4), (5)

$$\underline{\underline{M}} = \begin{pmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{xy} & W_{yy} & W_{yz} \\ W_{xz} & W_{yz} & W_{zz} \end{pmatrix} . \qquad (6)$$

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$$W = f \iiint_F \frac{dm}{e} \, dF \quad , \qquad (6a)$$

f is the gravitational constant, dm the element of the mass of the Earth, F is the volume of the Earth, and e is the mutual straight distance between the test point and the running point of the integration procedure covering the volume F.

During the recent years, the Marussi tensor has gained a special actuality in certain problems of the physical geodesy, [2] [3]. In the investigations on the uniqueness of the boundary value problem, the presupposition is introduced that the inverse matrix $\underline{\underline{M}}^{-1}$ is non-singular. Thus, the determinant derived from the Marussi tensor $\underline{\underline{M}}$, (6), has not to be equal to zero,

$$\det \underline{\underline{M}} = \begin{vmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{xy} & W_{yy} & W_{yz} \\ W_{xz} & W_{yz} & W_{zz} \end{vmatrix} \neq 0. \qquad (7)$$

The equation (7) is the so-called Marussi condition which is to be investigated now.

The equation (6) shows the Marussi tensor with regard to the rectangular x, y, z-system which is in relation to the equator plane.

For the following investigations, it is of advantage to transform the $\underline{\underline{M}}$ matrix into the local horizontal Cartesian u, v, w-co-ordinate system.

The w-axis shows into the exterior space, and it has the opposite direction of the gravitation force of the Earth. The u-axis is the intersection line of the horizontal plane and the plane of the z- and w-axis. The u-axis shows to the south, Fig. 1. The v-axis is directed to the east. The u, v-plane is the horizontal plane of the test point.

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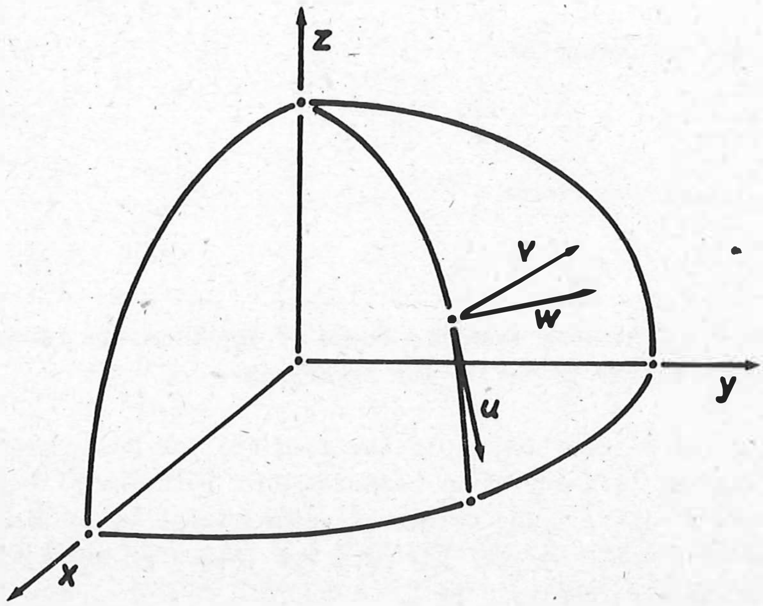


Fig. 1: The geocentric equatorial x, y, z-system and the local horizontal u, v, w-system. The base vectors $\underline{\underline{u}}$ and $\underline{\underline{v}}$ determine the horizontal plane.

In the u, v, w-system, the $\underline{\underline{M}}$ matrix has the following shape,

$$\underline{\underline{M}} = \begin{pmatrix} W_{uu} & W_{uv} & W_{uw} \\ W_{uv} & W_{vv} & W_{vw} \\ W_{uw} & W_{vw} & W_{ww} \end{pmatrix} . \quad (8)$$

The u, v, w system is fixed in the space.

The $\underline{\underline{M}}$ matrix can be derived from the potential W by the scalar multiplication of the nabla operator with the gradient,

$$\nabla \cdot \nabla W . \quad (9)$$

This form is free of the introduction of a special coordinate system.

The gravitation potential W fulfills the Laplace differential equation in the exterior space of the Earth,

$$\Delta W = 0 , \quad (10)$$

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or, in the x, y, z- co-ordinates ,

$$W_{xx} + W_{yy} + W_{zz} = 0 , \tag{11}$$

and, in the u, v, w- co-ordinates ,

$$W_{uu} + W_{vv} + W_{ww} = 0 . \tag{12}$$

The equations (11) (12) show that the trace of the $\underline{\underline{M}}$ matrix is equal to zero. The trace of a tensor is one of its **invariants**.

The elements of the Marussi tensor of the form (8) can be expressed by the entirety of all the five curvature parameters of both the plumbline and the level surface W = const. . The concerned mathematical deductions shall not be given here in detail, [1] [2] [3] [4]. The following expression for the Marussi tensor is the result,

$$\underline{\underline{M}} = - g \begin{pmatrix} \varkappa_1 & \tau & \vartheta_1 \\ \tau & \varkappa_2 & \vartheta_2 \\ \vartheta_1 & \vartheta_2 & -(\varkappa_1 + \varkappa_2) \end{pmatrix} . \tag{13}$$

The equation (13) is in keeping with the equation (10). g is the amount of the vector of the gravitation intensity. $\varkappa_1$ is the normal curvature of the level surface W = const. in the direction of the u-axis, i. e. in the north - south direction. $\varkappa_2$ is the corresponding value in the direction of the v-axis, i. e. in the east - west direction. A positive amount of $\varkappa_1$ or $\varkappa_2$ means convexity. $\tau$ is the amount of the torsion of the geodesic line which is traced on the level surface W = const. in the direction of the parallel, [3] [4]. The plumbline through the test point has the curvature component $\vartheta_1$ in the u, w-plane. $\vartheta_2$ is the analogous value in the v, w-plane. The curvature of the plumbline itself is $\sqrt{\vartheta_1^2 + \vartheta_2^2}$ .

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## 2. The criterion equation in case of a gravitational potential.

In case that the Marussi tensor is singular, the following equation is found from the formulas (7) and (13), accounting for the fact that g is never equal to zero,

$$0 = \begin{vmatrix} \varkappa_1 & \tau & \vartheta_1 \\ \tau & \varkappa_2 & \vartheta_2 \\ \vartheta_1 & \vartheta_2 & -(\varkappa_1 + \varkappa_2) \end{vmatrix} . \tag{14}$$

Now, the u, v, w-system is rotated around the w-axis till the u-axis and the v-axis show in the directions of the main curvature lines. Thus, (14) takes the following form, [1] [2] [3] [4],

$$0 = \begin{vmatrix} k_1 & 0 & l_1 \\ 0 & k_2 & l_2 \\ l_1 & l_2 & -(k_1 + k_2) \end{vmatrix} . \tag{15}$$

This matrix transformation - executed by a rotation - does not change the amount of det $\underline{\underline{M}}$. The reason is the fact that a rotation can be expressed by the multiplication with certain rotation matrices. These matrices have the character of orthonormal matrices. The determinants of such matrices are equal to the unity. And, the multiplication of a determinant with the unity does not change the value of it.

In the equation (15), $k_1$ and $k_2$ are the main curvatures, $l_1$ and $l_2$ are the components of the curvature of the plumbline, i. e. the projection of the curvature of the plumbline on the vertical planes of the main curvatures $k_1$ and $k_2$. The geodesic torsion in the direction of the main curvature lines is well-known to be equal to zero, [3] [4].

Hence, from (15), if $k_1$, $k_2$ are the main curvatures of the level surface W=const.,

$$- k_1 k_2 (k_1 + k_2) - k_1 l_2^2 - k_2 l_1^2 = 0. \tag{16}$$

(16) leads to

$$k_1 (k_2^2 + l_2^2) + k_2 (k_1^2 + l_1^2) = 0, \tag{17}$$

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or, finally,

$$\frac{k_1}{k_2} = - \frac{k_1^2 + l_1^2}{k_2^2 + l_2^2} \quad . \tag{18}$$

This equation (18) is the necessary and sufficient condition for the fulfillment of (14) and (15). In case, the formula (18) is right, the $\underline{\underline{M}}$ matrix is singular and the Marussi condition is not fulfilled. The four curvature parameters $k_1$, $k_2$, $l_1$, $l_2$ observe the relation (18), if the Marussi condition is violated.

The equation (18) leads to the following sufficient condition for the determinant det $\underline{\underline{M}}$ being zero,

$$\frac{k_1}{k_2} < 0 \quad . \tag{19}$$

This inequation describes the fact that $k_1$ must have the inverse sign of $k_2$. Thus, the geometrical shape of the level surface W = const. at the test point is a saddle point. The Gauss curvature

$$K = k_1 \cdot k_2 \tag{20}$$

follows by (19) to be negative,

$$K < 0 \; . \tag{21}$$

The level surface of the test point has a hyperbolic curvature, Fig. 2. The curvature center of the main curvature $k_1$ is situated on the one side of the level surface, and the curvature center of the other main curvature $k_2$ is situated on the other side of this level surface.

Further on, the relation (18) is valid if

$$k_1 = l_1 = 0. \tag{22}$$

The relation (22) demands that one main curvature, $k_1$, has to be equal to zero. This is the condition for the existence of a point in which the level surface has parabolic curvature. Furthermore, the relation (22) demands that the curvature component of the plumbline in the direction of this main curvature, $l_1$, has to be equal to zero, too.

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If g is the gravitation foroe, and if dp is the element of length in the direction of the main curvature line of $k_1$, in this oase, $l_1$ has the following formula, [3] [4],

$$l_1 = \frac{\partial \ln g}{\partial p} \quad . \qquad (23)$$

In case of (22) and because of (23). the horizontal gradient of the gravitation force in the direction of the main curvature line attributed to $k_1$ has to be equal to zero. The horizontal gradient of the gravitation force g follows to be perpendicular to the direction of that main curvature line which is attributed to $k_1$. This gradient has the direction of the main curvature line which is attributed to $k_2$.

In the above derivations and in the formulation of the shape of the criterion determinant, (15), the whole amount of the gravitational potential W was involved. Sure, W can be replaced by the sum of the perturbation potential T and the standard potential. The sum of both these potentials is identical with W.

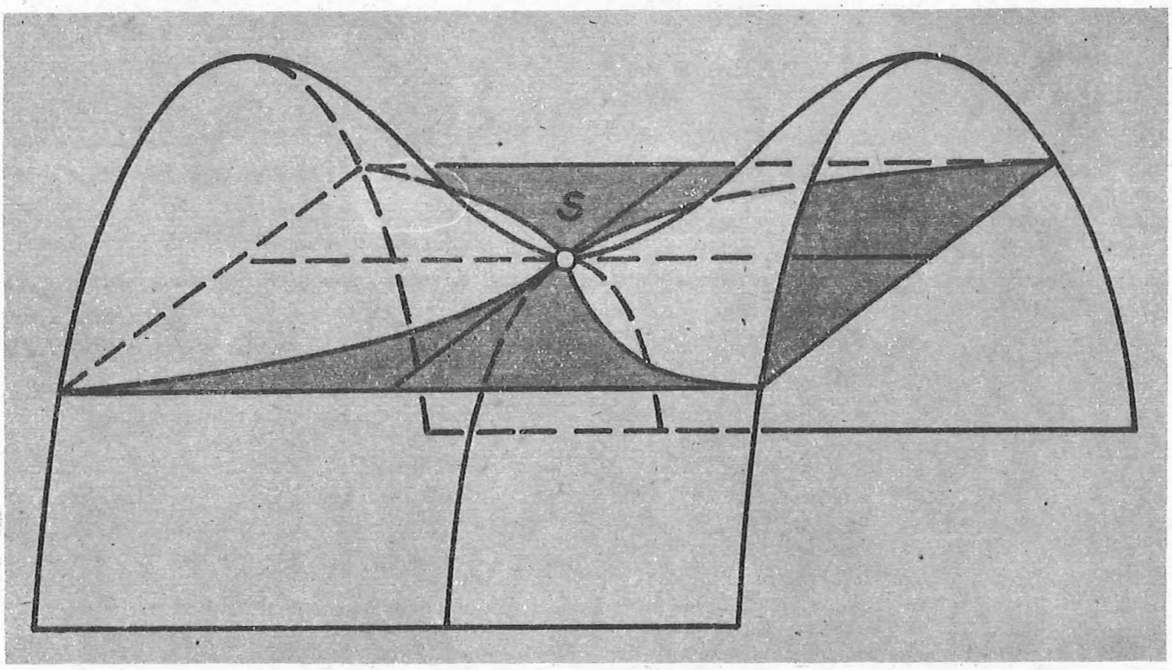


Fig. 2: In the point S, the level surfaoe has hyperbolio curvature, (saddle point).

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## 3. The criterion equations in case of a gravity potential.

If a rotating and gravitating Earth is considered, the gravitational potential must be supplemented by the potential of the centrifugal force Z. Instead of the Laplace differential equation, the Poisson differential equation is valid now, [1], in the exterior space,

$$\Delta V = 2\omega^2 \quad . \tag{24}$$

$$V = W + Z \quad , \tag{25}$$

$$Z = \frac{1}{2}\omega^2 (x^2 + y^2) \quad . \tag{26}$$

$\omega$ is the angular velocity of the rotation of the Earth, $\omega = 7.3 \cdot 10^{-5}$ [rad $s^{-1}$]. $\omega$ is considered constant. The Marussi tensor of the field of the V potential is denominated by $\underline{\underline{N}}$,

$$\underline{\underline{N}} = \begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix} . \tag{27}$$

In the new horizontal u, v, w-system which relates to the potential V, the criterion equation,

$$\det \underline{\underline{N}} = 0 \quad , \tag{28}$$

has the following shape, in analogy to (14), [1],

$$0 = \begin{vmatrix} \varkappa_1 & \tau & \vartheta_1 \\ \tau & \varkappa_2 & \vartheta_2 \\ \vartheta_1 & \vartheta_2 & -(\varkappa_1 + \varkappa_2) - 2\frac{\omega^2}{g} \end{vmatrix} \tag{29}$$

The tensor $\underline{\underline{N}}$ in the new horizontal u, v, w-system, is transformed by a rotation around the vertical w-axis, till the horizontal u- and v-axis coincide with the tangential vectors of the main curvature lines, [1] [2] [3] [4]. This are the main curvature lines which run along the level surface V = const. of the gravity potential V , (25).

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Thus, in case of a gravity potential, the criterion for the existence of a singularity of the Marussi tensor has the following shape,

$$0 = \begin{vmatrix} k_1 & 0 & l_1 \\ 0 & k_2 & l_2 \\ l_1 & l_2 & -(k_1 + k_2) - 2\,\frac{\omega^2}{g} \end{vmatrix} . \tag{30}$$

$k_1$, $k_2$, $l_1$, $l_2$ are again the curvature parameters, applying to the system of the main curvature lines. The determinant (30) leads to

$$k_1\, k_2\, (k_1 + k_2 + 2\,\frac{\omega^2}{g}) + k_1\, l_2^2 + k_2\, l_1^2 = 0. \tag{31}$$

In the relation (31), the values $\omega^2$ and g are always positive,

$$\omega^2 > 0 , \qquad g > 0. \tag{32}$$

Thus, both the main curvatures $k_1$ and $k_2$ cannot be positive simultaneously. Therefore, the level surface V = const. can not be convex.

Furthermore, $k_1 = 0$ necessitates either $k_2 = 0$ or $l_1 = 0$.

For the investigation of a type of special interest, the formula (31) is transformed to

$$\frac{k_1}{k_2} = - \frac{l_1^2 + k_1^2\,(1 + \frac{2}{g}\,\omega^2\,\frac{1}{k_1})}{l_2^2 + k_2^2} . \tag{33}$$

Obviously, this relation (33) leads to the following inequations,

$$k_1 > 0 , \qquad k_2 < 0 . \tag{34}$$

Therefore, (34), if one of the main curvatures is positive, e. g. $k_1 > 0$, the other main curvature has to observe the inequation $k_2 < 0$. Hence, the curvature of the level surface V = const. has a hyperbolic character, in case of (34).

The following curvature type, (35), remains to be considered,

$$k_1 < 0 , \qquad k_2 < 0 . \tag{35}$$

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The relations (35) describe a level surface type with concave curvature. Thus, (31) and (35) lead to the fact that the term in the braces of (31) has to have a positive value,

$$(k_1 + k_2 + 2\frac{\omega^2}{g}) > 0, \text{ if } k_1 < 0 \text{ and } k_2 < 0. \qquad (36)$$

Thus, from (36),

$$|k_1 + k_2| < 2\frac{\omega^2}{g} \; . \qquad (37)$$

(35) and (37) lead to

$$|k_1| < 2\frac{\omega^2}{g} = 10^{-11}\ \mathrm{cm}^{-1} \; . \qquad (38)$$

A similar inequation is valid for $k_2$.

The inequation (38) is equivalent to the following relation,

$$\left|\frac{1}{k_1}\right| > 160\ R \; . \qquad (39)$$

R is the radius of the globe. The curvature $k_2$ has a similar formula as (39) for $k_1$.

In case of (37) (38) (39), the two components of the plumbline deflections $\Theta$ alter along a horizontal distance of one kilometer by about -32". This fact is evidenced in the following way. If,in the surroundings of the test point, the level surface approximates a plane, in this case, the plumbline deflections alter by $\delta\Theta$ = -32."38 along the distance of one kilometer, it is on the strength of the curvature of the reference surface which is here the globe. Now, if the level surface in the near surroundings of the test point undergoes a downbuckling according to (35) (37) (38) (39), the amounts of both the main curvatures of the level surface have the possibility to diminish from zero to $-\frac{1}{160\ R}$ .So,the curvature type of the level surface becomes concave. This downbuckling of the level surface alters the plumbline deflections supplementary by the amount of - 0."20 within a kilometer.

Thus, the equations (31) and (35) lead to the following relation,

$$-32."58 < \delta\Theta < -32."38; \; k_1 < 0, \; k_2 < 0. \qquad (40)$$

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In (40), $\delta\Theta$ is the alteration of the plumbline deflection along the distance of one kilometer.

Summarizing, in case of a gravity potential, (24) (25), the Marussi tensor (27) is possible to be singular only if the relation (34) is fulfilled, i. e. the curvature of the level surface is hyperbolic, or, if (35) (37) or (40) is valid, i. e. the level surface has a slight concavity.

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## B. On the evaluation of the numerical amount of the residual term of the solution of the geodetic boundary value problem.

Contents



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Summary

The solution of the geodetic boundary value problem, for the surface of the Earth as boundary surface, consists in the addition of the plane topographic correction to the free-air anomaly in the Stokes solution. Under certain circumstances, a small term which is expressed by the height gradient of the Bouguer anomalies has to be added. Further, a residual term of very small amount generally to be neglected is to be considered. It has a closed mathematical expression in terms of the difference of two certain potentials: The potential of the spatial visible mountain masses (but having the standard density) minus the potential of these masses condensed at the globe in form of a spherical surface distribution of masses. In order to find the residuum by this difference potential, three explicit formulas are developed here. They allow the computation of this difference potential /B/" and of its vertical and horizontal derivatives.

Zusammenfassung

Die Lösung des geodätischen Randwertproblems für die Erdoberfläche besteht in der Addition der ebenen Geländereduktion der Schwere zu den Freiluftanomalien im Integral von Stokes. In bestimmten Fällen sollte ein kleiner Ausdruck hinzugenommen werden, der von dem vertikalen Gradienten der Bougueranomalien abhängt. Zu diesem Integral muss ein Restglied addiert werden. Dieses hat einen geschlossenen mathematischen Ausdruck, und es kann meistens vernachlässigt werden. Es ist sehr klein. Das Restglied wird ausgedrückt durch ein Differenzpotential. Dieses ist die Differenz zwischen dem Potential der sichtbaren Massen mit Standarddichte und dem Potential dieser Massen nach Kondensation an der Erdkugel. Für die Abschätzung der Grösse des Restgliedes werden ausführliche Rechenformeln entwickelt, und zwar für dieses Potential selbst und für seine vertikalen und horizontalen Ableitungen.

Аннотация

Решение геодезической краевой задачи для поверхности земли состоит в суммировании прямой топографической поправки гравитации с аномалией воздуха на открытой местности в интеграле Штока. В определенных случаях должно использоваться выражение, зависящее от вертикальных градиентов аномалий Бугура. В этом интеграле суммируется отстаточный член. Он представляет собой законченное математическое выражение и им можно в большинстве случаев пренебречь из-за его незначительной величины. Остаточный член выражается дифференциальным потенциалом, который является разницей между потенциалом видимых масс со стандартной плотностью и потенциалом этих масс после конденсации на земле. Для оценки величины остаточного члена разрабатываются исчерпывающие формулы расчёта, а именно как для самого потенциала, так и для его вертикальных и горизонтальных производных.


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## 1. The evaluation of the potential expression /B/".

The following 3 terms are considered in this chapter C,

$$\Xi_1 = /B/'' = B - B_{cond.} , \qquad (1)$$

$$\Xi_2 = \left[\frac{\partial B}{\partial r}\right]'' = \frac{\partial B}{\partial r} - \frac{\partial B_{cond.}}{\partial r} , \qquad (2)$$

$$\Xi_3 = \frac{\partial /B/''}{\partial x, y} . \qquad (2a)$$

At first, the potential expression is in the fore, (1).

According to the purposes of the following derivations, the expressions for $\Xi_1$ and $\Xi_2$, /1/ /2/ /3/, are to be modified in order to bring them into a shape convenient for routine numerical computations.

At first, the $\Xi_1$ term is considered. The potential of the visible mountain masses above the sea level, (with the standard density $\varrho$, $\varrho = 2.65$ /g cm$^{-3}$/), is

$$B = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r=R}^{R+h_Q} \left(\frac{1}{\varepsilon}\right)_P r^2 \sin\psi \, dr \, d\psi \, d\alpha , \qquad (3)$$

P specifies the test point at the surface of the Earth, Fig. 1. The integration over the geocentric radius r is divided into two steps. The first step has the interval: $R \leq r \leq (R + h_P)$. The second step is: $(R + h_P) \cdots r \cdots (R + h_Q)$. $h_P$ is the height of the test point P above the spherical reference sphere. $h_P$ is a fixed value within the course of the integration. $h_Q$ is the height of the running point Q at the surface of the Earth. The point Q moves over the Earth in the course of the integration, Fig. 1. Along these lines, the integral (3) changes to

$$B = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \left[ \int_{r=R}^{R+h_P} + \int_{R+h_P}^{R+h_Q} \right] \left(\frac{1}{\varepsilon}\right)_P r^2 \sin\psi \, dr \, d\psi \, d\alpha . \qquad (4)$$

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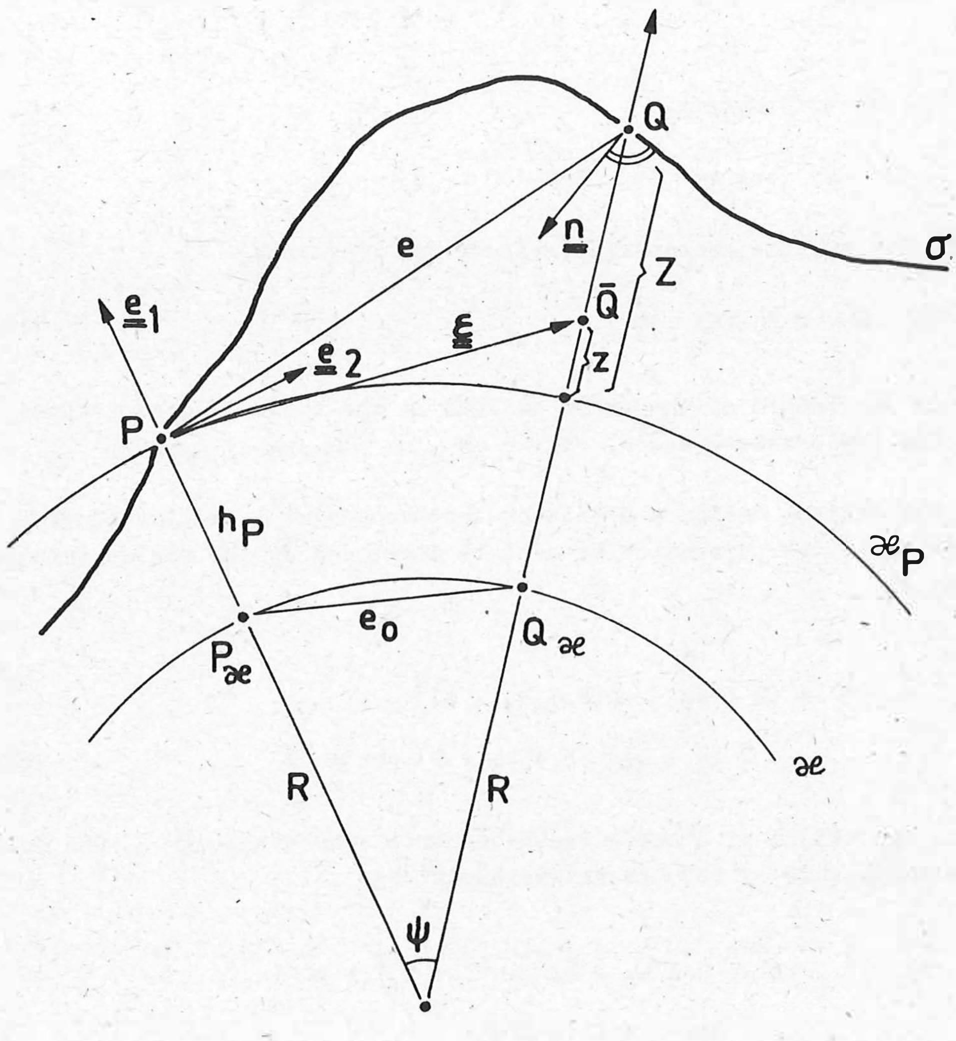


Fig. 1: The test point P and the running point Q on the surface of the Earth $\sigma$; their mutual distance and their height difference. The spheres $\varkappa_P$ and $\varkappa$ .

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The potential of the mountain masses (of the standard density $\varrho$ ) is now condensed at the surface of the globe $\varkappa$ with the radius R. In the test points $P_{\varkappa}$ of the globe $\varkappa$ , the potential of the condensed masses has the subsequent formula, Fig. 1,

$$B_{cond.} = 4\pi f \varrho R h_P + f \varrho R^2 \iint_{\omega} (h_Q - h_P)(\gamma e_0) \, d\omega . \quad (5)$$

$\omega$ is the unit sphere,

$$d\omega = \cos\varphi \cdot d\varphi \cdot d\lambda . \quad (6)$$

$\varphi$ and $\lambda$ are the geocentrio latitude and longitude,

$$e_0 = 2 R \sin \frac{\psi}{2} . \quad (7)$$

$e_0$ is the length of the chord defined by the foots of the perpendiculars of the two points P and Q, Fig. 1 ,it are the piercing points $P_{\varkappa}$ and $Q_{\varkappa}$.

The oblique distance $\varepsilon$ between the test point P and the running point $\bar{Q}$ of the perpendicular of Q is developed by the cosine theorem, Fig. 1,

$$\varepsilon^2 = (R + h_P)^2 + (R + h_P + z)^2 - $$
$$- 2 (R + h_P) (R + h_P + z) \cos\psi . \quad (8)$$

z is the height of $\bar{Q}$ above the geocentric sphere through P. The following rearrangements of (8) are self-explanatory,

$$\varepsilon^2 = R^2 + 2 R h_P + h_P^2 + R^2 + h_P^2 + z^2 +$$
$$+ 2 Rh_P + 2 Rz + 2 h_P z -$$
$$- 2 (R^2 + Rh_P + Rz + h_P R + h_P^2 +$$
$$+ h_P z) \cos\psi , \quad (9)$$

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$$\varepsilon^2 = 2\,R^2 + 4\,Rh_P + 2Rz + 2\,h_P^{\,2} + z^2 + 2\,h_Pz - 2\,(R^2 + 2\,Rh_P + Rz + h_P^{\,2} + h_Pz)\cos\psi \quad . \tag{10}$$

Obviously, it is recommended to replace the cosine function of $\psi$ by the sine function, using

$$\cos\psi = 1 - 2\,\sin^2\frac{\psi}{2} \quad . \tag{11}$$

(10) and (11) reveal,

$$\varepsilon^2 = z^2 + 4\,R^2\,\sin^2\frac{\psi}{2} + 4\,(2\,Rh_P + Rz + h_P^{\,2} + h_Pz)\,\sin^2\frac{\psi}{2} \quad . \tag{12}$$

Hence,

$$\varepsilon^2 = 4\,R^2\,\sin^2\frac{\psi}{2} + z^2 + 4\,R^2\,\sin^2\frac{\psi}{2}\left[2\,\frac{h_P}{R} + \frac{z}{R} + \frac{h_P^{\,2}}{R^2} + \frac{h_P\cdot z}{R^2}\right] . \tag{13}$$

(7) and (13) give,

$$\varepsilon^2 = e_o^{\,2} + z^2 + e_o^{\,2}\left[2\,\frac{h_P}{R} + \frac{z}{R} + \frac{h_P^{\,2}}{R^2} + \frac{h_Pz}{R^2}\right] , \tag{14}$$

or,

$$\varepsilon^2 = e_o^{\,2} + z^2 + e_o^{\,2}\left[\frac{2\,h_P + z}{R} + \frac{h_P^{\,2} + h_Pz}{R^2}\right] . \tag{15}$$

Further on,

$$\varepsilon^2 = e_o^{\,2} + z^2 + e_o^{\,2}z\left[\frac{1}{R} + \frac{h_P}{R^2}\right] + e_o^{\,2}\left[\frac{2\,h_P}{R} + \frac{h_P^{\,2}}{R^2}\right] \quad . \tag{16}$$

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Hence,

$$\varepsilon^2 = e_o^2 + z^2 + (e_o^2 + z^2)\left\{ \frac{e_o^2\, z}{e_o^2 + z^2}\left[\frac{1}{R} + \frac{h_P}{R^2}\right] + \right.$$

$$\left. + \frac{e_o^2}{e_o^2 + z^2}\left[\frac{2\,h_P}{R} + \frac{h_P^2}{R^2}\right]\right\} \quad . \tag{17}$$

The main term on the right hand side of (17) is

$$e_o^2 + z^2 \, . \tag{18}$$

It is factored out,

$$\varepsilon^2 = (e_o^2 + z^2)\left\{ 1 + \frac{e_o^2\, z}{e_o^2 + z^2}\left[\frac{1}{R} + \frac{h_P}{R^2}\right] + \right.$$

$$\left. + \frac{e_o^2}{e_o^2 + z^2}\left[\frac{2\,h_P}{R} + \frac{h_P^2}{R^2}\right]\right\} \quad . \tag{19}$$

With the abbreviations

$$D_1 = \frac{e_o^2}{e_o^2 + z^2} \cdot \frac{z}{R}\left[1 + \frac{h_P}{R}\right] \quad , \tag{20}$$

and

$$D_2 = \frac{e_o^2}{e_o^2 + z^2} \cdot \frac{2\,h_P}{R}\left[1 + \frac{h_P}{2\,R}\right] \quad , \tag{21}$$

follows,

$$\varepsilon^2 = (e_o^2 + z^2)\left[1 + D_1 + D_2\right] \quad . \tag{22}$$

Neglecting a relative error of about $10^{-3}$ to $10^{-4}$, the brackets of (20) and (21) can be omitted,

$$D_1 \cong \frac{e_o^2}{e_o^2 + z^2} \cdot \frac{z}{R} \quad , \tag{23}$$

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$$D_2 \cong \frac{e_o^2}{e_o^2 + z^2} \cdot \frac{2\,h_P}{R} \quad . \tag{24}$$

Because of

$$|D_1| \ll 1, \quad |D_2| \ll 1, \tag{25}$$

it is allowed to apply certain binomial series developments for the powers of the term in the brackets of (22). Thus,

$$\frac{1}{\varepsilon} = \frac{1}{\sqrt{e_o^2 + z^2}} \left[ 1 - \frac{1}{2} D_1 - \frac{1}{2} D_2 \right] , \tag{26}$$

with $D_1$ and $D_2$ according to (23) (24).

Later on, this formula for $\frac{1}{\varepsilon}$ shall be introduced in the integrand of (4). But, before doing so, the integration covering the interval

$$R \leqslant r \leqslant (R + h_P) \tag{27}$$

should be discussed separately. It will conduct in the vicinity of (5).

This relevant part of the integration according to (4) has the following form,

$$M = f\varrho \int_{\psi = 0}^{\pi} \int_{\alpha = 0}^{2\pi} \int_{r = R}^{R + h_P} \left( \frac{1}{\varepsilon_P} \right) r^2 \sin\psi \; dr \, d\psi \, d\alpha \quad . \tag{28}$$

The meaning of the potential M allows a simple interpretation. Since R and $h_P$ are constant values for the integration according to (28), M is the potential of a homogeneous shell of the standard density $\varrho$ and of the thickness $h_P$, the inner radius is R and the exterior is R + $h_P$. The test point P lies on the exterior margin of this shell. Further on, M can be interpreted also as the difference of two spherical potentials,

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$$M = M_1 - M_2 . \tag{29}$$

$M_1$ is the potential of a homogeneous sphere of the radius $R + h_P$. The test point is situated on the surface of this sphere. Thus,

$$M_1 = f \varrho \frac{4}{3} \pi \frac{(R + h_P)^3}{R + h_P} . \tag{30}$$

$M_2$ is the potential of a homogeneous sphere of the radius R. The test point has now the distance $R + h_P$ from the center, as in case of $M_1$,

$$M_2 = f \varrho \frac{4}{3} \pi \frac{R^3}{R + h_P} . \tag{31}$$

(29) (30) (31) reveal,

$$M = f \varrho \left[ \frac{\frac{4}{3} \pi (R + h_P)^3}{R + h_P} - \frac{\frac{4}{3} \pi R^3}{R + h_P} \right] = \frac{4}{3} \pi f \varrho \frac{1}{R + h_P} \left[ (R + h_P)^3 - R^3 \right] =$$

$$= \frac{4}{3} \pi f \varrho \frac{1}{R + h_P} \left[ R^3 + 3 \cdot R^2 h_P + 3 R h_P^2 + h_P^3 - R^3 \right] =$$

$$= 4 \pi f \varrho \frac{1}{R + h_P} \left[ R^2 h_P + R h_P^2 + \frac{1}{3} h_P^3 \right] =$$

$$= 4 \pi f \varrho \frac{1}{R + h_P} R^2 \left[ h_P + \frac{h_P^2}{R} + \frac{1}{3} h_P \left( \frac{h_P}{R} \right)^2 \right] . \tag{32}$$

The denominator of (32) allows a development into a binomial series,

$$(R + h_P)^{-1} = \frac{1}{R} \left(1 + \frac{h_P}{R}\right)^{-1} = \frac{1}{R} \left[ 1 - \frac{h_P}{R} + \left( \frac{h_P}{R} \right)^2 - + \ldots \right] . \tag{33}$$

The combination of (32) and (33) leads to

$$M = 4 \pi f \varrho R h_P \left[ 1 + \frac{h_P}{R} + \frac{1}{3} \left( \frac{h_P}{R} \right)^2 \right] \left[ 1 - \frac{h_P}{R} + \left( \frac{h_P}{R} \right)^2 \right] , \tag{34}$$

or,

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$$M = 4 \pi f \varrho R h_P \left[ 1 + \frac{1}{3} \left( \frac{h_P}{R} \right)^2 \right] . \qquad (35)$$

Therefore, the first part of the integral (4) can be replaced by (35).

In case, the reciprocal distance $\frac{1}{\varepsilon}$ is replaced by a development in Legèndre functions, (28), the relation (35) for M is corroborated. Because of the orthogonality relations of the Legendre functions, the Legendre function of the degree n = 0 only has to be taken into account in this problem.

Finally, the expression for M is introduced into (4), (see (28) (35)). The resulting expression for B is taken as the first term on the right hand side of (1).

The formula (5) replaces the second term on the right hand side of (1). Along these lines, the term $\Xi_1$ proves to have this form,

$$\Xi_1 = \left[ B \right]'' = B + B_{cond.} = 4 \pi f \varrho R h_P \left[ 1 + \frac{1}{3} \left( \frac{h_P}{R} \right)^2 \right] +$$

$$+ f \varrho \int_{\psi = 0}^{\pi} \int_{\alpha = 0}^{2\pi} \int_{r = R + h_P}^{R + h_Q} \left( \frac{1}{\varepsilon} \right)_P r^2 \sin \psi \, dr \, d\psi \, d\alpha -$$

$$- 4 \pi f \varrho R h_P - f \varrho R^2 \iint_{\omega} (h_Q - h_P) \frac{1}{e_0} d\omega . \qquad (36)$$

The first term on the right hand side of (36) has the expression

$$\frac{1}{3} \left( \frac{h_P}{R} \right)^2 \qquad (37)$$

in the brackets. It gives rise to an impact on the potential value of $\Xi_1$ by the term

$$N = \frac{4}{3} \pi f \varrho R h_P \left( \frac{h_P}{R} \right)^2 . \qquad (38)$$

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With

$$h_P = 3 \text{ km} \tag{39}$$

follows

$$N = G \cdot 0.3 \text{ mm} , \tag{40}$$

G is the global mean value of the gravity. Thus, the relation (40) shows that the reflection of N on the height anomaly is not more than 0.3 mm, a value that can be neglected.

The impact of N on the gravity is equal to

$$\frac{2}{R} N = 0.1 \ \mu\text{gal} . \tag{41}$$

This term is negligible, too.

Hence, (36) turns to

$$\Xi_1 = f \varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r=R+h_P}^{R+h_Q} \left(\frac{1}{\varepsilon}\right)_P r^2 \sin\psi \, dr \, d\psi \, d\alpha$$
$$- f \varrho R^2 \iint_{\omega} (h_Q - h_P) \frac{1}{e_o} \, d\omega . \tag{42}$$

Now, (42) undergoes further rearrangements. The reciprocal distance is replaced by (26). In the first integral on the right hand side of (42), the integration over the radius r gives, accounting for

$$h_Q = h_P + Z , \tag{43}$$

$$Y = \int_{r=R+h_P}^{R+h_P+Z} \left(\frac{1}{\varepsilon}\right)_P r^2 \, dr =$$

$$= \int_{r=R+h_P}^{R+h_P+Z} \left(\frac{1}{\varepsilon}\right)_P (R + h_P + z)^2 \, d z . \tag{44}$$

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$$Y = \int_{r = R + h_P}^{R + h_P + Z} \left(\frac{1}{\varepsilon}\right)_P (R + h_P)^2 \left[1 + \frac{z}{R + h_P}\right]^2 dz, \qquad (45)$$

and with

$$\left[1 + \frac{z}{R + h_P}\right]^2 \cong 1 + 2\,\frac{z}{R + h_P}\,, \qquad (46)$$

follows,

$$Y = \int_{z=0}^{Z} \left(\frac{1}{\varepsilon}\right)_P (R + h_P)^2\, dz + 2 \int_{z=0}^{Z} \left(\frac{1}{\varepsilon}\right)_P (R + h_P)^2 \frac{z}{R + h_P}\, dz, \qquad (47)$$

or,

$$Y = (R + h_P)^2 \int_{z=0}^{Z} \left(\frac{1}{\varepsilon}\right)_P dz + 2\,R \int_{z=0}^{Z} \left(\frac{1}{\varepsilon}\right)_P z\, dz\,. \qquad (48)$$

In (48), all the terms are neglected which are equivalent to a relative error of the order

$$\left(\frac{h_P}{R}\right)^2 \quad \text{or} \quad \left(\frac{z}{R}\right)^2 \qquad (49)$$

in the main term of (48). It is in keeping with the precision of the empirical determination of the location of the geodetic control points.

(48) takes the following shape, introducing (26) as the substitute for the reciprocal distance,

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$$Y = (R + h_P)^2 \int_{z=0}^{Z} \frac{1}{\sqrt{e_o^2 + z^2}} dz -$$

$$- \frac{(R + h_P)^2}{2 R} \int_{z=0}^{Z} \frac{e_o^2 z}{(e_o^2 + z^2)^{3/2}} dz -$$

$$- (R + h_P)^2 \frac{h_P}{R} \int_{z=0}^{Z} \frac{e_o^2}{(e_o^2 + z^2)^{3/2}} dz +$$

$$+ 2 R \int_{z=0}^{Z} \frac{1}{\sqrt{e_o^2 + z^2}} z\, dz. \tag{50}$$

Some simple modifications of (50) give

$$Y = (R + h_P)^2 \int_{z=0}^{Z} \frac{1}{\sqrt{e_o^2 + z^2}} dz -$$

$$- \frac{R e_o^2}{2} \int_{z=0}^{Z} \frac{z}{(e_o^2 + z^2)^{3/2}} dz - Rh_P e_o^2 \int_{z=0}^{Z} \frac{1}{(e_o^2 + z^2)^{3/2}} dz +$$

$$+ 2 R \int_{z=0}^{Z} \frac{1}{\sqrt{e_o^2 + z^2}} z\, dz. \tag{51}$$

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With

$$w = \frac{Z}{e_o} = \frac{h_Q - h_P}{e_o} \,, \tag{52}$$

and

$$U_1 = \int_{z=0}^{Z} \frac{1}{\sqrt{e_o^2 + z^2}} \, dz = \ln\left[ w + \sqrt{1 + w^2} \right] =$$

$$= \operatorname{arsinh} w \,, \tag{53}$$

$$U_2 = \int_{z=0}^{Z} \frac{z}{(e_o^2 + z^2)^{3/2}} \, dz = -\frac{1}{e_o} \frac{1}{\sqrt{1 + w^2}} + \frac{1}{e_o} \,, \tag{54}$$

$$U_3 = \int_{z=0}^{Z} \frac{1}{(e_o^2 + z^2)^{3/2}} \, dz = \frac{1}{e_o^2} \frac{w}{\sqrt{1 + w^2}} \,, \tag{55}$$

$$U_4 = \int_{z=0}^{Z} \frac{1}{\sqrt{e_o^2 + z^2}} \, z \, dz = e_o \sqrt{1 + w^2} - e_o \,, \tag{56}$$

and

$$k_1 = (R + h_P)^2 \,, \tag{57}$$

$$k_2 = -\tfrac{1}{2} R \, e_o^2 \,, \tag{58}$$

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$$k_3 = - R\, h_P\, e_o^{\,2} \quad , \tag{59}$$

$$k_4 = 2\, R \quad , \tag{60}$$

follows for Y, (51),

$$Y = \sum_{i=1}^{4} k_i\, U_i \quad . \tag{61}$$

The above expression for Y is equal to the Y value of (44). It is introduced into the equation (42).

$$\Xi_1 = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} Y \sin\psi \; d\psi \; d\alpha \; -$$

$$- f\varrho\, R^2 \iint_{\omega} (h_Q - h_P) \frac{1}{e_o}\, d\omega \quad . \tag{62}$$

And with (61), (6),

$$\Xi_1 = f\varrho \iint_{\omega} \sum_{i=1}^{4} k_i\, U_i\, d\omega \; -$$

$$- f\varrho\, R^2 \iint_{\omega} (h_Q - h_P) \frac{1}{e_o}\, d\omega \quad . \tag{63}$$

(43), (52) and (63) reveal

$$\Xi_1 = f\varrho \iint_{\omega} \sum_{i=1}^{4} k_i\, U_i\, d\omega - f\varrho\, R^2 \iint_{\omega} w\, d\omega \; . \tag{64}$$

The second term on the right hand side of (64) is joined with the $k_i\, U_i$ expressions of the first term on the right hand side of (64),

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$$k_5 = - R^2 , \tag{65}$$

$$U_5 = w . \tag{66}$$

Thus, (64) turns to

$$\Xi_1 = f \varrho \iint_{\omega} \sum_{i=1}^{5} k_i U_i \cdot d\omega . \tag{67}$$

The equations (53) to (60), (65), (66), (67) can be combined,

$$\Xi_1 = f \varrho \iint_{\omega} \sum_{i=1}^{5} V_i \, d\omega , \tag{68}$$

$$V_1 = (R + h_p)^2 \text{ arsinh } w, \tag{69}$$

$$V_2 = \frac{1}{2} R e_o \left[ \frac{1}{\sqrt{1 + w^2}} - 1 \right] , \tag{70}$$

$$V_3 = - R h_p \frac{w}{\sqrt{1 + w^2}} , \tag{71}$$

$$V_4 = 2 R e_o \left[ \sqrt{1 + w^2} - 1 \right] \tag{72}$$

$$V_5 = - R^2 w . \tag{73}$$

The above formulas (68) to (73) are a representation of (1), immediately suitable for numerical computations. They are applied to the model computations in Chapter B of [3].

Further, in this context, a certain special property of the sum of

$$V_1 + V_5 \tag{74}$$

should be pointed out , since it is important in the numerical

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applications. In many cases, in especial if the distance $e_o$ is sufficient great, the quotient w, (52), fulfills the following inequation,

$$|w| \ll 1 \quad , \qquad (75)$$

(53) allows a convergent series development of (69),

$$V_1 = (R + h_P)^2 \left[ w - \frac{1}{6} w^3 + \frac{3}{40} w^5 - + \ldots \right] \qquad (76)$$

(76) is convergent if

$$|w| < 1 \quad . \qquad (77)$$

With

$$R^2 \cong (R + h_P)^2 \cong R^2 (1 + \frac{h_P}{R})^2 \qquad (78)$$

and with

$$\frac{h_P}{R} \ll 1 \quad , \qquad (79)$$

follows

$$V_1 \cong R^2 \left[ w - \frac{1}{6} w^3 + - \ldots \right] \quad . \qquad (80)$$

Hence, in case the constraint (75) is observed,

$$V_1 + V_5 \cong - \frac{1}{6} R^2 w^3 \quad . \qquad (81)$$

Thus, in the sum

$$\sum_{i=1}^{5} V_i \qquad (82)$$

of (68), the terms $V_1$ and $V_5$ cancel each other more or less.

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This phenomenon reflects in the subsequent inequations:
In case, (75) is valid, the inequations

$$\left| V_1 + V_5 \right| \ll \left| V_1 \right| \quad , \tag{83}$$

$$\left| V_1 + V_5 \right| \ll \left| V_5 \right| \quad , \tag{84}$$

are fulfilled.

## 2. The evaluation of the radial derivative $\left[ \frac{\partial B}{\partial r} \right]''$.

The second term which is to be developed here is $\Xi_2$, (2). Here, the radial derivative of the potential B, (3),

$$\frac{\partial B}{\partial r} , \tag{85}$$

and the radial derivative of the potential of the condensed masses, (5),

$$\frac{\partial B_{cond.}}{\partial r} , \tag{86}$$

are of importance. They have the following integral expressions,

$$\frac{\partial B}{\partial r} = f \varrho \int_{\psi = 0}^{\pi} \int_{\alpha = 0}^{2\pi} \int_{r' = R}^{R + h_Q} \left( \frac{\partial 1/\varepsilon}{\partial r} \right)_P r'^2 \sin \psi \, dr' \, d\psi \, d\alpha \quad , \tag{87}$$

and, (see Fig. 2),

$$\left[ \frac{\partial B_{cond.}}{\partial r} \right]_{\overline{P} \rightarrow P_{\varkappa}} = f \, \varrho \, R^2 \iint_{\omega} h_Q \left[ \frac{\partial 1/e^*}{\partial r} \right]_{\overline{P} \rightarrow P_{\varkappa}} d\omega \quad . \tag{88}$$

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r is in (87) and (88) the geocentric radius of the test point P resp. $\bar{P}$, r' is the geocentric radius of the running volume element of the integration (87). $P_{\varkappa}$ is the foot of the perpendicular of the surface test point P taken in the level of the globe with the radius R, Fig. 1. $\bar{P}$ is a test point in the exterior of this globe, Fig. 2. In (88), $e^{*}$ is the straight distance between the point $\bar{P}$ and the running point on the globe. The relation for the exterior normal derivative of the potential of a surface distribution transforms (88) into the following expression which is valid for surface test points, [4] [5],

$$\left(\frac{\partial B_{cond.}}{\partial r}\right)_{P_{\varkappa}} = -2\pi f \rho h_P + f \rho R^2 \iint_{\omega} h_Q \frac{\partial 1/e_0}{\partial r} d\omega \quad , (89)$$

with

$$\frac{\partial 1/e_0}{\partial r} = \left(\frac{\partial 1/e^{*}}{\partial r}\right)_{\bar{P} \to P_{\varkappa}} \quad . \qquad (90)$$

The term $\Xi_2$ is the difference of (87) and (89), (see (2)),

$$\Xi_2 = \left[\frac{\partial B}{\partial r}\right]'' = \frac{\partial B}{\partial r} - \left(\frac{\partial B_{cond.}}{\partial r}\right)_{P_{\varkappa}} \quad . \qquad (91)$$

At first, the integral (87) undergoes a certain transformation. The integration over the radius r' from the lower bound R to the upper bound R + $h_Q$ is subdivided into two steps, (3) (4). The first step is again the interval R $\leq$ r' $\leq$ (R + $h_P$), and the second is (R + $h_P$) ... r' ... (R + $h_Q$). The first step contributes to (87) by

$$A = f \rho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r'=R}^{R+h_P} \left(\frac{\partial 1/\varepsilon}{\partial r}\right)_P r'^2 \, dr' \, d\omega \quad . \qquad (92)$$

The expression A has a simple interpretation. If a homogeneous shell of the density $\rho$ and of the boundary spheres with the radius r' = R and r' = R + $h_P$ is considered, the expression of -A is the gravitation intensity that this shell exerts on the points of the upper boundary sphere, (r' = R + $h_P$). Thus, -A is equivalent to the gravitation

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intensity that two balls exert on a test point of the radius $R + h_P$, the one ball has the radius $R + h_P$ and the density $\varrho$ , the other has the radius R and the density $- \varrho$ . Consequently,

$$A = - f \varrho \frac{4}{3} \pi \; (R + h_P)^3 \frac{1}{(R + h_P)^2} +$$

$$+ f \varrho \frac{4}{3} \pi \; R^3 \frac{1}{(R + h_P)^2} \; . \qquad (93)$$

The following rearrangements of (93) are self-explanatory,

$$A = - \frac{4}{3} \pi \; f \varrho \frac{1}{(R + h_P)^2} \left[ (R + h_P)^3 - R^3 \right] \; , \qquad (94)$$

$$\frac{1}{(R + h_P)^2} = \frac{1}{R^2} \; \frac{1}{(1 + \frac{h_P}{R})^2} =$$

$$= \frac{1}{R^2} \left[ 1 - 2 \frac{h_P}{R} + 3 \left( \frac{h_P}{R} \right)^2 - + \ldots \right] , \qquad (95)$$

$$(R + h_P)^3 - R^3 = R^3 + 3 R^2 h_P + 3 R h_P^2 + h_P^3 - R^3 \; , \qquad (96)$$

$$(R + h_P)^3 - R^3 = R^3 \left[ 3 \frac{h_P}{R} + 3 \left( \frac{h_P}{R} \right)^2 + \left( \frac{h_P}{R} \right)^3 \right] , \qquad (97)$$

$$A = - \frac{4}{3} \pi f \varrho \; R \left[ 1 - 2 \frac{h_P}{R} + 3 \left( \frac{h_P}{R} \right)^2 \right] \cdot$$

$$\cdot 3 \frac{h_P}{R} \left[ 1 + \frac{h_P}{R} + \frac{1}{3} \left( \frac{h_P}{R} \right)^2 \right] , \qquad (98)$$

$$A = - 4 \pi \; f \varrho \, h_P \left[ 1 - \frac{h_P}{R} + \frac{4}{3} \left( \frac{h_P}{R} \right)^2 \right] \qquad (99)$$

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The oombination of (87) and (92) gives,

$$\frac{\partial B}{\partial r} = A + f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r'=R+h_P}^{R+h_Q} \left[\frac{\partial 1/\varepsilon}{\partial r}\right]_P r'^2 \, dr' \, d\omega . \quad (100)$$

Further on, (100) and (89) are introduoed into (91). Thus, the subsequent relation is obtained, (43),

$$\Xi_2 = - 4\pi f \varrho h_P \left[1 - \frac{h_P}{R} + \frac{4}{3}\left(\frac{h_P}{R}\right)^2\right] +$$

$$+ f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r'=R+h_P}^{R+h_P+Z} \left[\frac{\partial 1/\varepsilon}{\partial r}\right]_P r'^2 \, dr' \, d\omega +$$

$$+ 2\pi f \varrho h_P - f \varrho R^2 \iint_{\omega} h_Q \frac{\partial 1/e_o}{\partial r} \, d\omega . \quad (101)$$

In the last term, E, on the right hand side of (101), $h_Q$ is to be replaoed by $h_P$ and Z, (43). Hence,

$$E = F - f\varrho R^2 \iint_{\omega} Z \frac{\partial 1/e_o}{\partial r} \, d\omega , \quad (102)$$

$$F = - f \varrho R^2 h_P \iint_{\omega} \frac{\partial 1/e_o}{\partial r} \, d\omega . \quad (103)$$

The computation of the integral of (103) includes the computation of the oblique distanoe $e^*$ , (90). The one of the two end points of this straight line $e^*$ - the lower - is situated in the level of the globe with the radius R, (Fig. 2, (7) (8) (11) (90)), and the other end point - the upper - is the point P, lying in a certain height above the globe.

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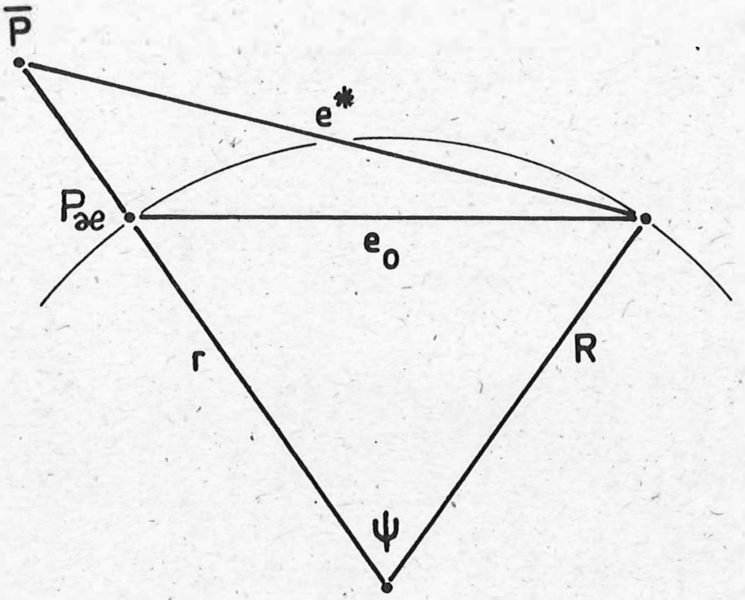


Fig. 2: Geometrical relations about the oblique distance $e^*$ .

The subsequent deductions are self-explanatory, Fig. 2,

$$e^{*2} = r^2 + R^2 - 2\,rR\cos\psi \quad , \tag{104}$$

$$2\,e^*\,de^* = 2\,rdr - 2\,R\cos\psi \cdot dr, \tag{105}$$

$$r \longrightarrow R \quad , \tag{106}$$

$$e_o\,de_o = R\,(1 - \cos\psi)\,dr, \tag{107}$$

$$\frac{\partial e_o}{\partial r} = \frac{1}{e_o}\,R\,(1 - \cos\psi) \quad , \tag{108}$$

$$\frac{\partial e_o}{\partial r} = \frac{1}{e_o}\,2\,R\sin^2\frac{\psi}{2} \quad , \tag{109}$$

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$$\frac{\partial e_o}{\partial r} = \frac{1}{2\,R}\, e_o \,, \tag{110}$$

$$\frac{\partial \frac{1}{e_o}}{\partial r} = -\frac{1}{e_o^2}\frac{\partial e_o}{\partial r} = -\frac{1}{2\,Re_o} = -\frac{1}{4\,R^2 \sin\frac{\psi}{2}} \,. \tag{111}$$

$$\iint_\omega \frac{\partial 1/e_o}{\partial r}\, d\omega = -\frac{1}{4\,R^2}\int_{\alpha=0}^{2\pi}\int_{\psi=0}^{\pi}\frac{1}{\sin\frac{\psi}{2}}\, d\omega =$$

$$= -\frac{1}{4\,R^2}\int_{\alpha=0}^{2\pi} d\alpha \int_{\psi=0}^{\pi}\frac{1}{\sin\frac{\psi}{2}}\sin\psi\; d\psi =$$

$$= -\frac{1}{4\,R^2}\cdot 2\pi \int_{\psi=0}^{\pi} 2\cos\frac{\psi}{2}\, d\psi = -\frac{1}{4\,R^2}\cdot 2\pi \cdot 4 \,. \tag{112}$$

(112) is introduced into (103), giving

$$F = 2\,\pi\, f\, \varrho\, h_P \,. \tag{113}$$

Hence, the last term E on the right hand side of (101)turns to

$$E = 2\,\pi\, f\,\varrho\, h_P - f\,\varrho\, R^2 \iint_\omega Z\,\frac{\partial\, 1/e_o}{\partial\, r}\, d\omega \,. \tag{114}$$

(101), (111) and (114) are combined to

$$\Xi_2 = f\varrho \int_{\psi=0}^{\pi}\int_{\alpha=0}^{2\pi}\int_{r'=R+h_P}^{R+h_P+Z}\left[\frac{\partial\, 1/\varepsilon}{\partial\, r}\right]_P r'^2\, dr'd\omega \; +$$

$$+\frac{1}{2} f\varrho\, R \iint_\omega Z\,\frac{1}{e_o}\, d\omega + 4\,\pi\, f\,\varrho\, h_P\,\frac{h_P}{R} \,. \tag{115}$$

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In (115), the quadratic term in the brackets of the first expression on the right hand side of (101),

$$- 4 \pi f \rho h_P \cdot \frac{4}{3} \left(\frac{h_P}{R}\right)^2 , \qquad (116)$$

is omitted. For $h_P$ = 3000 m, (116) contributes by not more than 0.2 μgal.

The next step is the development of the term

$$\left[\frac{\partial 1/\varepsilon}{\partial r}\right]_P , \qquad (117)$$

appearing in the first integrand of (115).

The vector $\underline{\underline{\varepsilon}}$ points from the test point P to the running point $\bar{Q}$ which has the height z above the sphere $\varkappa_P$ of the test point P. The amount of $\underline{\underline{\varepsilon}}$ is $\varepsilon$ . The vector $\underline{\underline{\varepsilon}}$ can be expressed by the orthogonal base vectors $\underline{\underline{e}}_1$ and $\underline{\underline{e}}_2$, Fig. 1, Fig. 3.

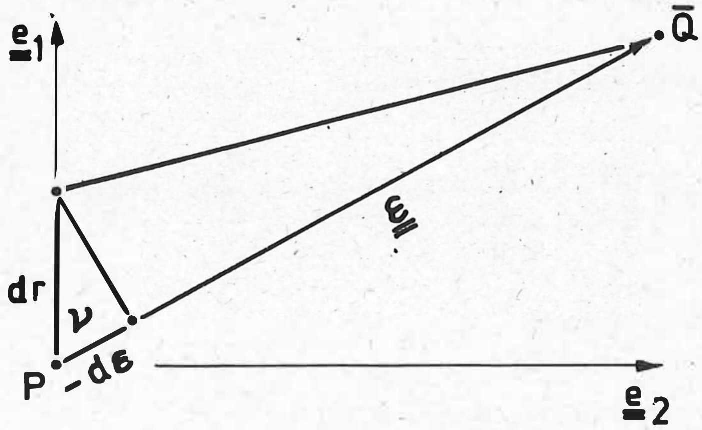


Fig. 3: The differentiation of $\varepsilon$ .

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The differential quotient (117) leads to

$$\frac{\partial}{\partial r}\frac{1}{\varepsilon} = -\frac{1}{\varepsilon^2}\frac{\partial\varepsilon}{\partial r} \quad . \tag{118}$$

A look on Fig. 3 shows that

$$\frac{\partial\varepsilon}{\partial r} = -\cos\gamma = -\frac{\underline{\underline{\varepsilon}}\,\underline{\underline{e}}_1}{\varepsilon} \quad . \tag{119}$$

Thus,

$$\frac{\partial}{\partial r}\frac{1}{\varepsilon} = \frac{\underline{\underline{\varepsilon}}\,\underline{\underline{e}}_1}{\varepsilon^3} \quad . \tag{120}$$

According to Fig. 1 and 3, the vector $\underline{\underline{\varepsilon}}$ has the following expression,

$$\underline{\underline{\varepsilon}} = \Big[(R + h_P + z)\cos\psi - (R + h_P)\Big]\,\underline{\underline{e}}_1 + $$

$$+ (R + h_P + z)\sin\psi \cdot \underline{\underline{e}}_2 \; . \tag{121}$$

The length $\varepsilon$ of the vector $\underline{\underline{\varepsilon}}$ is already represented by (19). $D_1$ and $D_2$ come from (23) and (24). But in this context, the chord $e_o$ situated in the sea level is replaced now by the chord e' situated in the level of the test point P

$$e' = 2\,(R + h_P)\sin\frac{\psi}{2} \quad . \tag{122}$$

The equations (7) and (122) relate $e_o$ and e',

$$e_o^{\,2} = e'^2\,\frac{R^2}{(R + h_P)^2} = e'^2\,\frac{1}{(1 + \frac{h_P}{R})^2} \cong e'^2\,(1 - 2\,\frac{h_P}{R}) . \tag{123}$$

The relation (22) and (123) lead to

$$\varepsilon^2 = \Big[e'^2\,(1 - 2\,\frac{h_P}{R}) + z^2\Big](1 + D_1 + D_2) \cong$$

$$= (e'^2 + z^2 - 2\,e_o^{\,2}\,\frac{h_P}{R})\,(1 + D_1 + D_2) ,$$

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$$\varepsilon^2 = \left[ e'^2 + z^2 - \frac{e_o^2}{e'^2 + z^2} (e'^2 + z^2)\, 2 \frac{h_P}{R} \right] (1 + D_1 + D_2) =$$

$$= (e'^2 + z^2)\,(1 - 2 \frac{e_o^2}{e'^2 + z^2} \frac{h_P}{R} + D_1 + D_2) \quad . \qquad (124)$$

With the $D_1$ and $D_2$ terms of (23) and (24), the development (124) reveals,

$$\varepsilon^2 = (e'^2 + z^2)\,(1 + D_3) \quad , \qquad (125)$$

$$D_3 = \frac{e_o^2}{e_o^2 + z^2} \cdot \frac{z}{R} \quad . \qquad (126)$$

Within the neglections connected with (23) - i. e. as long as relative errors of about $10^{-7}$ in the distances between the control points are neglected - the amount of $D_3$ is equal to that of $D_1$, (23) (126). Because of the inequation

$$| D_3 | \ll 1 \quad , \qquad (127)$$

it is admitted to have binomial series developments for the powers of

$$1 + D_3 \quad . \qquad (128)$$

Returning back to the relation (120), the denominator on the right hand side of (120) can be substituted by

$$\varepsilon^{-3} = (e'^2 + z^2)^{-3/2}\,(1 - \frac{3}{2} D_3) \quad . \qquad (129)$$

The nominator of the right hand side of (120) is obtained by the scalar multiplication of (121) with $\underline{e}_1$,

$$\underline{\varepsilon}\, \underline{e}_1 = (R + h_P + z) \cos\psi - (R + h_P) \quad . \qquad (130)$$

(130) and (129) are substitutes for the nominator and denominator of (120),

$$\frac{\partial}{\partial r} \frac{1}{\varepsilon} = \frac{(R + h_P + z) \cos\psi - (R + h_P)}{(e'^2 + z^2)^{3/2}} (1 - \frac{3}{2} D_3) \quad . \qquad (131)$$

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The nominator of (131) is subjected to a certain rearrangment, (11),

$$(R + h_P + z) \cos\psi - (R + h_P) =$$

$$(R + h_P)\,(1 - 2 \sin^2 \frac{\psi}{2}) - (R + h_P) + z\,(1 - 2 \sin^2 \frac{\psi}{2}) =$$

$$= z - 2\,(R + h_P + z) \sin^2 \frac{\psi}{2} \quad . \qquad (132)$$

With

$$K = R + h_P \quad , \qquad (133)$$

$$r' = K + z \quad , \qquad (134)$$

and regarding (131) (132), the expression (115) gets the following shape,

$$\Xi_2 = f\varrho \iint\limits_{\omega} d\omega \int\limits_{z=0}^{Z} (K + z)^2 \frac{z - 2\,(K + z) \sin^2 \frac{\psi}{2}}{(e'^2 + z^2)^{3/2}} \,(1 - \frac{3}{2} D_3)\, dz \; +$$

$$+ \frac{1}{2} f \varrho\, R \iint\limits_{\omega} Z \frac{1}{e_o}\, d\omega + 4\pi f \varrho\, h_P \frac{h_P}{R} \quad . \qquad (135)$$

$D_3$ is characterized by (126) and (127).

For the further deductions, the term $\Xi_2$ is compared with the plane topographic reduction of the gravity, C; [1] [3] [4]. C can be expressed by the following formula,

$$C = f\varrho \int\limits_{\psi=0}^{\beta} d\psi \int\limits_{\alpha=0}^{2\pi} d\alpha \cdot \psi \cdot K^2 \int\limits_{z=0}^{Z} \frac{z\,dz}{(z^2 + K^2 \cdot \psi^2)^{3/2}} \quad . \qquad (136)$$

It is well-known about the computation of the plane topographic reduction, the integration over the $\psi$ parameter must not be extented up to $\psi = \pi$. The term $\beta$ in (136) is understood as a sufficient great upper bound for the $\psi$ parameter. Beyond of $\beta$, the integration by (136) has no perceptible impact on C. The plane surface element which is introduced in

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(136) has the shape, (133),

$$K^2 \psi \; d\alpha \; d\psi \quad , \qquad (137)$$

it is the surface element of a plane polar coordinate system.

The first integral on the right hand side of (135) has an integrand that can be divided into two parts. The first part is

$$\Xi_{2.1} = f\rho \int_{\psi=0}^{\pi} d\psi \int_{\alpha=0}^{2\pi} d\alpha \; \sin\psi \int_{z=0}^{Z} (K+z)^2 \frac{z}{(e'^2 + z^2)^{3/2}} (1 - \tfrac{3}{2} D_3) \, dz, \qquad (138)$$

with

$$d\omega = \sin\psi \; d\psi \; d\alpha \quad . \qquad (139)$$

In (136), the $\beta$ value will not be greater than about $1^o$. Therefore, a comparison of (136) and (138) shows that these two expressions have corresponding terms of similar amounts. The analogy is,

$$\psi \longrightarrow \sin\psi \quad , \qquad (140)$$

$$K^2 \longrightarrow (K+z)^2 \; , \qquad (141)$$

$$(z^2 + K^2 \psi^2)^{3/2} \longrightarrow (z^2 + e'^2)^{3/2} \quad , \qquad (142)$$

$$1 \longrightarrow (1 - \tfrac{3}{2} D_3) \quad . \qquad (143)$$

Obviously, the difference between the integrands of (138) and of (136) is much more small than the amounts of the individual integrands of (138) or (136). An impressive compensation effect does work if the difference

$$\Xi_{2.1} - C \qquad (144)$$

is treated. With

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$$L = \sin\psi \; (K + z)^2 \frac{z}{(z^2 + e'^2)^{3/2}} (1 - \frac{3}{2} D_3) -$$

$$- \psi K^2 \frac{z}{(z^2 + K^2 \psi^2)^{3/2}} , \qquad (145)$$

follows

$$\Xi_{2.1} - C = f\varrho \int_{\psi=0}^{'} d\psi \int_{\alpha=0}^{2\pi} d\alpha \int_{z=0}^{Z} L \quad dz \quad . \qquad (146)$$

The first and the seoond term on the right hand side of (145) paralyze each other nearly.

The combination of (135) (136) (138) (146) leads to

$$\Xi_2 - C = \Xi_{2.1} - C + \Xi_{2.2} \quad . \qquad (147)$$

$\Xi_{2.2}$ has the following expression,

$$\Xi_{2.2} = - f\varrho \iint_{\omega} d\omega \int_{z=0}^{Z} (K + z)^2 \frac{2\,(K + z) \sin^2 \frac{\psi}{2}}{(e'^2 + z^2)^{3/2}} (1 - \frac{3}{2} D_3)\, dz +$$

$$+ \frac{1}{2} f\varrho \; R \iint_{\omega} Z \frac{1}{e_o} d\omega + 4\pi f\varrho \; h_P \frac{h_P}{R} \quad . \qquad (148)$$

The first and the seoond term on the right hand side of (148) paralyse each other oonsiderably, too.

In this context, the integrand of the first term of (148) undergoes a rearrangement. The subsequent deduotions are self-explanatory, (7),

$$2\,(K + z) \sin^2 \frac{\psi}{2} = 2\,(K + z) \frac{e_o^2}{4\,R^2} =$$

$$= \frac{1}{e_o} \frac{1}{2} e_o^3 \,(K + z) \frac{1}{R^2} \quad . \qquad (149)$$

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Thus, (149) transforms the first integrand of (148) into the following form,

$$- \frac{1}{2} f \varrho R \frac{1}{e_o} \frac{(K + z)^3}{R^3} \frac{e_o^{\,3}}{(e'^2 + z^2)^{3/2}} (1 - \frac{3}{2} D_2)\, d\omega\, dz. \qquad (150)$$

The abbreviation

$$S = \int_{z=0}^{Z} \left[ 1 - \frac{(K + z)^3}{R^3} \frac{e_o^{\,3}}{(e'^2 + z^2)^{3/2}} (1 - \frac{3}{2} D_2) \right] dz, \qquad (151)$$

reveals, (148),

$$\Xi_{2.2} = \frac{1}{2} f \varrho R \iint_{\omega} \frac{1}{e_o} S \, d\omega + 4 \pi f \varrho h_P \frac{h_P}{R} . \qquad (152)$$

Finally, (146) (147) (152) lead to

$$\Xi_2 - C = f\varrho \int_{\psi=0}^{\pi} d\psi \int_{\alpha=0}^{2\pi} d\alpha \int_{z=0}^{Z} L \, dz +$$

$$+ \frac{1}{2} f \varrho R \iint_{\omega} \frac{1}{e_o} S \, d\omega + 4 \pi f \varrho h_P \frac{h_P}{R} . \qquad (153)$$

Finally, in the construction of the residual term of the solution of the geodetic boundary value problem according to [1] [2] [3], the term

$$\Xi_2 - C = \left[ \frac{\partial B}{\partial r} \right]'' - C \qquad (154)$$

has to be supplemented by the expression for

$$\frac{2}{R} [B]'' . \qquad (155)$$

Thus, the expression in the relevant residual term has the following accomplished form, (156) . This relevant residual term is obtained in the chapter B of [3] , equation ([illegible]5), by the integrand of the $\chi_4$ term, and it is treated also by the equations (89) ([illegible]7) of the same chapter of [3] .

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$$\left[\frac{\partial B}{\partial r}\right]'' + \frac{2}{R}\,[B]'' - C = \left[\frac{\partial B}{\partial r} + \frac{2}{R}\,B\right]'' - C =$$

$$= \Xi_2 - C + \frac{2}{R}\,[B]'' = \Xi_2 - C + 2\,f\varrho\,\frac{1}{R}\iint_{\omega}\sum_{i=1}^{5} V_i\,d\omega \;. \qquad (156)$$

The term

$$\Xi_2 - C \qquad (157)$$

comes from (153), and the $V_i$ terms from (68) to (73).

In [3], the numerical amounts of (157) and of

$$[B]'' \text{ and } \frac{2}{R}\,[B]'' \qquad (158)$$

are computed for some mountain models. They proved to be negligible; (see [3], chapter B, section 7.1. and 7.2.).

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## 3. The evaluation of the horizontal derivatives $\frac{\partial [B]''}{\partial x, y}$.

In the formula for the determination of the height anomalies $\zeta$, the residual term contains the value of $[B]''$, (1); (see also [1] [2] [3] and the equations (1) (2) of the chapter D of this publication). In a similar way, the residual term in the formula for he surface plumb-line deflections $\xi$ and $\eta$ involves the amount of, [1] [2] [3], (2a),

$$\Xi_3 = \frac{\partial [B]''}{\partial x, y} = \left(\frac{\partial \quad B}{\partial x, y}\right)_\sigma - \left(\frac{\partial B_{cond.}}{\partial \bar{x}, \bar{y}}\right)_\varkappa . \qquad (159)$$

At the surface test point P, dx and dy are the horizontal arc elements which are introduced for the differentiation in the south - north and in the west - east direction. The differentiation with regard to dx or dy necessitates that the concerned function which is to be differentiated has to be defined not only along the oblique surface of the Earth, but, further on, even along the parts of the horizontal plane lying near by the point P.The function B has to be known for the horizontal plane area of the vicinity of P; Fig. 4.

However, the differentiation with regard to $d\bar{x}$ and $d\bar{y}$ happens along the surface of the globe $\varkappa$, (159); Fig. 4.

Thus, $\Xi_3$ has the following expression, (4) (5), [1] [2] [3],

$$\Xi_3 = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \left[ \int_{r=R}^{R+h_P} + \int_{r=R+h_P}^{R+h_P+Z} \right] \left(\frac{\partial 1/\varepsilon}{\partial x, y}\right)_P r^2 \sin\psi \, dr d\psi \, d\alpha$$

$$- f\varrho \, R^2 \iint_\omega \left(\frac{\partial 1/e_o}{\partial \bar{x}, \bar{y}}\right)_{P \, \varkappa} (h_P + Z) \, d\omega \quad . \qquad (160)$$

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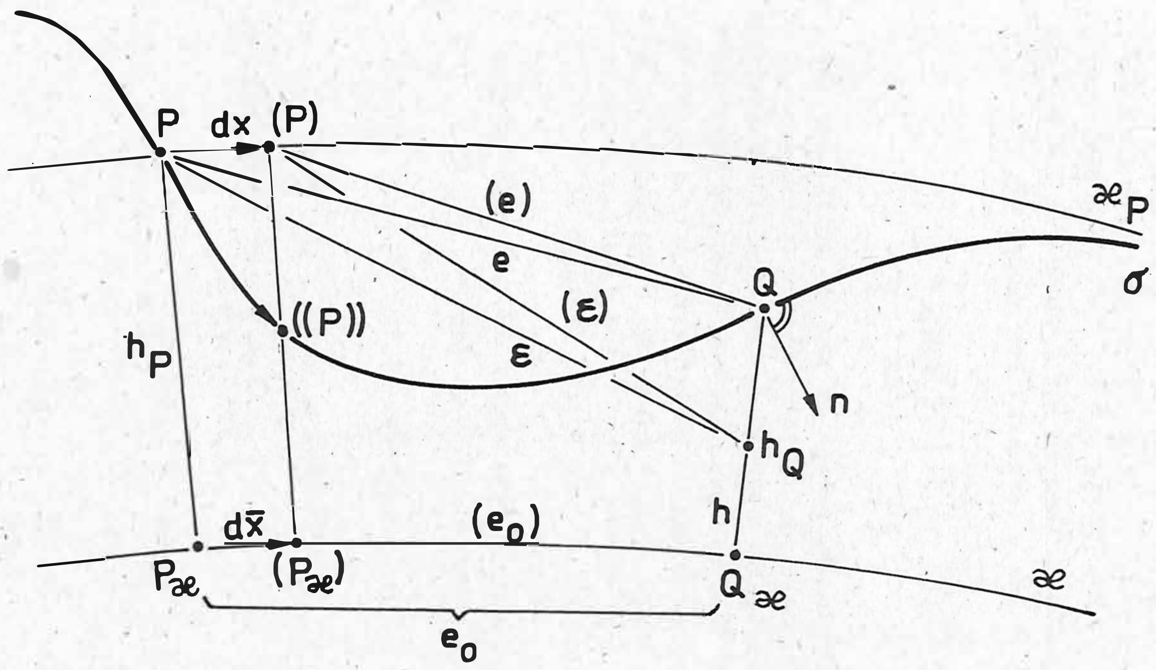


Fig. 4: The horizontal derivations of the straight distances e, $e_o$, ε.



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r is the radius of the running integration point.

In the first integral on the right hand side of (160), the integration over the radius r within the interval

$$R \leqslant r \leqslant (R + h_P) , \qquad (161)$$

leads to the expression for the horizontal components of the gravitation force of a homogeneous spherical shell with the width $h_P$. These components are well-known to be equal to zero, it is obvious.

An analogous property is found for the horizontal derivatives of the potential of a homogeneous spherical surface distribution. Therefore, [4], and with $h_P$ = const.,

$$0 = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r=R}^{R+h_P} \left[\frac{\partial 1/\varepsilon}{\partial x,\ y}\right]_P r^2 \sin\psi \, dr d\psi d\alpha \ , \qquad (162)$$

and

$$0 = f \varrho R^2 h_P \iint_{\omega} \left(\frac{\partial 1/e_o}{\partial \bar{x},\ \partial \bar{y}}\right)_{P_{\varkappa}} d\omega \ . \qquad (163)$$

For the computation of the horizontal derivations of a spherical surface distribution, (163), any special jump relation - which show up in case of the normal derivations - have not to be taken into account.

(162) and (163) transform (160) into the subsequent form,

$$\Xi_3 = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r=R+h_P}^{R+h_P+Z} \left[\frac{\partial 1/\varepsilon}{\partial x,\ y}\right]_P r^2 \sin\psi \, dr d\psi \, d\alpha \ -$$

$$- f \varrho R^2 \iint_{\omega} \left(\frac{\partial 1/e_o}{\partial \bar{x},\ \bar{y}}\right)_{P_{\varkappa}} Z \, d\omega \ . \qquad (164)$$

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The differential quotient with regard to the arc element dx or dy is found as the limit value of the concerned difference quotient,

$$\frac{\partial \varepsilon}{\partial x} = \lim \left( \frac{(\varepsilon) - \varepsilon}{|\overline{P\,,(P)}|} \right)_{\overline{P\,,(P)} \to 0} \quad . \tag{165}$$

The value $\varepsilon$ which figures in (165) has the following expression, (16),

$$\varepsilon^2 = e_o^2 + z^2 + e_o^2\, z\, \frac{1}{R} + e_o^2\, \frac{2\, h_P}{R} \quad . \tag{166}$$

In (166), a relative error of the order of $10^{-7}$ in the distances $\varepsilon$ is neglected, it is in keeping with the noise of the coordinates of the control points. In (166), $e_o^2$ is factored out,

$$\varepsilon^2 = z^2 + e_o^2 \left[ 1 + \frac{z}{R} + \frac{2\, h_P}{R} \right] . \tag{167}$$

An analogous formula is valid for $(\varepsilon)^2$, see Fig. 1 and Fig. 4.

$$(\varepsilon)^2 = z^2 + (e_o)^2 \left[ 1 + \frac{z + 2\, h_P}{R} \right] \quad . \tag{168}$$

The passage to the limit of (165) reveals,

$$(\varepsilon) = \varepsilon + \frac{\partial \varepsilon}{\partial x}\, dx \quad , \tag{169}$$

and, in a similar way, Fig. 4,

$$(e_o) = e_o + \frac{\partial e_o}{\partial \bar{x}}\, d\bar{x} \quad . \tag{170}$$

The difference of (167) and (168) gives,

$$(\varepsilon)^2 - \varepsilon^2 = \left[ (e_o)^2 - e_o^2 \right] \left[ 1 + \frac{z + 2\, h_P}{R} \right] . \tag{171}$$

The difference on the left hand side of (171) is expressed by the differential quotient of $\varepsilon$ , using (169).

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The difference on the right hand side of (171) is in relation to the differential quotient of $e_o$, see (170). The following lines are self-explanatory,

$$(\varepsilon)^2 - \varepsilon^2 = \left[(\varepsilon) + \varepsilon\right]\left[(\varepsilon) - \varepsilon\right] = \left[(\varepsilon) + \varepsilon\right]\frac{\partial \varepsilon}{\partial x}\,dx \cong$$

$$\cong 2\,\varepsilon\,\frac{\partial \varepsilon}{\partial x}\,dx \quad , \tag{172}$$

$$(e_o)^2 - e_o{}^2 = \left[(e_o) + e_o\right]\frac{\partial e_o}{\partial \bar{x}}\,d\bar{x} \cong$$

$$\cong 2\,e_o\,\frac{\partial e_o}{\partial \bar{x}}\,d\bar{x} \quad , \tag{173}$$

and, by combination with (171),

$$2\,\varepsilon\,\frac{\partial \varepsilon}{\partial x}\,dx = 2\,e_o\,\frac{\partial e_o}{\partial \bar{x}}\,d\bar{x}\left[1 + \frac{z + 2\,h_P}{R}\right] \quad , \tag{174}$$

and with Fig. 4,

$$dx^2 = (d\bar{x})^2\,(1 + 2\,\frac{h_P}{R}) \quad , \tag{175}$$

hence, (174) (175),

$$\frac{\partial \varepsilon}{\partial x}\,dx = \frac{\partial e_o}{\partial \bar{x}}\,\frac{e_o}{\varepsilon}\left[1 + \frac{z + h_P}{R}\right] dx \quad , \tag{176}$$

$$\frac{\partial \frac{1}{\varepsilon}}{\partial x} = -\frac{1}{\varepsilon^2}\,\frac{\partial \varepsilon}{\partial x} \quad . \tag{177}$$

$$\frac{\partial 1/\varepsilon}{\partial x} = -\frac{\partial e_o}{\partial \bar{x}}\,\frac{e_o}{\varepsilon^3}\left[1 + \frac{z + h_P}{R}\right] \quad , \tag{178}$$

$$\frac{\partial e_o}{\partial \bar{x}} = \frac{de_o}{d\psi}\,\frac{\partial \psi}{\partial \bar{x}} \quad . \tag{179}$$

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The infinitesimal spherical triangle of Fig. 5 and the relation (7) show that

$$\frac{de_o}{d\psi} = R \cos \frac{\psi}{2} \quad , \tag{180}$$

$$\frac{R \, \partial\psi}{\partial \bar{x}} = - \cos \alpha \quad , \tag{181}$$

$\alpha$ is the azimuth, counted clockwise from the north.

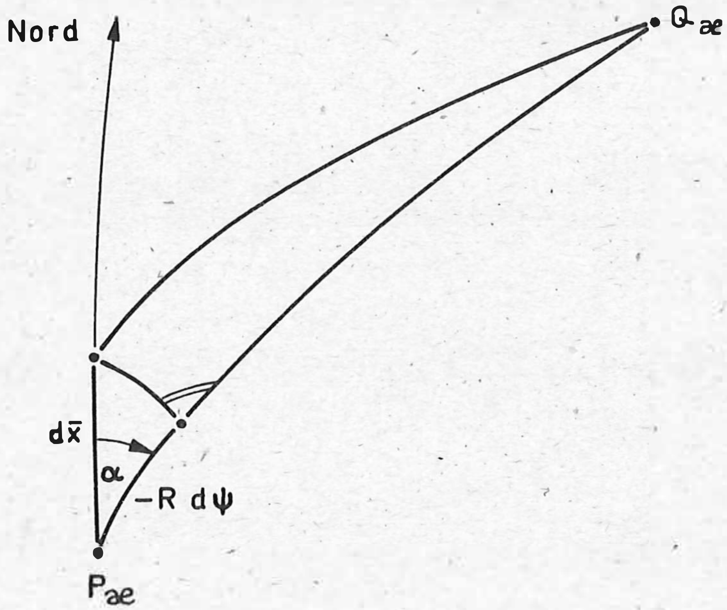


Fig. 5: The south-north derivative of the spherical distance $\psi$ . The infinitesimal spherical triangle is plotted on the globe with the radius R.

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Thus, from (179) (180) (181),

$$\frac{\partial e_o}{\partial \bar{x}} = -\cos\frac{\psi}{2}\cos\alpha \quad , \qquad (182)$$

and with (178),

$$\frac{\partial \frac{1}{\varepsilon}}{\partial x} = \cos\frac{\psi}{2}\cdot\cos\alpha\cdot\frac{e_o}{\varepsilon^3}\left[1+\frac{z+h_P}{R}\right] . \qquad (183)$$

Further on,

$$\left(\frac{\partial \frac{1}{e_o}}{\partial \bar{x},\bar{y}}\right)_{P_{æ}} = -\frac{1}{e_o^2}\frac{\partial e_o}{\partial \bar{x},\bar{y}} = \frac{1}{e_o^2}\cos\frac{\psi}{2}\begin{Bmatrix}\cos\alpha\\ \sin\alpha\end{Bmatrix} . \qquad (184)$$

(183) and (184) are introduced into (164),

$$\Xi_3 = f\varrho\int_{\psi=0}^{\pi}\int_{\alpha=0}^{2\pi}\int_{r=R+h_P}^{R+h_P+Z}\cos\frac{\psi}{2}\begin{Bmatrix}\cos\alpha\\ \sin\alpha\end{Bmatrix}\frac{e_o}{\varepsilon^3}\left[1+\frac{z+h_P}{R}\right]r^2\sin\psi\, dr\, d\psi\, d\alpha\ -$$

$$-f\varrho R^2\int_{\psi=0}^{\pi}\int_{\alpha=0}^{2\pi}\frac{1}{e_o^2}\cos\frac{\psi}{2}\cdot Z\cdot\begin{Bmatrix}\cos\alpha\\ \sin\alpha\end{Bmatrix}d\omega . \qquad (185)$$

The first integrand on the right hand side of (185) contains the product

$$\cos\frac{\psi}{2}\cdot e_o\cdot\sin\psi = \cos\frac{\psi}{2}\cdot 2R\sin\frac{\psi}{2}\cdot 2\sin\frac{\psi}{2}\cos\frac{\psi}{2} =$$

$$= 4R\sin^2\frac{\psi}{2}\cos^2\frac{\psi}{2} = R\sin^2\psi . \qquad (186)$$

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The formulas (22) up to (26) give,

$$\left(\frac{1}{\varepsilon}\right)^3 = (e_o{}^2 + z^2)^{-\frac{3}{2}} (1 - \tfrac{3}{2} D_4) \quad , \qquad (187)$$

$$D_4 = \frac{e_o{}^2}{e_o{}^2 + z^2} \cdot \frac{z + 2\, h_P}{R} \quad , \qquad (188)$$

$$| D_4 | \ll 1 \quad . \qquad (189)$$

With (133) (134), - if the prime at r is no more taken along -, follows,

$$r^2 = (K + z)^2 = K^2 \left(1 + \frac{z}{K}\right)^2 = K^2 \left(1 + 2\frac{z}{K}\right) = K^2 (1 + D_6) \quad , \qquad (190)$$

$$D_6 \cong 2 \frac{z}{R} \quad . \qquad (191)$$

A product : the first integrand of (185) follows to be equal to

$$\left(\frac{1}{\varepsilon}\right)^3 r^2 \left[1 + \frac{z + h_P}{R}\right] =$$

$$= \frac{K^2}{(e_o{}^2 + z^2)^{3/2}} \left[1 - \frac{3}{2} \cdot \frac{e_o{}^2}{e_o{}^2 + z^2} \cdot \frac{2\, h_P + z}{R} + \right.$$

$$\left. + 2 \frac{z}{R} + \frac{h_P + z}{R} \right] \quad . \qquad (192)$$

(185) (186) and (192) give,

$$\Xi_3 = f \varrho R \int_{\psi = 0}^{\pi} \sin^2\psi \; d\psi \int_{\alpha = 0}^{2\pi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} d\alpha \cdot X -$$

$$- f \varrho R^2 \int_{\psi = 0}^{\pi} \int_{\alpha = 0}^{2\pi} \frac{1}{e_o{}^2} \cos \frac{\psi}{2} \cdot Z \cdot \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} d\omega \quad . \qquad (193)$$

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$$X = \int_{r = R + h_P}^{R + h_P + Z} \left(\frac{1}{\varepsilon}\right)^3 \left[1 + \frac{z + h_P}{R}\right] r^2 \, dr \quad . \tag{194}$$

(192) is considered as a substitute for the integrand in (194). With

$$dr = dz \tag{195}$$

follows,

$$X = \int_{z = 0}^{Z} \frac{K^2}{(e_o^2 + z^2)^{3/2}} \left[1 - \frac{3}{2} \frac{e_o^2}{e_o^2 + z^2} \cdot \frac{2 h_P + z}{R} + \right.$$

$$\left. + \frac{2 z}{R} + \frac{h_P + z}{R}\right] dz. \tag{196}$$

Since it is intended to carry out the integrations on the right hand side of (196) along the lines of the analytical integrations, the integrand is developed into four terms which are identical with standard integrands which can be found in the integration tables,

$$X = \sum_{j = 1}^{4} X_j \quad , \tag{197}$$

$$X_1 = K^2 \left(1 + \frac{h_P}{R}\right) \int_{z = 0}^{Z} \frac{1}{(e_o^2 + z^2)^{3/2}} \, dz \quad . \tag{198}$$

(133) gives,

$$1 + \frac{h_P}{R} = \frac{K}{R} \quad , \tag{199}$$

thus,

$$X_1 = \frac{K^3}{R} \int_{z = 0}^{Z} \frac{1}{(e_o^2 + z^2)^{3/2}} \, dz \quad . \tag{200}$$

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$$X_2 = 3 \frac{K^2}{R} \int_{z=0}^{Z} \frac{z}{(z^2 + e_o^2)^{3/2}} dz \quad . \tag{201}$$

$$X_3 = -3 K^2 e_o^2 \frac{h_P}{R} \int_{z=0}^{Z} \frac{dz}{(z^2 + e_o^2)^{5/2}} \quad . \tag{202}$$

$$X_4 = -\frac{3}{2} K^2 e_o^2 \frac{1}{R} \int_{z=0}^{Z} \frac{z}{(z^2 + e_o^2)^{5/2}} dz \quad . \tag{203}$$

The second integral on the right hand side of (193) undergoes some self-explanatory rearrangements. It is,

$$\frac{1}{e_o^2} \cos \frac{\psi}{2} \sin \psi = \frac{1}{e_o^3} e_o \cos \frac{\psi}{2} \sin \psi =$$

$$= \frac{1}{e_o^3} 2 R \sin \frac{\psi}{2} \cos \frac{\psi}{2} \sin \psi = R \frac{1}{e_o^3} \sin^2 \psi \quad . \tag{204}$$

With (204), the second integral of (193) turns to

$$f \varrho R \int_{\psi=0}^{\pi} \sin^2\psi \, d\psi \int_{\alpha=0}^{2\pi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} d\alpha \left[ - R^2 \frac{1}{e_o^3} Z \right] \quad . \tag{205}$$

A comparison of (193) (197) and (205) reveals that a further function $X_5$ can be defined,

$$X_5 = - R^2 \frac{1}{e_o^3} Z \quad . \tag{206}$$

Hence, (193)-

$$\Xi_3 = f \varrho R \int_{\psi=0}^{\pi} \sin^2\psi \, d\psi \int_{\alpha=0}^{2\pi} \sum_{j=1}^{5} X_j \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} d\alpha \quad . \tag{207}$$

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$X_5$ has a satisfactory expression, (206). For $X_j$, (j = 1, 2, 3, 4), a look on the integration tables indicates, within the here introduced approximations,

$$X_1 = \frac{K^3}{R} \cdot \frac{Z}{e_o^2 \, (e_o^2 + Z^2)^{1/2}} \quad , \qquad (208)$$

$$X_2 = 3 \, R \left[ \frac{1}{e_o} - \frac{1}{(e_o^2 + Z^2)^{1/2}} \right] , \qquad (209)$$

$$X_3 = - \, 3 \, R \, h_P \left[ Z + \frac{2}{3} \frac{Z^3}{e_o^2} \right] \frac{1}{(e_o^2 + Z^2)^{3/2}} \quad , \qquad (210)$$

$$X_4 = \frac{1}{2} \, R \, e_o^2 \left[ \frac{1}{(e_o^2 + Z^2)^{3/2}} - \frac{1}{e_o^3} \right] . \qquad (211)$$

It is important that a comparison of $X_1$ and $X_5$, (208) and (206), reveals that these terms compensate and paralyze each other, more or less, as long as $e_o$ is much more great than the absolute amount of Z.

The formulas (207), (208) to (211), (206) represent a solution for

$$\frac{\partial \, [B]''}{\partial \, x, \, y} \quad , \qquad (212)$$

(see (159) ), which is convenient for an application in the numerical computations.

The publication [3] contains a numerical estimation of the amounts of $X_1$, $X_2$, $X_3$, $X_4$, $X_5$, and, thus, also an evaluation of the amount of (212), the above equation; (see [3], chapter B, equation (118) to (122)). The first model mountain has the following parameters: $h_P = 0$, Z = 2 km, the base surface of the mountain is equal to $R^2 \, \Delta\omega = (4 \text{ km})^2$, the distance between the test point

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and the center of the mountain is $R\psi = e_o = 5$ km. The computations resulted

$$\Xi_3 \cdot \frac{\rho''}{G} = \frac{\partial[B]''}{\partial x} \cdot \frac{\rho''}{G} = 0.''3 \quad . \tag{212a}$$

The direction from the test point to the center of the model mountain has the azimuth $\alpha = 0^o$. Consequently, in extreme situations in the midst of the high mountains, it is possible that the amount of

$$\Xi_3 \cdot \frac{\rho''}{G} \tag{212b}$$

effects the amounts of the plumb-line deflections computed from the free-air anomalies by more than 0."1. In such a situation, the term (212b) has to be included into the expression for the deflections, in order to complete the theory; (see [3], chapter B, equation (66)):

$$\left.\begin{matrix}\xi\\ \eta\end{matrix}\right\} = \frac{1}{4\pi\gamma} \iint\limits_{\omega} \left[\Delta g_T + C\right] \frac{dS}{d\psi} \left\{\begin{matrix}\cos\alpha\\ \sin\alpha\end{matrix}\right\} d\omega \; - \; -\frac{1}{\gamma} \cdot \sum_{i=4,6,7,8} \frac{\partial}{\partial x, \; \partial y} \Big| \chi_i \; , \tag{212c}$$

with

$$\frac{\partial}{\partial x, \; \partial y} \chi_4 \cdot \frac{\rho''}{G} = \Xi_3 \cdot \frac{\rho''}{G} \; , \tag{212d}$$

being equal to, (212),

$$\frac{\partial[B]''}{\partial x, \; y} \cdot \frac{\rho''}{G} \quad . \tag{212e}$$

The second model mountain has the parameters: $h_P = 0$, $Z = 3$ km, $R^2 \Delta\omega = (40 \text{ km})^2$, $R\psi = e_o = 100$ km. These values lead to an

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amount of not more than

$$\Xi_3 \cdot \frac{\rho''}{G} = \frac{\partial [R]''}{\partial x} \cdot \frac{\rho''}{G} = 3'' \cdot 10^{-4} \quad , \qquad (212f)$$

for $\alpha = 0^{\circ}$. Thus, it can be taken for granted that the effect exerted on (212) (212b) by distant mountains is insignifioant.

## 4. The residual terms for the fixed and for the free boundary value problem.

As to the basing theoretical conception that is behind the here discussed boundary value problem, it must be stated that the heights $h_P$ and $h_Q$ are the heights above the sphere; or, considering the flattening of the Earth, they are the heights above the ellipsoid, (see Fig. 1). Thus, if the here introduced h values are considered to be a priori given values, the basing conception has the character of a boundary value problem for the real surface of the Earth. In case, the h values are definitely known, the problem has the charaoter of a fixed boundary value problem.

But in reality, the h values - the heights above the sphere - are not known, a priori. The reason is that the height anomalies $\zeta$ are the unknown values of the problem. The heights above the sphere, h, consist of the sum of the known normal heights, h', and the a priori unknown height anomalies $\zeta$ ,

$$h = h' + \zeta \quad . \qquad (213)$$

The value of $\zeta$ is unknown, a priori; it is the value to be determined. Therefore, the shape of the surface of the Earth, being the boundary surfaoe, is also unknown a priori - at least within the limits of the amounts of $\zeta$ -. Consequently, the problem presents itself in our applioations by the character of a free boundary value problem.

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However, this free boundary value problem - for the free-air anomalies $\Delta g_T$ as boundary values along the unknown surface of the Earth - is placed in the very near vicinity of the fixed boundary value problem for the telluroid as given boundary surface,-for the free-air anomalies as boundary values, too. This close vicinity is founded on the fact that the surface of the Earth and of the telluroid differ by relative small and smoothed vertical point shifts, being the height anomalies $\zeta$ .

Thus, the transition from the free boundary value problem (for the surface of the Earth) to the fixed boundary value problem (for the telluroid as boundary surfaoe) happens by the introduction of h' instead of h in the above derivations presented by the above equations, from (1) to (212).

The effect this transition takes on the final result is to be discussed now. The closed solution for the boundary value problem has the following shape, [1] [2] [3], (see also chapter D, equation (1), of this publication),

$$T = \frac{r_P}{4\pi\gamma} \iint_{\omega} \left[\Delta g_T + C + C_1\right] S(\psi)\, d\omega + \Xi_1 \quad . \qquad (214)$$

The free-air anomalies $\Delta g_T$ do not change if the boundary surface changes from the surface of the Earth to the telluroid, i. e. if h changes to h'. In the mathematical developments of [1] [2] [3], the surface of the Earth was introduced as the boundary surface shaped by the heights h above the sphere. Hence, the supplementary terms C, $C_1$, $\Xi_1$ of (214) are understood to be expressed by these h values. Consequently, these supplementary terms C, $C_1$, $\Xi_1$ will be effected by the transition from the h values to the h' values.

In this context, in the computation of the plane topographic reduction of the gravity C, (being a term which has to be added to the free-air anomalies in the Stokes integral (214)), h' has to be introduced instead of h. But, there is no doubt, the interchange of h with h' by

$$h \cong h' \qquad (215)$$

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takes no perceptible impact on the C values. Of course, the precise version of (215) is (213). Since the C value depends on the differences of the heights in the surroundings of about 100 km radius around the test point, (136), therefore, the disregard of the differences of the $\zeta$ values in these surroundings will effect the error in the C value. This error is caused by the interchange of h with h', in (136). The $\zeta$ values are smoothed. In the mountains, the differences between neighboring $\zeta$ values of a mutual distance of not more than 100 km are witnin about 10 m. And this value is often within the noise of the plotted heights in the maps. Thus, the interchange of the h values with the h' values has no perceptible effect on the oomputations of C according to (136) and (153), since even the height differences and the differences of the $\zeta$ values are effective here, only. The transition from the h values to the h' values changes the C values by not more than some $\mu$gal, as an uncomplicated computation does show. This value can be neglected.

Further on, in the constituent $\Xi_1$ of the residual term of the solution of the geodetic boundary value problem, (214), a certain term appears that exerts a change of the resulting perturbation potential T by an amount of the form of

$$\frac{h_P}{R} B_c \quad , \tag{216}$$

(see /1/ /2/ /3/, and the equation (2) of the chapter D of this publication). Consequently, if h' is taken as a substitute for h, the final T value undergoes a change by an error of

$$\frac{\zeta}{R} B_c \quad . \tag{217}$$

The corresponding error effect in the final $\zeta$ value is obtained by a division of (217) through the mean global gravity value. Thus, the resulting error in the $\zeta$ value will not surpass a centimeter. It can be neglected in most cases (e.g. $\zeta$=100 m, $B_c$/G=600 m).

The exchange of h with h' influences also the $r_P$ value which appears before the integral of (214). Also this fact will effect the final T value by a negligible error which is of the order of

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the amount of the term (217). Here, the term (217) is replaced by $\zeta \cdot 1/R \cdot T$.

Obviously, the here discussed transition from h to h' changes also the term $C_1$, (214), (see also the equation (11) of the chapter D of this publication). Since the amount of $C_1$ is not greater than about 1 mgal, the difference

$$- \frac{\partial \Delta g_{Bouguer}}{\partial r} (h_Q - h_P) - \frac{\partial \Delta g_{Bouguer}}{\partial r} (h'_Q - h'_P) \qquad (218)$$

which yields to be effective here will always be by far a negligible term. It is obvious.

Principally, the three values relating to the real surface of the Earth

$$r_P,\ h_P,\ h_Q \qquad (219)$$

which appear in the supplementary terms on the right hand side of (214) can be computed by a further iteration step. This approach is equivalent to the procedure to compute the $\zeta$ value on the right hand side of the equation for h, (213), by an iteration process. For a first step, the $\zeta$ values in the formula (213) can be taken from the global maps of the $\zeta$ values already obtained by cosmic and terrestrial methods. This first step of this procedure will lead to an approximation of the h values that is better than the replacement of h by h'.

Thus, in the geodetic applications, it makes no difference whether the real surface of the Earth or the telluroid is introduced as the boundary surface. There is no essential difference between the fixed and the free boundary value problem. The free boundary value problem can be replaced by the fixed one and vice versa, in our applications at all events.

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C. The solution of the first mixed boundary value problem of the geodesy as an optimal method for the computation of the altimetry - gravimetry problem.

Contents



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Summary

Within the course of the computation of the solution of the altimetry - gravimetry problem, it is possible to account for the altimetry data on the oceans by a two-step-method: At first, the oceanic gravity anomalies are computed from the observed altimeter data by the inverse Stokes equation. Then, the height anomalies are obtained from the free-air anomalies, integrating the Stokes integral over whole the globe.

On the other hand, considering the equations that determine the solution of the first mixed boundary value problem, it is possible to express the height anomalies directly in terms of the oceanic altimeter data and of the continental free-air anomalies. This is the direct way.

The results of the direct way are more precise than those obtained by the indirect method, as numerical computations show.

The biases caused by the sea surface topography can be eliminated by an adaptation of the resulting height anomalies to some Doppler-derived height anomalies.



Zusammenfassung

Die ozeanischen Altimeterdaten können bei der Bestimmung des globalen Potentialfeldes in verschiedener Weise herangezogen werden. Mit der inversen Stokes-schen Gleichung kann man zunächst die ozeanischen Freiluftanomalien der Schwere errechnen. Aus dem damit bekannten globalen Feld dieser Anomalien erhält man mit dem Stokes-schen Integral das globale Potentialfeld.

Die Lösung des ersten gemischten Randwertproblems führt dagegen direkt von den Altimeterdaten zu dem globalen Potentialfeld. Es zeigt sich, dass dieser direkte Weg genauere Ergebnisse bringt als der indirekte.

Die Topographie des Meeres ruft kleine systematische Fehler in dem erhaltenen Potentialfeld hervor. Diese Fehler können durch Anfelderung an solche Höhenanomalien bestimmt werden, die aus Dopplerbeobachtungen von Satelliten gewonnen wurden.


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Резюме

Океанические данные альтиметрии могут использоваться в различных вариантах при определении глобального потенциального поля. С помощью инверсного равенства Штока можно сначала рассчитать океанические аномалии гравитации воздуха на открытой местности. Из поля этих аномалий, известного таким образом через интеграл Штока получают глобальное потенциальное поле.

Решение же первой смешанной геодезической краевой задачи ведет, напротив, непосредственно от данных альтиметрии к значению глобального потенциального поля. Оказалось, что этот прямой путь дает более точные результаты, чем косвенный.

Топография моря вызывает систематические ошибки в полученном потенциальном поле. Эти ошибки могут определяться путем наложения таких аномалий высот, которые получают из эффекта Доплера наблюдаемого со спутников.

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## 1. Introduction.

The here discussed problem does base on an observational material of the mixed type. The boundary values on the oceans are the altimeter data. Along the continents, the free-air anomalies of the gravity serve as the empirically given data.

At the beginning of the considerations, the sea surface topography is neglected. The impact the sea surface topography takes on the result is later discussed, at the end of the investigations.

Starting from the above described observational data, the problem consists in the construction of a global representation of the gravity field or of the potential field. These final field data have to be determined as precise as possible, [1] [2] [3] [6] [9] [10] [12] [13] [14].

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2. The indirect solution by the inverse Stokes equation.

The inversion of the fundamental differential equation of the physical geodesy, [6] [9],

$$-\frac{\partial T}{\partial r} - \frac{2}{r} T = \Delta g_F \quad , \tag{1}$$

leads to an expression that gives the height anomalies $\zeta$ in terms of the free-air anomalies $\Delta g_F$ of the gravity, (see also chapter B, equation (214), of this publication),

$$\zeta = \frac{r_P}{4\pi \gamma} \iint_{\omega} \left[ \Delta g_F + C \right] S(\psi) \; d\omega + \varepsilon \quad . \tag{2}$$

T is the surface perturbation potential, r is the geocentric radius, $\omega$ is the unit sphere, $S(\psi)$ the Stokes function, $\psi$ is the spherical distance between the test point and the point moving within the integration procedure. C is the plane terrain correction of the gravity, R is the mean radius of the Earth. $\gamma$ is the standard gravity at the surface test point P. The interdependence of T, $\zeta$ , $\gamma$ is expressed by

$$\zeta = \frac{T}{\gamma} = \frac{(T)_P}{\gamma_P} \quad . \tag{3}$$

The residual term $\varepsilon$ will rare reach more than about some centimeters, it can be neglected in view of the present standard of the gravity nets, [5] [6]. $\varepsilon$ is in the vicinity of the height gradient of the Bouguer anomalies, (see chapter D of this publication).

$$\varepsilon \cong - \frac{R}{4\pi \gamma} \iint_{\omega} (h_Q - h_P) \frac{\partial \Delta g_{Bouguer}}{\partial h} S(\psi) \; d\omega \quad , \tag{4}$$

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$h_Q$ is the height of the point moving within the integration, and $h_P$ is the height of the test point, [5] [6].

For the fixation of the subsequent ideas, it is recommended to transform the relation (2) into the matrix shape. In this context, the whole surface of the Earth is divided into a number X of finite surface elements of the constant size $\Delta \omega$ . Thus, the relation (2) turns to

$$\underline{\underline{z}} = \underline{\underline{S}}\,\underline{\underline{g}} \tag{5}$$

The mean values of $\zeta$ for the finite surface elements are the components of the vector $\underline{\underline{z}}$,

$$\underline{\underline{z}} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \cdots \\ \zeta_u \\ \cdots \\ \zeta_X \end{pmatrix} . \tag{6}$$

In an analogous way, the vector $\underline{\underline{g}}$ refers to the ( $\Delta g_F$ + C) values,

$$\underline{\underline{g}} = \begin{pmatrix} (\Delta g_F + C)_1 \\ (\Delta g_F + C)_2 \\ \cdots \\ (\Delta g_F + C)_u \\ \cdots \\ (\Delta g_F + C)_X \end{pmatrix} , \tag{7}$$

$$u = 1, 2, \ldots, X. \tag{8}$$

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In the relation (5), the meaning of the matrix $\underline{\underline{S}}$ derives from (2), it stands for the kernel function

$$\frac{R}{4\pi\gamma} \, S(\psi) \cdot \Delta\omega \quad , \tag{9}$$

in a self-explanatory way. The $\varepsilon$ term of (2) is neglected in (5).

The two vectors $\underline{\underline{z}}$ and $\underline{\underline{g}}$ consist of continental and oceanic components. Thus, they can be devided into a continental part and into an oceanic one, /3/ /6/, (5),

$$\underline{\underline{z}} = \begin{pmatrix} \underline{\underline{z}}_c \\ \underline{\underline{z}}_s \end{pmatrix} \quad , \tag{10}$$

$$\underline{\underline{g}} = \begin{pmatrix} \underline{\underline{g}}_c \\ \underline{\underline{g}}_s \end{pmatrix} \quad . \tag{11}$$

The matrix $\underline{\underline{S}}$ can be devided in a similar way, (5),

$$\underline{\underline{S}} = \begin{pmatrix} \underline{\underline{S}}_{c,c} & , & \underline{\underline{S}}_{c,s} \\ \underline{\underline{S}}_{s,c} & , & \underline{\underline{S}}_{s,s} \end{pmatrix} \quad , \tag{12}$$

or

$$\underline{\underline{S}} = \begin{pmatrix} \underline{\underline{S}}_c \\ \underline{\underline{S}}_s \end{pmatrix} \tag{13}$$

$\underline{\underline{z}}_s$ in (10) and $\underline{\underline{g}}_c$ in (11) are the given data. $\underline{\underline{z}}_c$ and $\underline{\underline{g}}_s$ are the unknown data, they can be determined along the subsequent developments.

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In the sub-space of the spherical hermonics of the 2nd and higher degree, the inversion of (5) is well-defined, /3/ /9/ /11/,

$$\underline{\underline{g}} = \underline{\underline{S}}^{-1} \; \underline{\underline{z}} \quad . \tag{14}$$

The matrix relation (14) represents the following integral relation,

$$\Delta g + C = \iint_{\omega} \left[ \zeta_Q - \zeta_P \right] \; S^{-1}(\psi) \quad d\omega \quad . \tag{15}$$

The function $S^{-1}$ depends on $\psi$ , it is the well-known inverse Stokes function. In the relation (15), $\zeta_Q$ is the $\zeta$ value for the running point Q, and $\zeta_P$ is the $\zeta$ value for the test point P which is fixed within the course of this integration. The explicit expression of (15) is, /9/,

$$\Delta g + C = - \frac{\gamma}{R} \; \zeta_P \; - \frac{\gamma}{2\pi} \; R^2 \iint_{\omega} \frac{\zeta_Q - \zeta_P}{l^3} \; d\omega \; , \tag{16}$$

$$l = 2 \; R \sin \frac{\psi}{2} \quad . \tag{17}$$

In the here discussed problem, the vector part $\underline{\underline{g}}_c$ is known. But, $\underline{\underline{g}}_s$ is an unknown vector. Therefore, the matrix relation (14) is applied to the oceanic test points only. With the symbolism

$$\underline{\underline{S}}^{-1} = \begin{pmatrix} \underline{\underline{S}}_c^{-1} \\ \\ \underline{\underline{S}}_s^{-1} \end{pmatrix} \quad , \tag{18}$$

and

$$S^{-1} = \begin{pmatrix} \underline{\underline{S}}_{c.c}^{-1} & , & \underline{\underline{S}}_{c.s}^{-1} \\ \\ \underline{\underline{S}}_{s.c}^{-1} & , & \underline{\underline{S}}_{s.s}^{-1} \end{pmatrix} \tag{19}$$

follows symbolically, (13) (14) (18),

$$\underline{\underline{g}}_s = \underline{\underline{S}}_s^{-1} \; \underline{\underline{z}} \quad . \tag{20}$$

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Along the lines of (16), it is possible to compute, point for point, the local values of the components of $\underline{g}_s$, i. e. the local values of ( $\Delta$g + C) along the oceans. Averaging over these local values situated within a certain compartment, these local values lead to the mean values of ( $\Delta$g + C) for the introduced compartments. The relations (15) (16) are well-known to be instable, since the amount of the kernel function of (16) increases enormously if the distance l to the test point diminishes.

The relations (16) and (20) permit to determine the mean oceanic values of ( $\Delta$g + C) of the compartments $\Delta\omega$. They have a standard error of about ± 2 to ± 5 mgal, if the compartments of 200 km x 200 km size are introduced, /7/ /11/, (See also: Rapp, R. H.; Detailed gravity anomalies and sea surface heights derived from GEOS - 3/ SEASAT altimeter data. Ohio State Univ., Dept. geod. Sci., Rep. 365 (1985)). Thus, for the subsequent model computations, it is allowed to introduce a standard error of

$$\mu = \pm 4 \text{ mgal} \tag{21}$$

for the here discussed average of the gravity anomalies of the $1^0$ x $1^0$ compartments; these gravity anomalies are viewed as computed by the inverse Stokes relation, (15) (16).

Now, the computation models are to be described. The ocean is represented by a square of $7^0$ x $7^0$ side length. The center of this ocean square has the geographical position: $\varphi = 0^0$, $\lambda = 0^0$. This square is subdivided into a grid of 49 compartments of $1^0$ x $1^0$ size. In the subsequent considerations, this $7^0$ x $7^0$ oceanic square is the integration area. The integration consists in a summing up over the 49 elements of $1^0$ x $1^0$ size which the ocean does consist of. The points $P_k$, (k = 1, 2, ..., 49), are the center points of the individual grid meshes.

Furthermore, the five test points $P_i$, (i = 1, 2, ..., 5), are introduced. This are the points for which the height anomalies are to be computed. These points have the following

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positions, Fig. 1,

$$P_1 \; ( \; \varphi_1 = 0^o, \quad \lambda_1 = 5^o) \quad , \tag{22}$$

$$P_2 \; ( \; \varphi_2 = 0^o, \quad \lambda_2 = 10^o) \quad , \tag{23}$$

$$P_3 \; ( \; \varphi_3 = 0^o, \quad \lambda_3 = 20^o) \quad , \tag{24}$$

$$P_4 \; ( \; \varphi_4 = 0^o, \quad \lambda_4 = 40^o) \quad , \tag{25}$$

$$P_5 \; ( \; \varphi_5 = 0^o, \quad \lambda_5 = 70^o) \quad . \tag{26}$$

The effect that the integration over this model ocean of $7^o \times 7^o$ size does exert on the continental height anomalies at these 5 test points, (22) to (26), even this is the problem to be investigated here.

The computation model, the model ocean, the 49 oceanic compartments and the 5 test points are plotted in the figure 1.

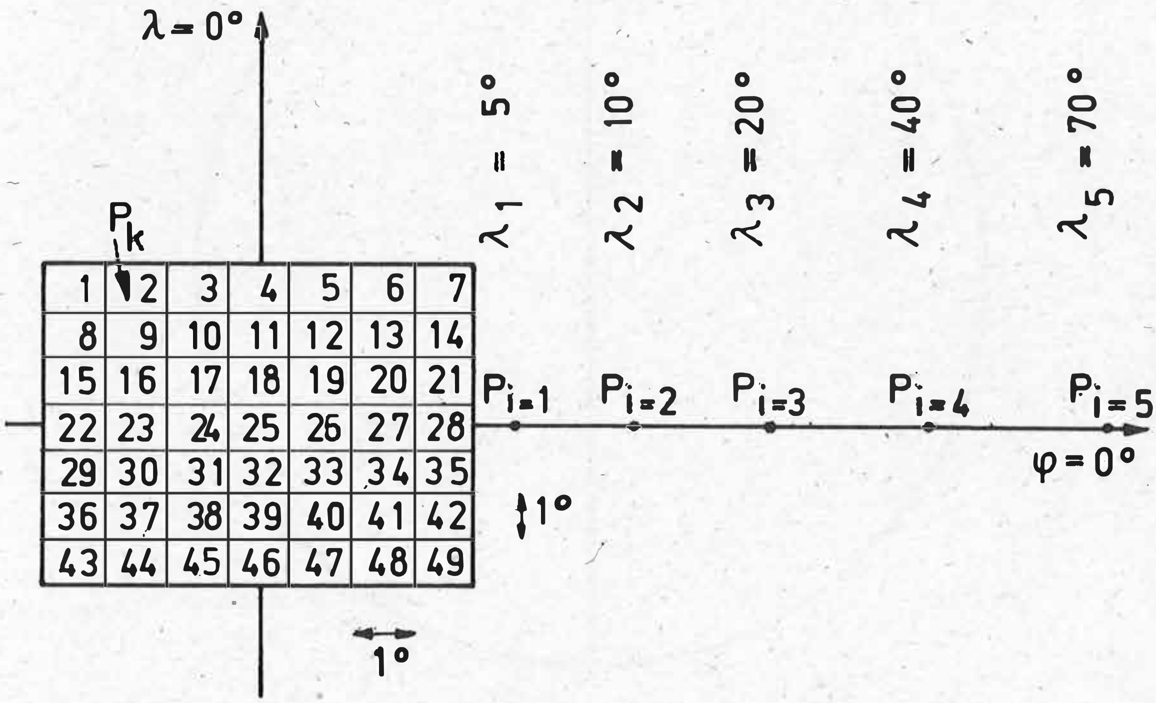


Fig. 1. The computation model, consisting of an ocean of $7^o \times 7^o$ square and of the 5 test points $P_i$.

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The figure 2 is a graph of the boundary values of the indirect way. Here, a circle serves as a substitute for the surface of the Earth. The letter s symbolizes the oceanic part of the Earth's surface, the letter c stands for the continental part. Along the oceanic part s, the values of the perturbation potential T are empirically given; on the continents c, the same is valid for the free-air anomalies.

The equations (14), (16) allow to compute the oceanic $\Delta g_F$ values from the oceanic T values.

Thus, the model of the figure 2 changes over to the model of the figure 3. The transition from the figure 2 over to the figure 3 is the first step of the indirect method.

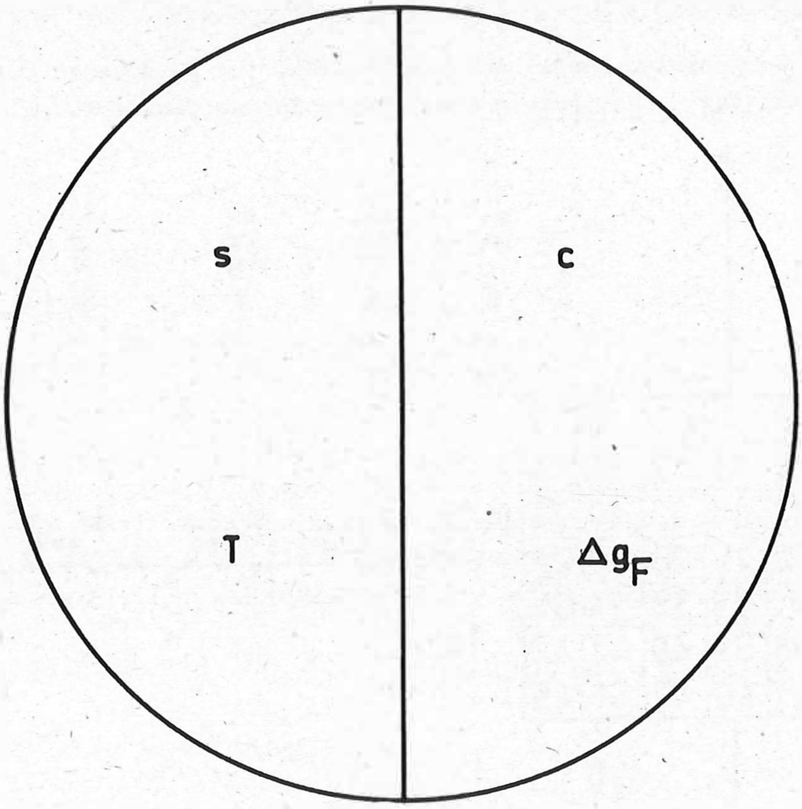


Fig. 2. The boundary values before the first step of the indirect method.

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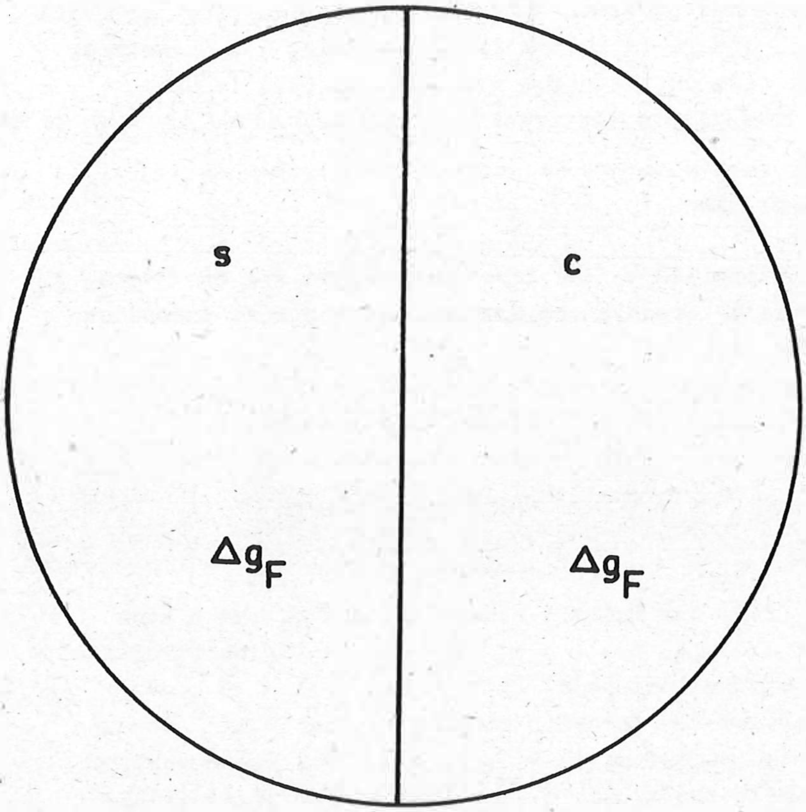


Fig. 3. The boundary values before the second step of the indirect method.

The second step of the indirect solution is nothing more than the integration over the globally given free-air anomalies, (see Fig. 3). It happens by the Stokes integral, (2). For the continental test points, the relations (5) (10) (11) (12) give

$$\underline{\underline{z}}_c = \underline{\underline{S}}_{c.c} \; \underline{\underline{g}}_c + \underline{\underline{S}}_{c.s} \; \underline{\underline{g}}_s \quad . \qquad (27)$$

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In the here discussed problem, only the integration over the model ocean of $7^o$ x $7^o$ square is in the fore, (see Fig. 1). Therefore, only the second term on the right hand side of (27) is here considered, to investigate the precision obtained along the second way,

$$\underline{\underline{z}}'_c = \underline{\underline{S}}_{c.s} \; \underline{\underline{g}}_s \quad . \qquad (28)$$

This relation is applied to the five test points $P_i$, (i = 1, 2, 3, 4, 5), and to the 49 oceanic compartments of the here introduced model ocean, Fig. 1.

Thus, the relations (2) and (6) and (28) lead to

$$(\delta\zeta')_{P_i,\,P_k} = \frac{R}{4\pi\gamma} \Delta\omega \cdot \mu \cdot S(\psi_{P_i,P_k}) \quad . \qquad (29)$$

$\mu$ comes from (21), the index i stands for one of the 5 test points, (i = 1, 2, 3, 4, 5). $(\delta\zeta')_{P_i,\,P_k}$ is the shift of the height anomaly at the test point $P_i$ caused by a shift of the free-air anomaly of the $1^o$ x $1^o$ oceanic compartment of the number k, (k = 1, 2, ..., 49). The amount of the gravity anomaly shift, (i. e. the independent shift), is equal to $\mu$, (21).

The four figures 4, 5, 6, and 7 show the values $(\delta\zeta')_{P_i,P_k}$ with regard to the four test points $P_1$, $P_2$, $P_3$, $P_4$, respectively, (i = 1, 2, 3, 4). The numbers plotted in the meshes of this grid represent the effect in mm that a 4 mgal shift of the free-air anomaly of such a $1^o$ x $1^o$ compartment takes on the height anomaly at the considered test point. Thus,

$$(\delta\zeta')_{P_1,\,P_{25}} = 17 \text{ mm} \quad , \qquad (30)$$

$$(\delta\zeta')_{P_2,\,P_{25}} = 9 \text{ mm} \quad , \qquad (31)$$

$$(\delta\zeta')_{P_3,\,P_{25}} = 3 \text{ mm} \quad , \qquad (32)$$

$$(\delta\zeta')_{P_4,\,P_{25}} = -0.1 \text{ mm} \quad . \qquad (33)$$

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Further,

$$(\delta\zeta')_{P_5,\ P_{25}} = -\ 1.4 \text{ mm} \quad . \qquad (34)$$

The nonplotted values of the white meshes of the four figures 4,5, 6,7 can be interpolated between the neighbouring data. The relations (30) to (34) refer to the central compartment of the model ocean, (k = 25).

The figures 4 to 7 give the impacts that the compartment values of $\Delta g_F = \mu = 4$ mgal exert on the $\zeta$ values at the test points, (29).

| | | | | | | |
|---|---|---|---|---|---|---|
| 10 | | | 15 | | | 23 |
| | | | | | | 29 |
| | | | | | | 36 |
| 11 | 12 | 14 | 17 | 21 | 28 | 40 |
| | | | | | | 36 |
| | | | | | | 29 |
| 10 | | | 15 | | | 23 |

Fig. 4.

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | | | 8 | | | 11 |
| | | | | | | 12 |
| | | | | | | 12 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | | | | | | 12 |
| | | | | | | 12 |
| 6 | | | 8 | | | 11 |

Fig. 5.

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| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | | | 3 | | | 4 |
| | | | | | | 4 |
| | | | | | | 4 |
| 3 | 3 | 3 | 3 | 4 | 4 | 4 |
| | | | | | | 4 |
| | | | | | | 4 |
| 3 | | | 3 | | | 4 |

Fig. 6.

| | | | | | | |
|---|---|---|---|---|---|---|
| -0.4 | | | -0.1 | | | 0.2 |
| | | | | | | 0.2 |
| | | | | | | 0.2 |
| -0.4 | -0.3 | -0.2 | -0.1 | 0 | 0.1 | 0.2 |
| | | | | | | 0.2 |
| | | | | | | 0.2 |
| -0.4 | | | -0.1 | | | 0.2 |

Fig. 7.

Fig. 4, 5, 6, 7. The results of the model computations for the indirect or two-step-method; - the altimeter data ⟶ the gravity anomalies ⟶ the continental height anomalies. The numbers in the individual compartments of these 4 grids give respectively the amounts in mm that a 4 mgal shift of the gravity anomaly of the concerned compartment does exert on the height anomaly $\zeta$ at the 4 test points $P_1$, $P_2$, $P_3$, $P_4$, respectively. Fig. 4 refers to the test point $P_1$, Fig. 5 to the test point $P_2$, Fig. 6 to $P_3$ and Fig. 7 to $P_4$.

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## 3. The direct solution by the first mixed boundary value problem.

Now, the direct method is to be studied. It works even by that solution of the altimetry - gravimetry problem which is also denominated as the first mixed boundary value problem of the geodesy. The mathematical aspects of this solution method was discussed in [3] [6] [12] [13] [14]. Fig. 8 represents the two different computation methods: The indirect or two-step-method on the one hand, 1- 2- 3, and the direct or one-step-method on the other hand. This latter procedure is represented by the step 1 - 3 of Fig. 8.

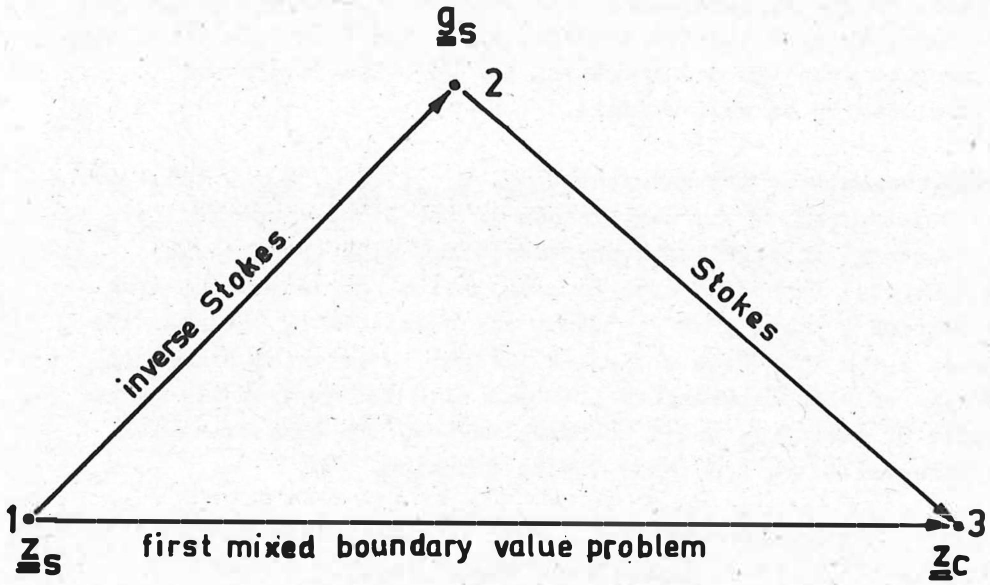


Fig. 8. The indirect way, running from the oceanic altimeter data $\underline{\underline{z}}_s$ via the oceanic gravity anomalies $\underline{\underline{g}}_s$ to the continental height anomalies $\underline{\underline{z}}_c$: 1- 2- 3. Further, the direct way, brought about by means of the first mixed boundary value problem: 1 - 3.

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A computation procedure for the solution of this first mixed problem was derived in [3] [6], it is especially recommended for numerical applications. For the considerations which are here in the fore, the basic ideas of this computation procedure are to be sketched now by some short lines.

First of all, the global equation (5) is divided into the oceanic and into the continental part, (10) (11) (12).

$$\underline{\underline{z}}_c = \underline{\underline{S}}_{c.c} \; \underline{\underline{g}}_c + \underline{\underline{S}}_{c.s} \; \underline{\underline{g}}_s \quad , \tag{35}$$

$$\underline{\underline{z}}_s = \underline{\underline{S}}_{s.c} \; \underline{\underline{g}}_c + \underline{\underline{S}}_{s.s} \; \underline{\underline{g}}_s \quad . \tag{36}$$

In our applications, $\underline{\underline{S}}_{s.s}$ is a positive definite, symmetrical and closed matrix; the elements of it are regular functions which cover the oceans. Thus, the rang defect of $\underline{\underline{S}}_{s.s}$ is equal to zero. In our applications, the elements of this matrix can be considered to have limited amounts, since the Stokes function must be averaged over the compartments, [3] [6]. The inverse of $\underline{\underline{S}}_{s.s}$ follows to be well-defined.

The elements of the matrices $\underline{\underline{S}}_{s.s}$, $\underline{\underline{S}}_{c.c}$, $\underline{\underline{S}}_{c.s}$, $\underline{\underline{S}}_{s.c}$, (35)(36), are proportional to the mean values of the Stokes function for the running integration compartments. Or, with other words, the spherical harmonics are, in imagination, not summed up over the degrees $2 \leqslant n \leqslant \infty$, (36b), as in case of S, but over the degrees $2 \leqslant n \leqslant N$. N is a limited integer, N is in keeping with the size of the compartments $\Delta\omega$ and with the empirically given details of the $\Delta g_F$ and T boundary values. The summation over the interval $2 \leqslant n \leqslant N$ leads to the function, [9],

$$\overline{S} = \sum_{n=2}^{N} \frac{2n+1}{n-1} P_n (\cos \psi) \quad , \tag{36a}$$

whereas the extension of the summation to the infinity

$$S = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n (\cos \psi) \tag{36b}$$

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gives the original Stokes function. $P_n$ are the Legendre polynomials, /9/. $\overline{S}$ is a regular and finite function, for all values of $\psi$ . $\overline{S}$ fulfills all the theoretical prerequisites for the inversion of the kernel function (symmetrical, positive definite, continuous, finite, closed, the defect is equal to zero in the subspace of the functions of the degrees $2 \leqslant n \leqslant N$). However, there is no doubt, in sufficient approximation, the averaging of $\overline{S}$ over a compartment $\Delta\omega$ yields approximatively the same amount as an averaging of S in such a way. In the S function, the degrees which are greater than N are averaged out in this way. Thus,

$$\underline{\underline{S}}_{s.s} \cong \overline{\underline{\underline{S}}}_{s.s} \ , \quad \underline{\underline{S}}_{c.c} \cong \overline{\underline{\underline{S}}}_{c.c} \ ,$$
$$\underline{\underline{S}}_{c.s} \cong \overline{\underline{\underline{S}}}_{c.s} \ , \quad \underline{\underline{S}}_{s.c} \cong \overline{\underline{\underline{S}}}_{s.c} \ . \qquad (36c)$$

Hence, summarizing finally, there comes no trouble from the fact that $S(\psi) \longrightarrow \infty$, if $\psi \longrightarrow 0$. This singularity is removable.

Before the background of the above lines, it is allowed to inverse the matrix $\underline{\underline{S}}_{s.s}$, and to formulate the following matrix relation, (36),

$$\underline{\underline{g}}_s = \underline{\underline{S}}_{s.s}^{-1} \left[ \underline{\underline{z}}_s - \underline{\underline{S}}_{s.c} \ \underline{\underline{g}}_c \right] \ . \qquad (37)$$

The relation (37) is introduced into (35), and the final form for the solution of the first mixed boundary value problem is obtained. It has the following shape,

$$\underline{\underline{z}}_c = \underline{\underline{S}}_{c.s} \underline{\underline{S}}_{s.s}^{-1} \underline{\underline{z}}_s + \left[ \underline{\underline{S}}_{c.c} - \underline{\underline{S}}_{c.s} \underline{\underline{S}}_{s.s}^{-1} \underline{\underline{S}}_{s.c} \right] \underline{\underline{g}}_c \ . \qquad (38)$$

In the following calculations, only the oceanic part on the right hand side of (38) is considered, it is the first term on the right side of (38). This scientific approach is in keeping with the way along which the indirect method was followed up, (28) (29). Hence, from (38),

$$\underline{\underline{z}}_c'' = \underline{\underline{S}}_{c.s} \underline{\underline{S}}_{s.s}^{-1} \underline{\underline{z}}_s \ . \qquad (39)$$

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(39) represents the impact which the oceanic altimeter data take on the continental height anomalies. It is the impact to be investigated.

In the subsequent computations about the 5 models, sketched by Fig. 1, a standard error of

$$\nu = \pm 0.3 \text{ m} \qquad (40)$$

is introduced now for the mean $1^{o} \times 1^{o}$ compartment values of the surface function $\zeta$ , (see Fig. 1). This $\nu$ value is compatible with the empirically obtained results of the satellite altimetry, [10]; (see also the yournal "Marine Geodesy", Vol. 8, 1 - 4, (1984)).

For the ensuing detailed computations, it is convenient to introduce now the abbreviation

$$\underline{\underline{A}} = \underline{\underline{S}}_{c.s} \; \underline{\underline{S}}^{-1}_{s.s} \quad . \qquad (41)$$

$\underline{\underline{A}}$ is free of instabilities, since $\underline{\underline{S}}_{c.s}$ has in (41) the efficiency of a stabilizer. (39) and (41) lead to

$$\underline{\underline{z}}''_{c} = \underline{\underline{A}} \; \underline{\underline{z}}_{s} \quad . \qquad (42)$$

The explicit shape of (42) is as follows, (see (29)), (40),

$$(\delta\zeta'')_{P_i,\, P_k} = a_{i.k} \cdot \nu \quad , \qquad (43)$$

with

$$\underline{\underline{A}} = \left\{ a_{i.k} \right\} = \begin{pmatrix} a_{1.1} & a_{1.2} & \cdots & a_{1.49} \\ a_{2.1} & a_{2.2} & \cdots & a_{2.49} \\ \cdots & \cdots & \cdots & \cdots \\ a_{5.1} & a_{5.2} & \cdots & a_{5.49} \end{pmatrix} , \qquad (44)$$

$$i = 1, 2, 3, 4, 5; \qquad (45)$$

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$$k = 1, 2, \ldots, 49 \tag{46}$$

The sequence of the i subindices refers to the test points, and that of the k subindices to the running $1^o \times 1^o$ compartments of the ocean model, Fig. 1. $a_{i.k}$ is a twofold table. The first suffix i, (44), specifies the row, the second suffix k specifies the column.

Before the results of the computations are described by the concerned elements $a_{i.k}$ of the matrix $\underline{\underline{A}}$, it seems to be convenient to give an insight into the structure of the matrix $\underline{\underline{S}}_{s.s}$ and of $\underline{\underline{S}}_{s.s}^{-1}$. The multiplication with the stabilizing matrix $\underline{\underline{S}}_{c.s}$ leads from $\underline{\underline{S}}_{s.s}^{-1}$ to the matrix $\underline{\underline{A}}$, (41). $\underline{\underline{S}}_{s.s}$ is a square matrix,

$$\underline{\underline{S}}_{s.s} = \left\{ u_{k.k'} \right\} \quad , \tag{47}$$

$$k, k' = 1, 2, \ldots, 49 \quad . \tag{48}$$

$$\left\{ u_{k.k'} \right\} = \begin{pmatrix} u_{1.1} & u_{1.2} & \cdots & u_{1.49} \\ u_{2.1} & u_{2.2} & \cdots & u_{2.49} \\ \cdots & \cdots & \cdots & \cdots \\ u_{49.1} & u_{49.2} & \cdots & u_{49.49} \end{pmatrix} \quad . \tag{49}$$

The first row of this matrix, (49), relates to the upper left mesh of the $1^o \times 1^o$ net that does cover the $7^o \times 7^o$ square of the model ocean, Fig. 1, (k = 1). The sequence of the elements of this row

$$u_{1.1}, u_{1.2}, \ldots, u_{1.49} \tag{50}$$

consists of the following amounts, disregarding a common multiplier,

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$$\begin{matrix} 406, & 125, & 65, & 45, & 34, & 28, & 23, \\ 125, & 90, & 59, & 43, & 33, & 27, & 23, \\ 65, & 59, & 47, & 38, & 31, & 26, & 22, \\ 45, & 43, & 38, & 33, & 28, & 24, & 21, \\ 34, & 33, & 31, & 28, & 25, & 22, & 20, \\ 28, & 27, & 26, & 24, & 22, & 20, & 18, \\ 23, & 23, & 22, & 21, & 20, & 18, & 17. \end{matrix} \tag{51}$$

The row of the number 25 of the matrix described by (49) refers to the center compartment of the model ocean, ($k = 25$). It has the following sequenoe,

$$u_{25.1}, u_{25.2}, \ldots, u_{25.49} \tag{52}$$

The amounts of (52) are as follows, disregarding again the same multiplier as in case of (51),

$$\begin{matrix} 33, & 38, & 43, & 45, & 43, & 38, & 33, \\ 38, & 47, & 59, & 65, & 59, & 47, & 38, \\ 43, & 59, & 90, & 125, & 90, & 59, & 43, \\ 45, & 65, & 125, & 406, & 125, & 65, & 45, \\ 43, & 59, & 90, & 125, & 90, & 59, & 43, \\ 38, & 47, & 59, & 65, & 59, & 47, & 38, \\ 33, & 38, & 43, & 45, & 43, & 38, & 33. \end{matrix} \tag{53}$$

The amounts of the elements of the two sets (51) and (53) are directly oomputed as the values of the Stokes funotion $S(\psi)$, (36b). Thus, they are precisely equal to the amounts of this function $S(\psi)$ for the arguments $\psi_{k.k'}$, whereat the suffixes k and k' specify the concerned compartment oenter points which are the endpoints of this spherical arc, $\psi_{k.k'}$; (47) (48), [9]; e. g. $S(\psi = 1^0) = 125$. Therefore, the amounts of (51) and (53) can be taken immediately and unchanged from the table of W. D. Lambert and F. W. Darling, (see: Tables for determining the form of the geoid and its indireot effeot on gravity, U. S. Coast and Geodetic Survey, Special Publ. No. 199, Washington 1936). In the expression $\psi_{k.k'}$ and in (47), the first suffix k specifies the row, the second the column, k'.

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As to the inverse, this is the matrix

$$\underline{\underline{S}}^{-1}_{s.s} = \left\{ v_{k.k'} \right\} \quad , \tag{54}$$

$$k, k' = 1, 2, \ldots, 49 \, . \tag{55}$$

$$\left\{ v_{k.k'} \right\} = \begin{pmatrix} v_{1.1} & v_{1.2} & \cdots & v_{1.49} \\ v_{2.1} & v_{2.2} & \cdots & v_{2.49} \\ \cdots & \cdots & \cdots & \cdots \\ v_{49.1} & v_{49.2} & \cdots & v_{49.49} \end{pmatrix} . \tag{56}$$

The first row of (56) has the following amounts, (k = 1), disregarding a common multiplier,

$$\begin{matrix} +2956, & -626, & -74, & -43, & -26, & -19, & -19, \\ -626, & -164, & -33, & -18, & -11, & -8, & -11, \\ -74, & -33, & -20, & -12, & -8, & -6, & -9, \\ -43, & -18, & -12, & -9, & -6, & -5, & -8, \\ -26, & -11, & -8, & -6, & -5, & -4, & -7, \\ -19, & -8, & -6, & -5, & -4, & -4, & -6, \\ -19, & -11, & -9, & -8, & -7, & -6, & -8. \end{matrix} \tag{57}$$

The 25th row of (56) is as follows, disregarding again the same common multiplier as in (57), (k = 25),

$$\begin{matrix} -9, & -8, & -11, & -12, & -11, & -8, & -9, \\ -8, & -10, & -18, & -15, & -18, & -10, & -8, \\ -11, & -18, & -141, & -508, & -141, & -18, & -11, \\ -12, & -15, & -508, & +3270, & -508, & -15, & -12, \\ -11, & -18, & -141, & -508, & -141, & -18, & -11, \\ -8, & -10, & -18, & -15, & -18, & -10, & -8, \\ -9, & -8, & -11, & -12, & -11, & -8, & -9. \end{matrix} \tag{58}$$

Now, the equation (43) is applied to the 5 models of the figure 1 which are already treated in the discussion of the indirect method. These 5 models differ by the position of the test points

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$P_i$ only, (i = 1, 2, 3, 4, 5), Fig. 1. In (43), the index k assumes all the elements of the sequence: k = 1, 2, ..., 49. Thus, all the compartments of the oceanic grid are considered.

Fig. 9 shows the amounts of $a_{i.k} \cdot \nu$ , for the $\nu$ value according to (40), and for i = 1, and for the index k running through all the numbers 1, ..., 49. In Fig. 9, these amounts are written in all the concerned $1^o \times 1^o$ compartments of the grid. They show the impacts that the shifts of the oceanic $1^o \times 1^o$ mean altimeter data exert on the height anomaly at the test point $P_1$, (i = 1), for $\nu$ = 0.3 m, Fig. 9.

Fig. 10 gives the corresponding values of $a_{2.k} \cdot \nu$ for the test point $P_2$, (i = 2). Fig. 11 and Fig. 12 show these amounts for i = 3 and i = 4. All these amounts which are represented by the figures 9, 10, 11, 12 are measured in mm. They base on a shift by $\nu$ = 0.3 m, for the mean $1^o \times 1^o$ compartment value of the $\zeta$ function.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.5 | | | 1.9 | 2.6 | 3.7 | 9.2 |
| | | | | | | 12.0 |
| | | | | | | 18.0 |
| 0.9 | 0.6 | 0.8 | 1.1 | 1.9 | 3.6 | 23.0 |
| | | | | | | 18.0 |
| | | | | | | 12.0 |
| 1.5 | | | 1.9 | 2.6 | 3.7 | 9.2 |

Fig. 9.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.5 | | | 1.6 | 2.0 | 2.7 | 5.6 |
| | | | | | | 4.4 |
| | | | | | | 4.4 |
| 0.9 | 0.6 | 0.6 | 0.7 | 1.0 | 1.5 | 4.5 |
| | | | | | | 4.4 |
| | | | | | | 4.4 |
| 1.5 | | | 1.6 | 2.0 | 2.7 | 5.6 |

Fig. 10.

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| | | | | | | |
|---|---|---|---|---|---|---|
| 0.7 | | | 0.7 | | | 2.1 |
| | | | | | | 1.5 |
| | | | | | | 1.4 |
| 0.4 | 0.2 | 0.3 | 0.3 | 0.4 | 0.5 | 1.4 |
| | | | | | | 1.4 |
| | | | | | | 1.5 |
| 0.7 | | | 0.7 | | | 2.1 |

Fig. 11.

| | | | | | | |
|---|---|---|---|---|---|---|
| −0.3 | | | 0 | | | 0.2 |
| | | | | | | 0.2 |
| | | | | | | 0.2 |
| −0.2 | −0.1 | 0 | 0 | 0 | 0 | 0.2 |
| | | | | | | 0.2 |
| | | | | | | 0.2 |
| −0.3 | | | 0 | | | 0.2 |

Fig. 12.

Fig. 9, 10, 11, 12. The results of the model computations for the direct method according to the first mixed boundary value problem. The numbers in the individual compartments of these 4 grids give the amounts (in mm) of the impact that a shift of the regarded $1^o \times 1^o$ mean altimeter data (by $\nu = 0.3$ m) does exert on the height anomaly $\zeta$ at the 4 test points $P_1$, $P_2$, $P_3$, $P_4$, respectively.

As to the 5th test point $P_5$, the effect of $\nu$ on the $\zeta$ value at $P_5$ is $a_{5.k} \cdot \nu$ , (43), (k = 1, 2, ..., 49). In dependence upon the parameter k, this effect ranges between - 0.6 mm and - 0.1 mm within the $7^o \times 7^o$ model ocean, Fig. 1. For the center compartment of the ocean, k = 25, this impact is equal to - 0.13 mm.

Now, the central $1^o \times 1^o$ compartment, (k = 25), is brought into the fore. A data shift of $\zeta$, within this compartment, has a certain impact on the $\zeta$ values at the 5 test points.
- The concerned values obtained formerly by the indirect or two-step-method are tabulated by the equations (30) to (34) -. The analogous amounts found along the lines of the direct or one-step-method of the first mixed boundary value problem are as follows, (43), Fig. 9, 10, 11, 12; ( $\nu$ = 0.3 m).

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$$(\delta\zeta'')_{P_1, P_{25}} = 1.1 \text{ mm} , \tag{59}$$

$$(\delta\zeta'')_{P_2, P_{25}} = 0.7 \text{ mm} , \tag{60}$$

$$(\delta\zeta'')_{P_3, P_{25}} = 0.3 \text{ mm} , \tag{61}$$

$$(\delta\zeta'')_{P_4, P_{25}} = 0 \text{ mm} , \tag{62}$$

$$(\delta\zeta'')_{P_5, P_{25}} = -0.13 \text{ mm} . \tag{63}$$

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4. The direct way in comparison with the indirect procedure.

A comparison of the relations (30) to (34), on the one hand, with the relations (59) to (63), on the other hand, shows that the latter amounts are much more small than the values of (30) to (34). Thus, obviously, the resulting continental height anomalies $\zeta$ at the five test points are much more precise if the direct method of the first mixed boundary value problem was preferred, instead of the indirect method proceeding along the roundabout way via the gravity anomalies as an intermediary system (approximating $\underline{\underline{z}}'_c$ to $\underline{\underline{z}}''_c$, (28)(39)).

Preferring the direct method instead of the indirect method, the amounts of the standard deviations of the resulting height anomalies are lowered down by the multiplier $æ_i$, $(i = 1, 2, 3, 5)$. For the different test points $P_i$, $(i = 1, 2, 3, 5)$, $æ_i$ has the following amounts:

(30) and (59) give,

$$æ_1 = \frac{1.1}{17} = 0.06 \quad ; \tag{64}$$

(31) and (60) lead to

$$æ_2 = \frac{0.7}{9} = 0.08 \quad ; \tag{65}$$

(32) and (61) have the consequence,

$$æ_3 = \frac{0.3}{3} = 0.1 \quad ; \tag{66}$$

(34) and (63) yield a diminution coefficient of

$$æ_5 = \frac{0.13}{1.4} = 0.09 \quad . \tag{67}$$

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Of course, a glance on (64) (65) (66) and (67) does it show impressively, the precision of the results is greatly improved by the transition from the indirect method to the direct one.

The coefficient $\varkappa_4$ is not considered here because the Stokes function is equal to zero if $\psi$ is in the vicinity of $40^{\circ}$. Indeed, the spherical distances between the test point $P_4$ and the $1^{\circ} \times 1^{\circ}$ meshes of the model ocean are in the vicinity of $40^{\circ}$, Fig. 1.

A look at the 4 figures 9, 10, 11, 12 demonstrates the fact that there is not a trace of an instability, (but; (15) has instabilities). The guessed instabilities are not corroborated, as for (39), [14].

Sure, it is one of the main tasks of the theoretical geodesy to find out such evaluation methods which give the most precise and optimal results from the geodetic observations. A geodetic procedure consists, among others, of the observations and of the subsequent mathematical evaluations. On principle, this procedure can be compared with a chain. This chain cannot be stronger than the weakest link of it. The evaluation method should not be the weak point of the procedure.

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## 5. The difference method.

As to the practical applications of the relation (38), it seems to be convenient to introduce a difference method determining not the full amounts of the continental height anomalies $\zeta$ but the differences of these height anomalies relative to the height anomaly at a reference point situated in the considered continent. The transition from the absolute $\zeta$ values to the relative amounts of them is a procedure that will bring about a clear relief to the numerical computations.

In this case, now, the differences of certain vectors and matrices (attached to the test points) are to be considered, i. e. $\underline{\underline{z}}_c \longrightarrow \delta \underline{\underline{z}}_c$, $\underline{\underline{S}}_{c.s} \longrightarrow \delta \underline{\underline{S}}_{c.s}$, $\underline{\underline{S}}_{c.c} \longrightarrow \delta \underline{\underline{S}}_{c.c}$, (38).

Hence,

$$\delta \underline{\underline{z}}_c = \delta \underline{\underline{S}}_{c.s} \, \underline{\underline{S}}_{s.s}^{-1} \, \underline{\underline{z}}_s + \left[ \delta \underline{\underline{S}}_{c.c} - \delta \underline{\underline{S}}_{c.s} \, \underline{\underline{S}}_{s.s}^{-1} \, \underline{\underline{S}}_{s.c} \right] \underline{\underline{g}}_c . \quad (68)$$

$\delta \underline{\underline{z}}_c$ will contain the differences by which the continental $\zeta$ values differ from the one $\zeta$ value at the selected reference test point, [3] [6].

In order to have a fixation of the ideas connected with this difference method, a short example is sketched now. Along these lines, the differences

$$\zeta_A - \zeta_S = \zeta_{AS} \quad (68a)$$

and

$$\zeta_V - \zeta_S = \zeta_{VS} \quad (68b)$$

can be computed by the solution of the first mixed boundary value problem, introducing the relation (68). Here, the suffix A symbolizes Cape Arcona on the Rügen island in the Baltic Sea, S stands for Sopron and V denotes Varna at the Black Sea. The

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$\zeta_{AS}$ and $\zeta_{VS}$ values on the right hand side of (68a) and (68b) are computed by (68), integrating over the continental free-air anomalies and over the oceanic altimeter data. Even these values $\zeta_{AS}$ and $\zeta_{VS}$ can be improved by an adjustment procedure, in the course of a further treatment. Indeed, these computed $\zeta_{AS}$ and $\zeta_{VS}$ values can be anchored on the discrete altimeter data offshore the coasts of the GDR and of Bulgaria, i. e. $\zeta_A$ and $\zeta_V$, [10]. This possibility leads to the following condition equation,

$$\zeta_{AS} - \zeta_{VS} = \zeta_A - \zeta_V \quad . \qquad (69)$$

The two terms on the left hand side of (69) come by computation from the first mixed boundary value problem, integrating over the observed boundary values on the continents and oceans by (68). The reflection of the boundary values of the areas very distant from Europe on the left hand side of (69) will be very small. This is the essential advantage of the difference method. It allows to simplify the integrations over the distant areas by formula (68), [6].

The right hand side of (69) comes directly from the maps of the altimeter data, by a simple interpolation, [10].

Thus, finally, (69) leads to an adjustment procedure according to the method of least squares.

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## 6. The influence of the sea surface topography on the solution of the first mixed boundary value problem.

The sea surface topography is the height of the ocean surface above the geoid. It is denominated by q. Thus, the q values do exist along the oceans only. The satellite altimetry data are termed by $\zeta_{Altimetry}$. The height anomalies are denoted by $\zeta$, the $\zeta$ values derive from the surface perturbation potential by the relation (3). Consequently, these 3 terms have the subsequent equation,

$$\zeta + q = \zeta_{Altimetry} . \tag{70}$$

The equation (70) is valid along the oceans only. (70) gives an expression for q,

$$q = \zeta_{Altimetry} - \zeta . \tag{71}$$

The q values can be determined empirically by a low-low mission of satellite - to - satellite tracking, for instance; (see [4], p. 420, 421). Under the supposition that these q values are known by these independent methods, (discussed in [4], or along another way), in this case, it is possible to free the altimeter data from the impact of the sea surface topography by

$$\zeta = \zeta_{Altimetry} - q . \tag{72}$$

Strictly speaking, the thus obtained $\zeta$ values are the data which are understood to be introduced as the components of the $\underline{\underline{z}}_s$ vector in the earlier discussed two-step and one-step method for the determination of the continental height anomalies, $\underline{\underline{z}}_c$; (5) (6) (10) (14) (15) (16) (35) (36) (37) (38) (39) (42). Thus, if the sea surface topography q is known, this effect can be subtracted from the altimeter data in order to obtain the needed $\zeta$ values at sea, (72). In this case, there is no further trouble about any influence which the q values possibly would take on the continental $\zeta$ values, computed by (38) or (68).

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However, the q values are not yet determined in a reliable way. Thus, it is recommended to sketch another method for the elimination of the biases in the continental $\zeta$ values computed by (38) or (68). These biases are understood as a reflection of the q values. It is the $z_s$ vector of (38) and (68) which is directly influenced by the q values. The propagations of this influence on the $z_c$ values are these biases, caused by the neglection of the q values at sea.

From different sources which must not be discussed here, it is possible to have a certain idea of the order of the amplitudes and wave lengthes in the field of the oceanic q values. Several publications are devoted to this question, /8/. The knowledge of the main structures in the q value field is an aid for the evaluation of their influence on the continental $\zeta$ values which are computed along the lines of the first mixed boundary value problem.

The publication /8/ does contain the results of such an estimation. The influence of the sea surface topography on the results of the first mixed boundary value problem, (according to (38) or (68)), - i. e. the $\zeta$ values on the continents, - is now denominated by

$$f = f(\varphi, \lambda) \quad , \qquad (73)$$

it is a function of the latitude and longitude. In the European area, the lines of constant values of f run in the east - west direction about, they are equidistant about; in the north - south direction, the gradient of the f field describes a change of the f values by about 0.1 m over 1000 km, /8/.

Therefore, in the European area, the f field can be approximated by an analytical expression which is linear in the differences of the latitude and longitude, $\varphi$ and $\lambda$,

$$f = f(\varphi, \lambda) \cong c_0 + c_1 \cdot \Delta\varphi + c_2 \cdot \Delta\lambda \quad . \qquad (74)$$

The three constant coefficients $c_0$, $c_1$, $c_2$ in (74) can be determined by the evaluation of three Doppler-determined $\zeta$ values. It happens along the following lines.

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In case of boundary values which are free of the q value impact, the solution vector is obtained by (38). In case, the altimeter data are falsified by the q values, the equation (38) leads to the following computation procedure,

$$\underline{\underline{z}}_c + \Delta \underline{\underline{z}}_c (q) = \underline{\underline{S}}_{c.s} \underline{\underline{S}}^{-1}_{s.s} \left[ \underline{\underline{z}}_s + \underline{\underline{q}} \right] + $$

$$+ \left[ \underline{\underline{S}}_{c.c} - \underline{\underline{S}}_{c.s} \underline{\underline{S}}^{-1}_{s.s} \underline{\underline{S}}_{s.c} \right] \underline{\underline{g}}_c \quad . \qquad (75)$$

$\underline{q}$ is the vector representation of the q values. $\Delta \underline{\underline{z}}_c$ (q) is the vector representation of the continental f function, (74). The right hand side of (75) is determined empirically by the altimeter data

$$\underline{\underline{z}}_s + \underline{\underline{q}} \qquad (76)$$

and by the continental free-air anomalies

$$\underline{\underline{g}}_c \quad . \qquad (77)$$

This right hand side of (75) is denoted by

$$\underline{\underline{z}}^*_c = \underline{\underline{S}}_{c.s} \underline{\underline{S}}^{-1}_{s.s} \left[ \underline{\underline{z}}_s + \underline{\underline{q}} \right] + \left[ \underline{\underline{S}}_{c.c} - \underline{\underline{S}}_{c.s} \underline{\underline{S}}^{-1}_{s.s} \underline{\underline{S}}_{s.c} \right] \underline{\underline{g}}_c , \qquad (78)$$

it is a known vector, in this context. The relations (75) and (78) give

$$\underline{\underline{z}}_c + \Delta \underline{\underline{z}}_c (q) = \underline{\underline{z}}^*_c \quad . \qquad (79)$$

Some discrete values of the field represented by $\underline{\underline{z}}_c$ can be determined by Doppler observations of satellites, $\zeta_{Doppler}$, (and precise levellings). The expression $\Delta \underline{\underline{z}}_c$ (q) in (79) is the vector representation of the field of the f values, (74). In (79), $\underline{\underline{z}}^*_c$ is the vector representation of the field of certain values which are now denominated by $\zeta^*$, they are obtained by the computation along the lines of the first mixed boundary value problem, (78). Thus, by the transitions

$$\underline{\underline{z}}_c \longrightarrow \zeta_{Doppler} \quad , \qquad (79a)$$

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$$\Delta \underline{\underline{z}}_c (q) \longrightarrow f(\varphi, \lambda) \quad , \tag{79b}$$

$$\underline{\underline{z}}_c^* \longrightarrow \zeta^* \quad , \tag{79c}$$

the vector relation (79) turns to the following equation for the components,

$$\zeta_{\text{Doppler}} + f(\varphi, \lambda) = \zeta^* \quad . \tag{80}$$

(74) and (80) lead to

$$\zeta_{\text{Doppler}} - \zeta^* = - c_o - c_1 \cdot \Delta\varphi - c_2 \cdot \Delta\lambda \quad . \tag{81}$$

The right hand side of (81) contains the three unknown coefficients $c_o$, $c_1$, $c_2$. If the $\zeta_{\text{Doppler}}$ values are measured at 3 different and adequately spaced points of the considered continental area, and if - further on - the $\zeta^*$ values are computed for even these 3 points, in this case, it is possible to find the values of $c_o$, $c_1$, $c_2$ by an evaluation of (81).

Thus, the function f ( $\varphi$ , $\lambda$ ) comes to be known. Consequently, also the vector representation $\Delta \underline{\underline{z}}_c$ (q) of the function f ( $\varphi$ , $\lambda$ ) follows to be known by the inversion of (79b). Even this fact leads to the possibility of computing the unbiased height anomalies, $\underline{\underline{z}}_c$, as a two-dimensional function which covers whole the considered continent. The concerned computation formula is - a continuous vector -

$$\underline{\underline{z}}_c = \underline{\underline{z}}_c^* - \Delta \underline{\underline{z}}_c (q) \quad , \tag{82}$$

see (79). The first term on the right hand side of (82) is computed by (78). And the second term is reached by an adjustment of (81), a procedure that gives at first the coefficients $c_o$, $c_1$, $c_2$ and then, consequently, the continuous function f ( $\varphi$ , $\lambda$ ) by means of (74); the inversion of (79b) leads to the wanted vector , $\Delta \underline{\underline{z}}_c(q)$ .

Further on, to be more oomplete, and to avoid misunderstandings, the relation (82) has the vectors $\underline{\underline{z}}_c$ , $\underline{\underline{z}}_c^*$ , $\Delta \underline{\underline{z}}_c(q)$ which can represent the vector shape of certain continuous funotions, well-defined even over whole the area of all the continents , - inoreasing the range of ideas in this way - ; but, in this case, the function f has to be defined in

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a new way, introducing several continents instead of one, (85)(91). It is, (6) (10), extending the scope to several continents,

$$\underline{\underline{z}}_c = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \dots \\ \zeta_p \\ \dots \\ \zeta_K \end{bmatrix} , \qquad (83)$$

K is the number of the continental compartments, K $\leqslant$X, (p = 1, 2, ..., K). Further, in (82), the continental vector $\underline{\underline{z}}_c^*$ must be identified with the right hand side of (75), (78). It is computed by the mixed boundary values, i. e. the values $\underline{\underline{z}}_s + \underline{\underline{g}}$ and $\underline{\underline{g}}_c$, (75) (78) ; ( $t_i = \zeta^*_i$ ) .

$$\underline{\underline{z}}_c^* = \begin{bmatrix} t_1 \\ t_2 \\ \dots \\ t_p \\ \dots \\ t_K \end{bmatrix} . \qquad (84)$$

The second vector $\Delta \underline{\underline{z}}_c$ (q) on the right hand side of (82) refers to the continuous function f ( $\varphi$ , $\lambda$ ), the argument domain of it is now the area of all the continents, as it is the case for $\underline{\underline{z}}_c$ and $\underline{\underline{z}}_c^*$ also, (82) (74),

$$\Delta \underline{\underline{z}}_c(q) = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_p \\ \dots \\ u_K \end{bmatrix} = \begin{bmatrix} f(\varphi, \lambda)_1 \\ f(\varphi, \lambda)_2 \\ \dots \\ f(\varphi, \lambda)_p \\ \dots \\ f(\varphi, \lambda)_K \end{bmatrix} . \qquad (85)$$

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From now, it is self-explanatory, the one single linear expression for f ( $\varphi$ , $\lambda$ ) of the shape of (74) cannot be valid for whole the area of all the continents. Naturally! Its validity is restricted to properly choosen partial areas of the continents, e. g. of $10^o$ x $10^o$ or $20^o$ x $20^o$ squares, [8]. (For a continuous enumeration, the meaning of the functions $f(\varphi, \lambda)_p$ is explained by (85) ).

In case, whole the continental area of the globe is considered, (85), in this case, the continents should be divided into a certain number of properly chosen partial areas, in total number L, which have the following mean values for the individual compartments,

$$f(\varphi, \lambda)_{l.v} \quad . \qquad (86)$$

The index l denotes the regarded partial system and v the running compartment within of it,

$$l = 1, 2, \ldots, L \quad , \qquad (87)$$

$$v = 1, 2, \ldots, K_l \quad , \qquad (88)$$

$$K_1 + K_2 + \ldots + K_L = K \quad . \qquad (89)$$

$K_l$ denominates the total number of all the compartments which divide up the regarded partial area, - i. e. the partial area distinguished by the suffix l -, (88).

Within each individual partial area, a linear expression for the function f ( $\varphi$ , $\lambda$ ) is supposed to be valid, (74) (86). The shape of such a function is

$$f(\varphi, \lambda)_l = c_{l.o} + c_{l.1} \cdot \Delta\varphi + c_{l.2} \cdot \Delta\lambda \quad . \qquad (90)$$

The relation (90) is valid within the partial area of the number l. Thus, the discrete values of the function f for all the compartments, ($v = 1, 2, \ldots, K_l$), which divide up a certain partial area (of the number l), are

$$f(\varphi, \lambda)_{l.v} = c_{l.o} + c_{l.1} \cdot (\Delta\varphi)_{l.v} + c_{l.2} \cdot (\Delta\lambda)_{l.v} . \qquad (91)$$

$$v = 1, 2, \ldots, K_l \quad ; \qquad (92)$$

$$l = 1, 2, \ldots, L \quad . \qquad (93)$$

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The terms $(\Delta\varphi)_{1.v}$ and $(\Delta\lambda)_{1.v}$ are the differences of latitude and longitude with regard to a certain central compartment, for this compartment the value of $f(\varphi,\lambda)_1$ is equal to $c_{1.o}$ .

The discrete values $f(\varphi,\lambda)_{1.v}$ can be determined for each partial system (distinguished by the index 1) separately and by themselves alone. For the partial area which is distinguished by the suffix 1, or for a fixed certain value of 1,

$$1 = \text{const.} \quad , \tag{94}$$

and for

$$v = 1, 2, \ldots, K_1 \quad , \tag{95}$$

the coefficients

$$c_{1.o}, c_{1.1}, c_{1.2} \tag{96}$$

can be determined - separated from the coefficients of the other partial areas - by empirical means from the Doppler determined $\zeta$ values, $\zeta_{Doppler}$. The procedure connected with the evaluation of (81) has to be applied here.

A combination of all the functions $f(\varphi,\lambda)_1$, (see (90)), leads to a global representation of the function $f = f(\varphi,\lambda)$, covering the continents. This combination procedure can happen by a unification along the principles of the anblocking method, for instance. There is no need to add another word about the details of this anblocking, since it is well-known and self-explanatory.

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## 7. Conclusions.

The preceding developments demonstrate that the solution of the altimetry - gravimetry problem should happen at a profit along the rules of the first mixed boundary value problem, since this way is of significant advantage for the precision of the resulting height anomalies on the continents.

As to the procedure of the two-step-method which computes the oceanic free-air anomalies as an intermediary system, this two-step-method leads to a solution of a clear inferiority in precision, in comparison with the solution of the first mixed boundary value problem.

If the one-step-method of the first mixed boundary value problem is preferred instead of the indirect two-step-method, in this case, this exchange of the methods will be accompanied by a diminution of the standard deviation of the resulting continental height anomalies. A reducing multiplier of about 0.1 does work here.

Further on, a method is sketched which allows the elimination of the impact the sea surface topography takes on the resulting height anomalies. The introduction of some Doppler - determined height anomalies is essential for this method.

I am indebted to Eng. Helga Jurczyk for her essential co-operation in the electronic computer calculations.

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Korrigendum

In the below cited publication (ARNOLD 1984), page 350, line 7,8 and 9, is an erratum. The symbol T has to be replaced by $\delta T$, and similarly $\Delta g_F$ by $\delta\Delta g_F$, in order to be right. $\delta T$ and $\delta\Delta g_F$ are the residuals before the adjustment:

Observed values minus the computed ones. The precise shape of $\delta T$ and $\delta\Delta g_F$ can be found in the brackets of the equation (4) of the publication (ARNOLD 1981). But, for the computation of $\delta T$ and $\delta\Delta g_F$ and their r.m.s. values $\mu_T$ and $\mu_g$, the Stokes constants $T_n$ appearing in these brackets are taken from a beforehand given approximate harmonics development. The right procedure how to compute the residuals $\delta T$ and $\delta\Delta g_F$ and their r.m.s. values $\mu_T$ and $\mu_g$ is described by the passage appearing between the equations (6) and (7) of the publication (ARNOLD 1981).

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## L. Gravity disturbances as boundary values on the surface of the Earth.

Contents



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## Summary

The vertical derivatives of the perturbation potential at the surface of the Earth serve as boundary values. They are identical with the gravity disturbances. The perturbation potential is superposed by the model potential of the mountain masses. The solution of this boundary value problem turns out to have the shape of an integral representation for the perturbation potential at the surface of the Earth. The Hotine function is the kernel function. Two supplements must be added to the gravity disturbances. The first term is the plane topographical reduction of the gravity which depends in the main on the square of the height differences. The second term is rather small and often negligible, it is proportional to the height gradient of the Bouguer anomalies. Thus, this term depends chiefly from the geological density anomalies and the isostatic mountain roots. The final solution has the character of a closed expression, free of series developments of bad or dubious convergence, it is free of uninvestigated residual terms of certain series developments.



## Zusammenfassung

Die vertikalen Ableitungen des Störpotentials an der Erdoberfläche werden als Randwerte eingeführt. Man nennt diese Werte auch die Schwerestörungen. Das Störpotential wird mit dem Modellpotential der Gebirgsmassen superponiert. Es ergibt sich eine Integraldarstellung für das Störpotential an der Erdoberfläche. Die Funktion von Hotine dient als Kernfunktion. Zu den Schwerestörungen im Integranden treten zwei additive Grössen. Die erste ist die vor allem von den Quadraten der Höhenunterschiede abhängige ebene Geländereduktion der Schwere. Die zweite Grösse ist sehr klein und kann meistens vernachlässigt werden. Sie ist proportional dem Vertikalgradienten der Bougueranomalien, sie hängt damit von den geologischen Dichteanomalien und den isostatischen Gebirgswurzeln ab. Das Finalergebnis ist ein geschlossener mathematischer Ausdruck, Reihenentwicklungen mit schlechter oder unbewiesener Konvergenz erscheinen nicht.


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Резюме

Вертикальные производные мешающего потенциала на поверхности земли вводятся в качестве краевых значений. Эти значения также называют гравитационными нарушениями. Мешающий потенциал совмещается с модельным потенциалом массы горных пород. В результате этого появляется интегральное выражение для мешающего потенциала на поверхности земли. Функция Хотине служит в качестве основной функции. К мешающему потенциалу появляются две суммируемые величины. Первая величина, зависящая, прежде всего, от квадратов разницы высот, представляет собой прямую редукцию гравитации на пересеченной местности. Вторая величина, в большинстве случаев очень мала и ей можно пренебречь. Она пропорциональна вертикальным градиентам аномалий Бугура, она зависит тем самым от геологических аномалий плотности и изостатических корней горных пород. Конечный результат предоставляет собой законченное математическое выражение, где не появляются разложения в ряды с плохой или недоказанной сходимостью.

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## 1. Introduction.

In an earlier publication, the boundary value problem of Molodenskij was treated as a smoothed boundary value problem for a Bjerhammar sphere as boundary surface. The following closed solution was found, [1] [3],

$$T = \frac{r_P}{4\pi} \iint_{\omega} \left[ \Delta g_T + C + C_1 \right] S(\psi) \quad d\omega \; + \; \Xi_1 \quad , \qquad (1)$$

with

$$\Xi_1 = \left\{ [B]'' \right\} - \frac{R}{4\pi} \iint_{\omega} 2 \, B_c \frac{h_Q}{R^2} \; S(\psi) \; d\omega - \left\{ \frac{h_P}{R} B_c \right\} . \qquad (2)$$

The braces $\{ \}$ denote the fact that the shares of the spherical harmonics of the 0th and 1st degree should be subtracted in the expression (2) representing $\Xi_1$. T is the harmonic perturbation potential at the surface of the Earth $\sigma$ , it fulfills the Laplace differential equation,

$$\Delta T = 0 \quad , \text{ in } \Phi_a \; . \qquad (3)$$

$\Phi_a$ is the exterior space of the Earth. $r_P$ is the geocentric radius of the test point P for which the amount of the perturbation potential T is to be computed. $\omega$ is the unit sphere,

$$d\omega = \cos\varphi \cdot d\varphi \cdot d\lambda \quad , \qquad (4)$$

where $\varphi$ and $\lambda$ are the geocentric latitude and longitude. $\Delta g_T$ represents the free-air anomalies,

$$- \frac{\partial T}{\partial r} - \frac{2}{r} T = \Delta g_T = g - \gamma^{*} \quad . \qquad (5)$$

g is the vertical intensity of the gravity at the running point Q on the surface of the Earth. $\gamma^{*}$ is the standard gravity at the telluroid, $\gamma^{*}$ belongs to the telluroid point $Q^{*}$ which is situated vertical below the point Q, Fig. 1. The vertical distance from the point Q to the point $Q^{*}$ is equal to the height anomaly $\zeta$ ,

$$\zeta = \frac{T}{\gamma^{*}} \quad . \qquad (6)$$

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The amount of C in (1) is the plane topographical reduction of the gravity,

$$C = f\varrho \int_{\psi=0}^{\beta} d\psi \int_{\alpha=0}^{2\pi} d\alpha \cdot r_P{}^2 \psi \int_{a=0}^{Z} \frac{a \cdot da}{(a^2 + r_P{}^2 \psi^2)^{3/2}} \quad . \qquad (7)$$

f is the gravitational constant, $\varrho$ is the standard density of the Earth, $\varrho = 2.65 \left[g\ cm^{-3}\right]$. $\psi$ is the spherical distance. $\beta$ is a sufficient great value of $\psi$ ; if $\psi > \beta$ , the integration with regard to $\psi$ can be finished, since an eventual extension beyond of $\beta$ will have no effect on C. $\alpha$ is the azimuth. $r_P$ has the equation, (Fig. 1),

$$r_P = R + h_P \quad . \qquad (8)$$

$h_P$ is the height which the point P has above the globe. The flattening is neglected here. It is necessary to stress the fact that the $h_P$ value is not the height above the geoid, and not the normal height $h^*$ . The geocentric radius of any surface point has the relation, Fig. 1,

$$r = R + h = R + h^* + \zeta \quad . \qquad (9)$$

The r values of the test point P and of the running point Q differ by

$$Z = r_Q - r_P = h_Q - h_P = h_Q^* - h_P^* + \zeta_Q - \zeta_P \quad . \qquad (10)$$

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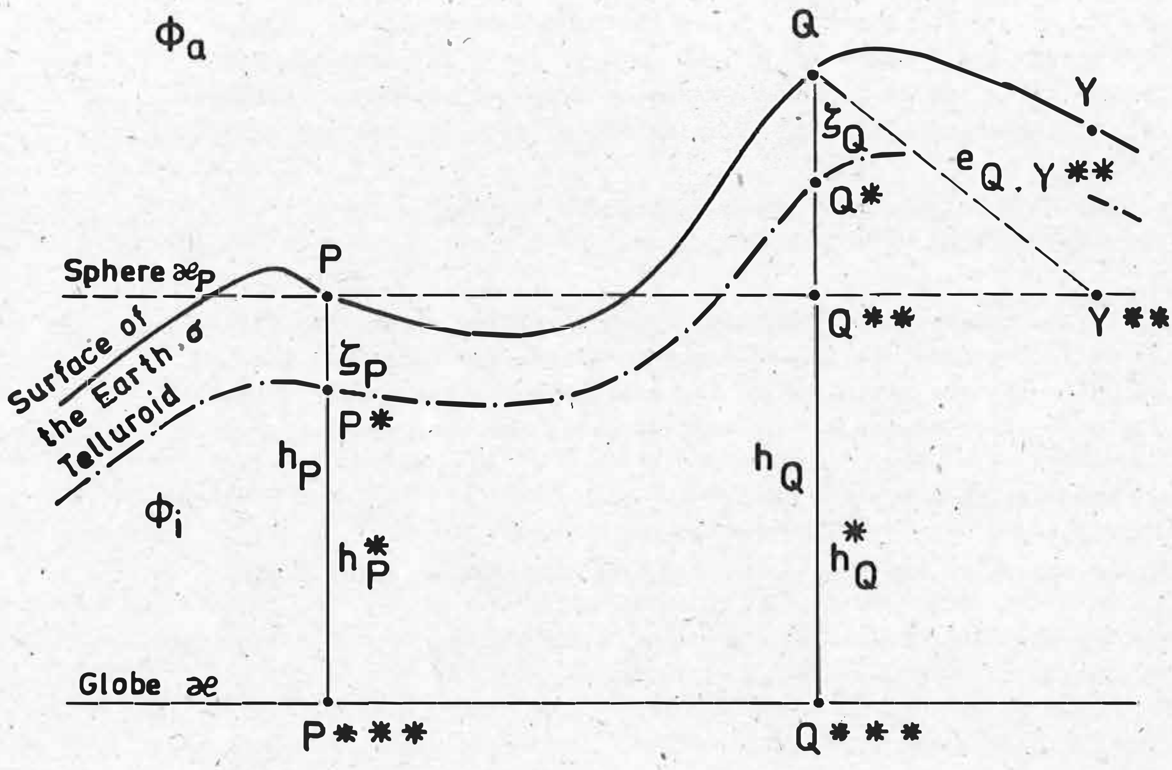


<u>Figure 1:</u> The test point P and the moving point Q, the height anomalies $\zeta$ , the normal heights $h^*$, and the heights h above the globe. The point $Q^{**}$ is the point where the plumb-line of Q meets the sphere $\varkappa_P$. This plumb-line meets the telluroid at the point $Q^*$.

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The term $C_1$ in the integrand of (1) derives - on the continents - from the Bouguer anomalies, $\Delta g_{Bouguer}$. Along the oceans, $C_1$ derives from the free-air anomalies. $C_1$ is proportional to the amount of the height gradient of the refined Bouguer anomalies - on the continents -, i.e. the familiar Bouguer anomalies supplemented by the C values. The amount of the $C_1$ term represents the change the refined Bouguer anomalies undergo if they are transported from the surface of the Earth $\sigma$ in the vertical direction to the level of the test point P, [3],

$$C_1 \cong - \frac{\partial \Delta g_{Bouguer}}{\partial r} (h_Q - h_P) \quad . \tag{11}$$

The relation (11) is a good representation of the $C_1$ term, as long as the following inequation is valid, ([3], chapter A),

$$\left| \alpha \right| = \left| \pi \sqrt{2} \frac{h_Q - h_P}{D} \right| << 1 \quad . \tag{12}$$

The length D is the horizontal extension of the area of the positive or negative Bouguer anomaly , see (100). The relation (12) is valid nearly at all places of the globe.

But, if the length D is understood to be the horizontal extension of the areas - on the continents - where the free-air anomalies have positive or negative amounts; in this case, the inequation (12) will be valid for lowland areas, only. The reason is found in the fact that the free-air anomalies show a strong correlation with the heights.

In case, the criterion (12) is fulfilled by the Bouguer anomalies, it is allowed to express the $C_1$ term by an integral which covers the Bouguer anomalies in the surroundings of the test point P,

$$C_1 \cong - (h_Q - h_P) \frac{1}{2\pi} \iint_{\varkappa} \frac{(\Delta g_{Bouguer})_Y - (\Delta g_{Bouguer})_Q}{e_0^3} \, d\varkappa_Y \tag{13}$$

As to (13), the following relations are valid,

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$$e_o = 2 R \sin \frac{\psi}{2} \quad , \tag{14}$$

and

$$d\varkappa = R^2 \, d\omega = R^2 \cos \varphi \; d\varphi \; d\lambda \quad . \tag{15}$$

Within the process of the integration according to (13), the point Y is the moving point ,( see Fig. 1 ).

In the relation (1), $S(\psi)$ represents the well-known Stokes function. The amount of $\Xi_1$ can be neglected in many cases. B is the model potential of the mountain masses which are situated above the ocean level. The standard density $\rho = 2.65 \left[ g\,cm^{-3} \right]$ is applied here. $B_c$ is the potential of these mountain masses condensed at the globe $\varkappa$ , Fig. 1, (see the equation (44), later on). In the relations (1) and (2), the term [B]" appears also. It is the difference between the Potential B at the surface point P and the potential $B_c$ at the point $P^{***}$, Fig. 1, [1] [2] [3].

The interpretation of the terms $\Delta g_T$, C, $C_1$ which appear in the Stokes integral is interesting.

As to $\Delta g_T$, this term depends on the global gravimetric measurements only, (5). It depends on the gravity measurements g at the surface of the Earth. Further on, the normal height $h^*$ is involved, since the standard gravity $\gamma^*$ at the telluroid point $Q^*$ has to be computed by $h^*$ , Fig. 1. And, $h^*$ is determined by the real potential values at the surface of the Earth which are obtained by a combination of the levellings with the gravity measurements.

As to the plane topographic reduction of the gravity C, (7), this term has the character of a gravity value.

Here, for the computation of C by (7), the differences of the rough data of the heights are of importance, with the precision of about some meters only. The height differences enter into the C term quadratic.

As to the $C_1$ term of (1), it depends on the isostatic mountain roots and on the density anomalies in the interior of the Earth, i. e. the amount by which the density of the geological masses deviates from the standard density $\rho$ . The $C_1$ term is proportional to the second vertical derivative of the potential of these density anomalies, (11) (13).

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With other words, $C_1$ is proportional to the first vertical derivative of the Bouguer anomalies - along the continental areas -. It has to be replaced by the first vertical derivative of the free-air anomalies, in case of oceanic areas.

Thus, the three terms $\Delta g_T$, C and $C_1$ have different sources and they have different characters, too.

As developed at other places, [3], the relation (1) is the solution of the boundary value problem of Molodenskij. This solution is of interest for the first mixed boundary value problem also, [2] [3]. - The other kind of solution (Green theorem), [1], does not involve any series development of bad or dubious convergence. In [1], the gravimetric or physical values do not move from the surface of the Earth to the globe, or to the sphere $\varkappa_P$, by series developments; but the geometric values, as the straight lines for instance, move from the sphere $\varkappa_P$ to the surface of the Earth by closed mathematical transformations, - e. g. the square of the chord has to be supplemented by the addition of the square of the height difference (Pythagoras), and by other terms, too - .

Now, a variant of the boundary value problem of Molodenskij is to be discussed. This variant is of importance also for the second mixed boundary value problem of the geodesy. This modified Molodenskij problem has not the free-air anomalies $\Delta g_T$, (5), as boundary values, but the gravity disturbances $\delta g$ serve now as boundary values at the surface of the Earth,

$$\delta g = -\frac{\partial T}{\partial r} = (g - \gamma)_Q = g - \gamma \quad . \qquad (16)$$

Here, in (16), the g value is again the measured gravity at the running point Q at the surface of the Earth $\sigma$ . The standard gravity $\gamma$ refers in (16) to even the same point Q. For the computation of the standard gravity at the point Q, the data of the heights $h_Q$ must be known empirically, Fig. 1. The point $Q^*$ is here not introduced, $Q^*$ enters into the computation of the free-air anomalies, (5).

The problem now intended to be solved consists in the determination of the perturbation potential T at the Earth's surface by means of the

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boundary values of (16), measured at the surface of the Earth. This are the gravity disturbances $\delta g$. Further on, the thus obtained global T values at the surface of the Earth will lead to the T values in the exterior space of the Earth, following the way given by the Dirichlet boundary value problem.

Introducing the $\delta g$ boundary values on the surface of the Earth, the investigation of the thus obtained boundary value problem will conduct to a solution by a kernel function, at least for the main term of the solution. Additionally, some small supplementary terms have to be added.

## 2. The Bjerhammar sphere.

It is allowed to introduce the globe $\varkappa$ with the radius R as a Bjerhammar sphere. The following ideas are connected with this procedure: In the exterior of the globe $\varkappa$, that is in the space $\Phi = \Phi_a + \Phi_i$, it is allowed to introduce a potential V. V is a harmonic function, regular in the space $\Phi$, Fig. 1. $\Phi_i$ is the space between the two surfaces $\sigma$ and $\varkappa$. $\Phi_a$ is the space exterior of $\sigma$. The amounts of the potential V at the surface of the Earth $\sigma$, and in the exterior space $\Phi_a$, are equal to the amounts of the perturbation potential T, at least within certain discrepancies the amounts of which are arbitrary small.

It is possible to apply the theorem of Keldysh-Lavrentiev on the potentials V and T:

There is given a function T which is harmonic in the exterior space of the surface of the Earth $\sigma$. Thus, it is harmonic in $\Phi_a$. This function T is regular in the exterior space $\Phi_a$ and on the surface $\sigma$. After this presupposition is fulfilled, it is allowed to introduce a harmonic function V which is harmonic and regular in the exterior space $\Phi$ of the globe $\varkappa$ (a regular function is unique and continuous).

$$\Delta V = 0 \quad , \text{in } \Phi . \qquad (16a)$$

The globe $\varkappa$ lies completely within the mass of the Earth. According to the theorem of Keldysh-Lavrentiev, this spatial function V approximates

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the spatial function T in $\Phi_a$ and on $\sigma$ in the following way, [4]:
It is possible to find an arbitrary small and positive number $\varepsilon_1$, thus, that

$$|T - V| < \varepsilon_1 \text{, in } \Phi_a \text{ and on } \sigma \text{ ,} \qquad (17)$$

with

$$\varepsilon_1 > 0 \text{ .} \qquad (18)$$

Thus, in the here discussed applications, it is admitted to equalize T and V in $\Phi$ and on $\sigma$. The geophysical meaning of the potential V in the space $\Phi_i$ must not be interpreted in this context, it is of no use for the subsequent developments.

This theorem of Keldysh-Lavrentiev is easily derived, e. g. by a procedure that employs the fact that the spherical harmonics series development for T is uniform convergent in $\Phi_a$ and along $\sigma$, [3]. The truncation of this series at a term of sufficient high degree and order leads to the neglection of an arbitrary small residual term. The truncated series is a sum, the validity of it can be extended downwards into the space $\Phi_i$. The truncated series represents the potential V.

Thus, the considerations about the theorem of Keldysh-Lavrentiev show that the area of validity of the harmonic potential T is allowed to be extended. The perturbation potential T can be considered as a function that is harmonic and regular in the exterior $\Phi$ of $\varkappa$. In $\Phi_a$ and along $\sigma$, T is the gravimetrically well-defined perturbation potential. In this context, the space $\Phi_i$ is considered as being free of masses. Hence, obviously,

$$\Delta T = 0 \text{, in } \Phi \text{ .} \qquad (19)$$

$$T = W - U \text{, in } \Phi_a \text{ and on } \sigma \text{ .} \qquad (20)$$

W is the real gravity potential and U is the standard potential of the level ellipsoid. The meaning of T in $\Phi_i$ must not be discussed in this context.

These ideas connected with the Bjerhammar sphere conduct to the following considerations.

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The test point P is situated on the sphere $\varkappa_P$ which has the radius $r_P$, Fig. 1. The surface of $\varkappa_P$ is placed completely in the space $\Phi$ which is the area of validity of the perturbation potential T, according to (19) and (20). $\varkappa_P$ serves now as the spherical boundary surface along which the gravity disturbances

$$\delta g = (\delta g)_{\varkappa_P} = (\delta g)_{Q^{**}} \tag{20a}$$

are distributed as boundary values. The following rigorous solution is easily found, /6/,

$$T_P = \frac{r_P}{4\pi} \iint\limits_{\omega} \delta g \cdot H(\psi) \cdot d\omega \quad , \tag{21}$$

with, (16),

$$\delta g = (\delta g)_{Q^{**}} = -\left(\frac{\partial T}{\partial r}\right)_{Q^{**}} \quad ; \text{ for } r = r_P \; . \tag{22}$$

The values of (22) are understood as valid for the points placed on the sphere $\varkappa_P$. The geophysical interpretation of the boundary values of (22) is not necessary. $H(\psi)$ is the Hotine function, /5/ /6/,

$$H(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi) =$$

$$= \operatorname{cosec}\frac{\psi}{2} - \ln\left(1 + \operatorname{cosec}\frac{\psi}{2}\right). \tag{23}$$

The Hotine function comprises the spherical harmonics of all degrees, the degrees n = 0 and n = 1 included. But, the Stokes function is free of these degrees of the numbers n = 0 and n = 1, (1).

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## 3. The model potential of the mountain masses.

The mountain masses which are situated above the ocean level are now in the fore of the considerations. In the subsequent model computations, the standard density $\varrho = 2.65 \left[ g\ cm^{-3} \right]$ is attributed to these masses. These model mountain masses are the gravitating sources of the potential B which is well-defined in the exterior space $\Phi_a$, Fig. 1,

$$B = I_a = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r=R}^{R+h} \frac{1}{\varepsilon} r^2 \sin\psi \, d\psi \, d\alpha \, dr. \qquad (24)$$

$\varepsilon$ is the oblique distance between the test point P and the mass element which is moving in the course of the integration.

It is now intended to apply the theorem of Keldysh-Lavrentiev to the potential B. A similar procedure was already considered with regard to the potential T, (19) (20): The potential B is introduced as a harmonic and regular function in the exterior space of the globe $\varkappa$ , this is the space $\Phi = \Phi_i + \Phi_a$. By the integral (24), B is a well-defined function along the surface $\sigma$ and in the exterior space $\Phi_a$. Since the theorem of Keldysh-Lavrentiev is intended to be applied, the space $\Phi_i$ is now considered to be free of gravitating masses. Therefore, the B potential is now defined in the following way, (24),

$$\Delta B = 0, \text{ in } \Phi . \qquad (25)$$

$$B = I_a, \text{ in } \Phi_a. \qquad (26)$$

The relation (26) describes the geophysical meaning of B in the space $\Phi_a$. The geophysical meaning of the function B, (25) and (26), in the space $\Phi_i$ is not discussed. Such a discussion would be of no use.

The values of the potential B at the surface of the Earth $\sigma$ can be obtained from the radial derivative of B on the sphere $\varkappa_P$. The following integral transformation performed by the Hotine kernel does solve this problem, (21),

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$$B_P = - \frac{r_P}{4\pi} \iint_{\omega} \frac{\partial B}{\partial r} H(\psi) \, d\omega \quad . \qquad (27)$$

The radial derivative in the integrand of (27),

$$- \frac{\partial B}{\partial r} = - \left(\frac{\partial B}{\partial r}\right)_{\varkappa_P} = - \left(\frac{\partial B}{\partial r}\right)_{Q^{**}} , \qquad (28)$$

is understood that it does lie on the surface of the sphere $\varkappa_P$ which has the radius $r_P$.

In the exterior space $\Phi_a$, the potential B can be represented by a uniform convergent spherical harmonics series development, /3/,

$$B = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^{n+1} P_{n.m}(\sin\varphi) \left[ b_{1.n.m} \cos m\lambda + b_{2.n.m} \sin m\lambda \right] , \text{ in } \Phi_a . \qquad (29)$$

Since the series of B, (29), is proved to be uniform convergent, it is possible to bring it into the following abbreviating shape,

$$B = B_N + B_{N.1} \quad , \quad \text{in} \quad \Phi_a \; , \qquad (30)$$

with

$$B_N = \sum_{n=0}^{N} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^{n+1} P_{n.m}(\sin\varphi) \left[ b_{1.n.m} \cos m\lambda + b_{2.n.m} \sin m\lambda \right] . \qquad (30a)$$

The following theorem describes the uniform convergence of (29): Corresponding to an arbitrary small positive number

$$\left| \varepsilon_N \right| > 0 \quad , \qquad (31)$$

it is possible to find an integer N such that the inequation

$$\left| B_{N.1} \right| < \left| \varepsilon_N \right| \qquad (32)$$

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is valid in the space $\Phi_a$.

The relation (30a) represents the function $B_N$ by a sum. This sum fulfills the Laplace differential equation, since the individual terms of it fulfill this equation. The term $|\varepsilon_N|$ is negligible, if N is sufficient great. Thus, B can be substituted by $B_N$, in sufficient approximation in the area of $\Phi_a$. Consequently, $B_N$ can take up the role of B in (25) and (26). Hence, it is self-explanatory, the validity of the theorem of Keldysh-Lavrentiev is proved in a rather simple way by the introduction of the convergent series development (29).

As to the perturbation potential T, it has also such a convergent series development in $\Phi_a$, as the form (29) of B.

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## 4. The superposition.

The difference of the two potentials T and B leads to the potential M,

$$M = T - B . \tag{33}$$

The vertical derivatives of the potentials T and B show a distinct correlation with the topographic heights. The vertical derivatives of the two potentials T and B yield certain gravity values which depend from the heights by a linear function. The concerned correlation coefficient is well-known by empirical means, the amount of it is about 0.1,

$$\delta g' = 0.1 \cdot h \quad , \tag{34}$$

The gravity variation $\delta g'$ is measured in mgal, h in meters. However, the correlation with the heights does no more exist in the difference of these two gravity values,

$$\frac{\partial M}{\partial r} = \frac{\partial T}{\partial r} - \frac{\partial B}{\partial r} \quad . \tag{35}$$

Indeed, the two individual terms on the right hand side of (35) show a distinct correlation with the topographic heights, but the left hand side of this equation is, on the whole, free of a correlation of this kind. The radial derivative of M is a smoothed function, it is as so smoothed as the Bouguer anomalies; and a clear correlation with the heights is no more existent, (70a).

The theorem of Keldysh-Lavrentiev is valid also for the potential M. Therefore, the potential M which is well-defined in the exterior space $\Phi_a$ obeys the following relations, (19) (20) (24) (25) (26) (33),

$$\triangle M = 0, \text{ in } \Phi \ , \tag{36}$$

$$M = W - U - I_a, \text{ in } \Phi_a \ . \tag{37}$$

The equation (37) shows the geophysical meaning of M in the exterior space, $\Phi_a$. The geophysical meaning of M in the interior space $\Phi_i$

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must not be discussed in this context.

The M potential can be expressed by the Hotine kernel function and the radial derivatives of M along the sphere with the radius $r_P$, (27).

$$M_P = - \frac{r_P}{4\pi} \iint_{\omega} \frac{\partial M}{\partial r} H(\psi) \, d\omega \quad . \tag{38}$$

The radial derivatives

$$- \frac{\partial M}{\partial r} = - \left( \frac{\partial M}{\partial r} \right)_{Q^{**}} \tag{39}$$

are again understood that they are valid for points placed at the surface of the sphere $\varkappa_P$ which has the radius $r_P$ ; Fig. 1.

The relation (38) is of fundamental importance for the following deductions.

$Q^{**}$ is the moving point at the sphere $\varkappa_P$ , Q is the corresponding point at the surface of the Earth $\sigma$ , the place of Q is vertical above $Q^{**}$; Fig. 1. The height of Q above $\varkappa_P$ is $h_Q - h_P$. Now, the radial derivatives according to the relation (39) must be expressed by these derivatives placed at the surface $\sigma$ , instead of the surface $\varkappa_P$ ,

$$- \left( \frac{\partial M}{\partial r} \right)_Q \quad . \tag{40}$$

The following relation can be formulated,

$$- \left( \frac{\partial M}{\partial r} \right)_{Q^{**}} = - \left( \frac{\partial M}{\partial r} \right)_Q + C_1 \quad , \tag{41}$$

or,

$$C_1 = - \left( \frac{\partial M}{\partial r} \right)_{Q^{**}} + \left( \frac{\partial M}{\partial r} \right)_Q \quad . \tag{42}$$

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The relations (33) (35) (41) turn the expression (38) into

$$T - B = - \frac{r_P}{4\pi} \iint_{\omega} \left\{ \left[ \frac{\partial T}{\partial r} - \frac{\partial B}{\partial r} \right]_Q - C_1 \right\} H(\psi) \; d\omega \quad . \qquad (43)$$

The integral (24) expresses the potential B at the surface $\sigma$ by the mountain masses of the standard density. In the further developments, the potential of these mountain masses condensed at the globe $\varkappa$ is introduced also. This potential is denoted by $B_c$. It has the following expression valid in test points at the globe $\varkappa$ which has the radius R, (4) (14),

$$B_c = f \varrho \; R^2 \iint_{\omega} h_Q \frac{1}{e_o} \, d\omega \quad . \qquad (44)$$

$B_c$ is the potential of a spherical surface distribution. For test points on $\varkappa$, the potential $B_c$ has the following limit value for its vertical derivative, if approaching the sphere $\varkappa$ from the exterior space, ($B_c$, $\partial B_c/\partial r$ without the suffix $P^{***}$ resp. $Q^{***}$ in the equations (45) to (49) ), [3] ,

$$\frac{\partial B_c}{\partial r} = - 4\pi \, f \varrho \; h_P - f \varrho \; R^2 \iint_{\omega} (h_Q - h_P) \frac{\sin \frac{\psi}{2}}{e_o^2} \, d\omega_Q \; . \qquad (45)$$

The potential B at $\sigma$ is now divided into two parts, (24) (44),

$$B = B_c + [B]'' \quad . \qquad (46)$$

The radial derivative of B at $\sigma$ is divided in a similar way,

$$\frac{\partial B}{\partial r} = \frac{\partial B_c}{\partial r} + \left[ \frac{\partial B}{\partial r} \right]'' \quad . \qquad (47)$$

The second terms on the right hand side of (46) and (47) are always much more small than the corresponding first terms on the right hand sides of these equations. The two expressions

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$$[B]'' \text{ and } \left[\frac{\partial B}{\partial r}\right]'' \qquad (47a)$$

depend on the squares of the heights, above all, [1] [2] [3]. Obviously, the following relation is right, (38), at points situated on the sphere $\varkappa$,

$$B_c = -\frac{R}{4\pi}\iint_{\omega} \frac{\partial B_c}{\partial r}\, H(\psi)\, d\omega \quad . \qquad (48)$$

The relations (46) and (47) and (43) give,

$$T - B_c - [B]'' =$$

$$= -\frac{r_P}{4\pi}\iint_{\omega}\left\{\left(\frac{\partial T}{\partial r}\right)_Q - \frac{\partial B_c}{\partial r} - \left[\frac{\partial B}{\partial r}\right]'' - C_1\right\} H(\psi)\, d\omega \quad . \qquad (49)$$

With

$$r_P = R + h_P \quad , \qquad (50)$$

and with (48) follows,

$$T = \frac{r_P}{4\pi}\iint_{\omega}\left\{-\left(\frac{\partial T}{\partial r}\right)_Q + \left[\frac{\partial B}{\partial r}\right]'' + C_1\right\} H(\psi)\, d\omega \, +$$

$$+ [B]'' + \frac{h_P}{4\pi}\iint_{\omega}\frac{\partial B_c}{\partial r}\, H(\psi)\, d\omega \, . \qquad (51)$$

An earlier publication shows, [1] [3],

$$\left[\frac{\partial B}{\partial r}\right]'' - C = f\varrho \int_{\psi=0}^{\pi} d\psi \int_{\alpha=0}^{2\pi} d\alpha \int_{a=0}^{Z} \Psi_1 \, da \, +$$

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$$+ \frac{1}{2} f \varrho \; R \iint_{\omega} \frac{1}{e_{0}} \; \Psi_2 \, d\omega \; + 4 \pi \; f \varrho \; h_P \frac{h_P}{R} . \qquad (52)$$

Z comes from (10). As it is evidenced by the computations in [3], the sum of the first and second term on the right hand side of (52) is smaller than 1 $\mu$gal for a neighbouring mountain of about 2 or 3 km height. The rather simple third term of (52) amounts to about 0.15 mgal for a height $h_P$ of 3 km.

The introduction of the relation

$$\left[ \frac{\partial B}{\partial r} \right]'' \cong C \quad , \qquad (53)$$

and of (16) and (48) transforms the equation (51) to,

$$T = \frac{r_P}{4\pi} \iint_{\omega} ( \delta g + C + C_1 ) \; H(\psi) \; d\omega \; + \; \Xi_2 \; , \qquad (54)$$

with,

$$\Xi_2 = [B]'' - \frac{h_P}{R} B_c \; . \qquad (54a)$$

Now, in order to be more precise, the braces $\{ \}$ are introduced in the expression (54a).

$$\Xi_2 = \left\{ [B]'' - \frac{h_P}{R} B_c \right\} =$$

$$= \left\{ [B]'' \right\} - \left\{ \frac{h_P}{R} B_c \right\} \cong$$

$$\cong [B]'' - \left\{ \frac{h_P}{R} B_c \right\} \; . \qquad (54b)$$

The braces $\{ \}$ are understood to have the meaning that the constituents contributed by the spherical harmonics of the 0th and 1st degree have to

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be subtracted.

As it is evidenced by an earlier publication, [3], the first term on the right hand side of (54a) amounts to not more than some millimeter, after it is divided through the global mean of the gravity G,

$$\frac{1}{G} [B]'' \quad . \tag{55}$$

The second term on the right hand side of (54a) is also very small. The relation (44) allows the precise evaluation of it. A rough estimation leads to

$$\frac{1}{G} \frac{h_P}{R} B_c \cong 6 \text{ cm} \quad , \tag{55a}$$

if $B_c = 200$ m $\cdot$ G, and $h_P = 2$ km. This is the amount by which the final result, the height anomaly $\zeta$, is shifted.

In extreme cases, it seems to be possible that the amount of $B_c = 1000$ m . can be reached. This value and an amount of $h_P = 4$ km lead to the considerable value of 0.7 m for the expression of (55a); (see: Drewes, H. et al.: Wirkung der Undulationen der Grenzflächen der Lithosphäre auf das Geoid. Veröff. d. Bayerischen Kommission f. d. Internationale Erdmessung. Astron. - Geod. Arbeiten, 48 (München) 1986).

Before this background, the term $\Xi_2$ on the right hand side of (54) is not considered, further on, in the following deductions. Therefore, the evaluation of the amount of the $C_1$ term, (42), remains as the task of the subsequent developments.

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## 5. The relation between the Bouguer anomalies and the model anomalies.

At first, the amount of the term

$$\left[\frac{\partial M}{\partial r}\right]_Q \tag{56}$$

on the right hand side of (42) is to be considered. The relations (5) (33) (35) give,

$$\frac{\partial M}{\partial r} = \frac{\partial T}{\partial r} - \frac{\partial B}{\partial r} \quad . \tag{57}$$

(45) and (47) lead to,

$$\frac{\partial B}{\partial r} = - 4 \pi f \varrho h_P - f \varrho R^2 \iint_{\omega} (h_Q - h_P) \frac{\sin \frac{\psi}{2}}{e_o^2} d\omega + \left[\frac{\partial B}{\partial r}\right]'' . \tag{58}$$

And with (14),

$$\frac{\sin \frac{\psi}{2}}{e_o^2} = \frac{1}{2 R e_o} \quad . \tag{59}$$

The introduction of (53) and (59) into (58) has the following result,

$$\frac{\partial B}{\partial r} = - 4 \pi f \varrho h_P - \frac{1}{2} f \varrho R \iint_{\omega} (h_Q - h_P) \frac{1}{e_o} d\omega + C. \tag{60}$$

(16), (57) and (60) are combined,

$$- \frac{\partial M}{\partial r} = \delta g - 4 \pi f \varrho h_P - \frac{1}{2} f \varrho R \iint_{\omega} (h_Q - h_P) \frac{1}{e_o} d\omega + C. \tag{61}$$

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With

$$\iint_{\omega} \frac{1}{e_o} \, d\omega = 4 \pi \frac{1}{R} \quad , \tag{62}$$

follows for (61),

$$-\frac{\partial M}{\partial r} = \delta g - 2 \pi f \varrho \, h_P - \frac{1}{2} f \varrho \, R \iint_{\omega} h_Q \frac{1}{e_o} \, d\omega + C. \tag{63}$$

Now, the relation representing the refined Bouguer anomalies $\Delta g_{Bouguer}$ is required, (see: Ledersteger, Astronomische und physikalische Geodäsie: Jordan/Eggert/Kneissl, Handbuch der Vermessungskunde, Band V, Stuttgart 1969.) , (8) (9) (1o) .

$$\Delta g_{Bouguer} = g + C - 2 \pi f \varrho \, h_P^* + 2 G \frac{h_P^*}{R} - \gamma_e \quad , \tag{64}$$

g is the gravity measured at the surface of the Earth. $h_P^*$ is the normal height. $\gamma_e$ is the standard gravity at the level ellipsoid. The gravity disturbance has the following expression, (6) (16),

$$\delta g = g - \gamma = g - \gamma_e + 2 G \frac{1}{R} (h_P^* + \zeta) \quad . \tag{65}$$

(64) and (65) give

$$\Delta g_{Bouguer} = \delta g - 2 \pi f \varrho \, h_P^* + C - \frac{2 G}{R} \zeta \quad , \tag{66}$$

and with (63)

$$\Delta g_{Bouguer} + \frac{\partial M}{\partial r} = 2 \pi f \varrho \zeta - \frac{2 G}{R} \zeta +$$

$$+ \frac{1}{2} f \varrho \, R \iint_{\omega} h_Q \frac{1}{e_o} \, d\omega \quad . \tag{67}$$

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The following relations are well-known, they define the attraction of the Bouguer plate and the free-air gradient of the gravity, /5/,

$$2 \pi f \rho = 0.1119 \left[ \text{mgal m}^{-1} \right] , \qquad (68)$$

and

$$2 \frac{G}{R} = 0.3086 \left[ \text{mgal m}^{-1} \right] . \qquad (69)$$

Thus,

$$\Delta g_{\text{Bouguer}} + \frac{\partial M}{\partial r} = \vartheta , \qquad (70)$$

or

$$\frac{\partial M}{r} = - \Delta g_{\text{Bouguer}} + \vartheta , \qquad (70a)$$

with, (67) (68) (69),

$$\vartheta = \vartheta_1 + \vartheta_2 , \qquad (71)$$

$$\vartheta_1 = - 0.2 \left[ \text{mgal m}^{-1} \right] \zeta , \qquad (72)$$

$$\vartheta_2 = \frac{1}{2} f \rho \; R \iint_{\omega} h_Q \frac{1}{e_o} \, d\omega . \qquad (73)$$

In the equation (72), $\vartheta_1$ is measured in mgal and $\zeta$ in meter.

Now, the amounts of $\vartheta_1$ and $\vartheta_2$ are evaluated for a mountain model. This model consists of a mountain massif of 2 km height and a horizontal extension of 50 km x 50 km. The amount of $\vartheta_2$ was computed up to a distance of 200 km from this mountain massif. The amount of $\vartheta_1$ was computed for the same area, and for a course of the $\zeta$ values which is in keeping with a plumb-line deflection of 20",

$$\frac{\partial \zeta}{\partial e_o} = 20'' \frac{1}{\rho''} = 10^{-4} . \qquad (74)$$

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Along these lines, the table 1 was computed.

Table 1

| $e_o$ [km] | $\zeta$ [m] | $\vartheta_1$ [mgal] | $\vartheta_2$ [mgal] | $\vartheta$ [mgal] |
|---|---|---|---|---|
| 0 | 0 | 0 | 0.5 | 0.5 |
| 40 | 4 | - 0.8 | 0.2 | - 0.6 |
| 60 | 6 | - 1.2 | 0.1 | - 1.1 |
| 80 | 8 | - 1.6 | 0.1 | - 1.5 |
| 100 | 10 | - 2.0 | 0.1 | - 1.9 |
| 140 | 14 | - 2.8 | 0.1 | - 2.7 |
| 160 | 16 | - 3.2 | 0.0 | - 3.2 |
| 180 | 18 | - 3.6 | 0.0 | - 3.6 |
| 200 | 20 | - 4.0 | 0.0 | - 4.0 |

In order to avoid misunderstandings, it is necessary to stress the fact that not the $\vartheta$ values themselves are of direct interest, they do not figure in the solution of this boundary value problem as terms to be added to any gravity value, by no means. It is the vertical derivative of the $\vartheta$ values that is here of direct interest. The effect that is of interest here that is the influence the $\vartheta$ values take on the $C_1$ term. Of course, (42)and(7o)and(7oa) can be combined to

$$C_1 = (\Delta g_{Bouguer} - \vartheta)_{Q**} - (\Delta g_{Bouguer} - \vartheta)_Q \, , \tag{75}$$

or

$$C_1 = C_{1.1} + C_{1.2} \, , \tag{76}$$

$$C_{1.1} = (\Delta g_{Bouguer})_{Q**} - (\Delta g_{Bouguer})_Q \, , \tag{77}$$

$$C_{1.2} = - \left[ \vartheta_{Q**} - \vartheta_Q \right] \, . \tag{78}$$

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The effect that the $\vartheta$ values exert on the $C_{1.2}$ term that is the matter of interest.

The amount of the $C_{1.2}$ term will be computed later, in context with the evalutation of the $C_{1.1}$ term, (125) (130) (133). Already now, it can be anticipated, the $\vartheta$ function of the structure given by the table 1 gives rise to a $C_{1.2}$ term that amounts to not more than about 0.06 mgal, (130). This value can be neglected.

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## 6. The height gradient of the Bouguer anomalies.

In the sucoeeding investigations, the vertical change of the Bouguer anomalies is in the fore, (77). The model potential M is divided into two parts, $M_1$ and $M_2$, (76), (42),

$$C_1 = C_{1.1} + C_{1.2} = - \left(\frac{\partial M_1}{\partial r}\right)_{Q^{**}} + \left(\frac{\partial M_1}{\partial r}\right)_Q -$$

$$- \left(\frac{\partial M_2}{\partial r}\right)_{Q^{**}} + \left(\frac{\partial M_2}{\partial r}\right)_Q \quad . \tag{79}$$

A comparison with the formulas from (75) to (78) shows that the following substitutions are right,

$$\Delta g_{Bouguer} = - \frac{\partial M_1}{\partial r} \quad , \tag{80}$$

and

$$\vartheta = \frac{\partial M_2}{\partial r} \quad . \tag{81}$$

In case, a spherical model Earth is considered, the vertical gradient of the free-air anomalies has a well-known integral representation, /1/ /5/,

$$\frac{\partial \Delta g_F}{\partial h} = \frac{R^2}{2\pi} \iint_\omega \frac{\Delta g_F - \Delta g_{F,0}}{e_0^3} \, d\omega \quad . \tag{82}$$

The essential part of the amount on the right hand side of (82) is obtained by an extension of the integration only over the surroundings of about A = 100 km, around the test point. Thus, a consideration of the vertical gradient which appears in (79) depends essentially on the struoture of the gravity field of the

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near surroundings of the test point, up to a distance of A = 100 km. In case, the following inequation is valid,

$$\frac{A}{R} \ll 1 \quad , \tag{83}$$

the here intended investigations will allow to neglect the curvature of the surface of the Earth. Therefore, the essential parts of the succeeding evaluation of $C_{1.1}$ and $C_{1.2}$ will undergo no modification if the spherical geometry is replaced by the plane geometry. This fact is corroborated by an investigation whioh uses spherical harmonics.

In this context, a rectangular Cartesian co-ordinate system is introduced. The x, y = plane is the horizontal plane of the point $Q^{**}$ in the level of the sphere $\text{æ}_p$. The z = axis shows perpendicular upwards, Fig. 1. In this situation, the potential function $M_1$, (79), which satisfies the Laplace differential equation,

$$\Delta M_1 = 0 \quad , \tag{84}$$

can be represented by an analytical expression of the following shape, (see chapter A of [3]),

$$M_1 = m_1 \left[ \cos 2\pi \frac{x}{p} \cdot \cos 2\pi \frac{y}{q} - \right.$$

$$\left. - \sin 2\pi \frac{x}{p} \cdot \sin 2\pi \frac{y}{q} \right] \cdot$$

$$\cdot \exp\left( - 2\pi z \sqrt{\frac{1}{p^2} + \frac{1}{q^2}} \right), \text{ in } \Phi \ . \tag{85}$$

In case of a global extension, $M_1$ can be developed in spherical harmonics, instead of the Fourier development according to (85).

Returning back to the relation (85), it is to be stressed that the representation (85) is valid for all the points above the x, y = plane and for a horizontal extension of the system of the

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x, y = co-ordinates of some hundred kilometers, - according to the theorem of Keldysh - Lavrentiev,(19)(20), - $m_1$ is the amplitude, p and q are the wave lengthes in the x = and in the y = direction. A symmetrical distribution is choosen,

$$p = q = L \quad . \tag{86}$$

The Bouguer anomalies which are determined by the model potential $M_1$, (85), are understood that they have a maximum for x = 0, y = 0; therefore, the sinus functions in the brackets of (85) are not taken into account,from now. Hence, combining (85) and (86), $M_1$ gets this shape,

$$M_1 = m_1 \cos 2\pi \frac{x}{L} \cos 2\pi \frac{y}{L} \cdot \cdot \exp\left(- 2\pi \sqrt{2} \frac{z}{L}\right) \quad . \tag{87}$$

(87) fulfills the Laplace differential equation.

The radial derivative of (87) is

$$\frac{\partial M_1}{\partial r} = - 2\pi \sqrt{2} \frac{1}{L} M_1 \quad . \tag{88}$$

(80) (87) (88) result the following Bouguer anomalies,

$$\Delta g_{\text{Bouguer}} = 2\pi \sqrt{2} \frac{1}{L} m_1 \cos 2\pi \frac{x}{L} \cos 2\pi \frac{y}{L} \cdot \cdot \exp\left(- 2\pi \sqrt{2} \frac{z}{L}\right) \quad . \tag{89}$$

For x = 0, y = 0, the Bouguer anomalies have a maximum value with an amplitude of the amount K,

$$K = K(z) = 2\pi \sqrt{2} \frac{1}{L} m_1 \cdot \exp\left(- 2\pi \sqrt{2} \frac{z}{L}\right) \quad . \tag{90}$$

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According to (89), the positive amounts of the Bouguer anomalies extend around the origin of the x, y = co-ordinate system up to a distance of $\frac{1}{4}$ L. In case that x or y has this amount, the Bouguer anomalies are equal to zero, (89). Hence, the total horizontal extension of the positive Bouguer anomalies amounts to

$$D = \frac{1}{2} L \quad . \tag{91}$$

The combination of (89) and (91) gives,

$$\Delta g_{Bouguer} = \pi \sqrt{2} \frac{1}{D} m_1 \cos \pi \frac{x}{D} \cos \pi \frac{y}{D} \cdot \exp\left(- \pi \sqrt{2} \frac{z}{D}\right) \quad , \tag{92}$$

and, with (87), for the potential $M_1$,

$$M_1 = m_1 \cos \pi \frac{x}{D} \cos \pi \frac{y}{D} \cdot \exp\left(- \pi \sqrt{2} \frac{z}{D}\right) . \tag{93}$$

The relations (90), (92) and (93) lead to

$$\Delta g_{Bouguer} = \pi \sqrt{2} \frac{1}{D} M_1 = K(z) \cos \pi \frac{x}{D} \cos \pi \frac{y}{D} \quad . \tag{94}$$

The repeated vertical derivative of the potential $M_1$ has the following recursion formula, (87) (88),

$$\frac{\partial^i}{\partial r^i} M_1 = \left(- \pi \sqrt{2} \frac{1}{D}\right)^i M_1, \quad (i = 1, 2, \ldots) \quad . \tag{95}$$

or, with (94),

$$\frac{\partial^i}{\partial r^i} M_1 = -\left(- \pi \sqrt{2} \frac{1}{D}\right)^{i-1} \Delta g_{Bouguer}, \quad (i = 1, 2, \ldots). \tag{96}$$

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The x, y = plane was introduced as the horizontal plane of the point $Q^{**}$, Fig. 1. Therefore, this point $Q^{**}$ is defined by the relation $x = y = z = 0$. The corresponding point Q has the relations, (77),

$$x = 0, \; y = 0, \; z = Z = h_Q - h_P. \tag{97}$$

(77) and (94) yield,

$$C_{1.1} = \left[ K(0) - K(Z) \right] \cos \pi \frac{x}{D} \cos \pi \frac{y}{D} , \tag{98}$$

or, with (90) and (91),

$$C_{1.1} = \pi \sqrt{2} \frac{1}{D} m_1 \cos \pi \frac{x}{D} \cos \pi \frac{y}{D} \left[ 1 - \right.$$

$$\left. - \exp \left( - \pi \sqrt{2} \frac{Z}{D} \right) \right] \tag{99}$$

Introducing the substitution

$$\alpha = \pi \sqrt{2} \frac{Z}{D} = \pi \sqrt{2} \frac{h_Q - h_P}{D} , \tag{100}$$

the following series development of the term in the brackets of (99) is a convergent expression for all the amounts of $\alpha$,

$$|\alpha| < \infty , \tag{101}$$

$$1 - \exp(-\alpha) = 1 - 1 + \alpha - \frac{1}{2!} \alpha^2 + \frac{1}{3!} \alpha^3 - + \ldots, \tag{102}$$

or,

$$1 - \exp(-\alpha) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} \alpha^n . \tag{102a}$$

Hence, (99),

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$$C_{1.1} = \pi\sqrt{2}\ \frac{1}{D}\, m_1 \cos \pi \frac{x}{D} \cos \pi \frac{y}{D} \sum_{n-1}^{\infty} (-1)^{n+1}\ \frac{1}{n!} \alpha^n . \quad (103)$$

The series development (103) is always convergent. In case, the amount of $|\alpha|$ is sufficient small,

$$|\alpha| \ll 1 \ , \quad (103a)$$

it is allowed to truncate the series (103) behind the term linear in $\alpha$ , (77) (99) (103),

$$C_{1.1} \cong \pi\sqrt{2}\ \frac{1}{D}\, m_1 \cos \pi \frac{x}{D} \cos \pi \frac{y}{D} \cdot \alpha \ , \ |\alpha| \ll 1. \quad (104)$$

A comparison of (92) and (104) yields (for $z = o$ in (92)) the following relation,

$$C_{1.1} = \left[\Delta g_{Bouguer}\right]_{Q^{**}} \cdot \alpha \ , \quad |\alpha| \ll 1. \quad (105)$$

The equation (105) is a representation of $C_{1.1}$ by the Bouguer anomalies which are distributed along the Bjerhammar sphere with the radius $r_p$ (with the running point $Q^{**}$). These anomalies are not directly measured. The directly measured Bouguer anomalies are placed in the points Q at the surface of the Earth $\sigma$ , Fig. 1. Thus, in the equation (105), the Bouguer anomalies in the points $Q^{**}$ should be lifted upwards by the height Z, they should be replaced by the corresponding anomalies in the points Q, - it is a thought that suggests itself. Following up this aim, the relations (92), (99), (100) lead to

$$C_{1.1} = \left[\Delta g_{Bouguer}\right]_Q \left[1 - \exp(-\alpha)\right] \exp(\alpha) \ , \quad (106)$$

or,

$$C_{1.1} = \left[\Delta g_{Bouguer}\right]_Q \left[\exp(\alpha) - 1\right] \ . \quad (107)$$

In this closed relation, the development (108) is introduced,

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$$\exp(\alpha) - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} \alpha^n, \quad |\alpha| < \infty \quad . \qquad (108)$$

The truncation of this series by taking along its first term only gives the succeeding expression, instead of (105),

$$C_{1.1} = \left( \Delta g_{Bouguer} \right)_Q \cdot \alpha \;, \quad |\alpha| << 1. \qquad (109)$$

Now, the Bouguer anomaly is replaced by the analytical expression of it, (94). The equations (94) and (109) are combined to

$$C_{1.1} = \alpha \cdot K(Z) \cdot \cos \pi \frac{x}{D} \cos \pi \frac{y}{D} \;, \quad |\alpha| \leqslant 1 \,. \qquad (110)$$

As the above relation does show, the amounts of the $C_{1.1}$ term are positive and negative ,( for $-D \leqslant x \leqslant +D, -D \leqslant y \leqslant +D$ ).

The amount of this $C_{1.1}$ term, (109), is intended to be evaluated for two models of Bouguer anomalies. The test points Q are situated on the z-axis, ($x = 0$, $y = 0$), in these computations. Consequently, a relative maximum of the $C_{1.1}$ values is considered, because the Bouguer anomalies spread over the x, y = plane have a maximum, if $x = 0$, $y = 0$, (see (110)). Therefore, the model computations can happen by the subsequent relation, with $x = y = 0$, $z = Z$, (110) (109),

$$C_{1.1} = \alpha \cdot K(Z) = ( \Delta g_{Bouguer} )_Q \cdot \alpha \,. \qquad (110a)$$

Along the oceans, the Bouguer anomalies have to be replaced by the free-air anomalies, in (110a).

The first model has the following parameters,

$$\Delta g_{Bouguer} = 20 \text{ mgal},$$

$$h_Q - h_P = 1 \text{ km} \;, \qquad (111)$$

$$D = 50 \text{ km}.$$

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They lead to these values, (100),

$$\alpha = 0.09 \ll 1 , \tag{112}$$

and, (110a),

$$C_{1.1} = 1.8 \text{ mgal} . \tag{113}$$

The second model is characterized by

$$\begin{aligned} \Delta g_{Bouguer} &= 40 \text{ mgal}, \\ h_Q - h_P &= 1 \text{ km}, \\ D &= 200 \text{ km}. \end{aligned} \tag{114}$$

The amounts

$$\alpha = 0.02 \ll 1, \tag{115}$$

and

$$C_{1.1} = 0.9 \text{ mgal} \tag{116}$$

result by the computations. The values of (113) and (116) computed for $C_{1.1}$ are the maximal values of this term, for the considered waves. The mean value for $C_{1.1}$ will be smaller.

The relations (113) and (116) show that the amounts of $C_{1.1}$ are within the precision of the global gravity net of the present state. The cosmic missions of the future, as satellite gradiometry and satellite - to - satellite tracking, are hoped to give a global gravity field with a standard deviation of about ± 2 mgal for the compartments of 500 x 500 km size. (See: Wichiencharoen, C.; Recovery of $1^o$ - mean anomalies in a local region from a low - low satellite - to - satellite tracking mission. Ohio State Univ., Dept. geod. Sci., Rep. 363 (1985) ) .

The formula (109) results by a spectral representation of the field of the Bouguer anomalies. In the practical applications,

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such a spectral analysis of the Bouguer anomalies is not current in use in the routine computations. But, the continuous two-dimensional function of the Bouguer anomalies along the surface of the Earth is well-known by the concerned maps. Therefore, it is useful to show a way that leads to the determination of the amount of $C_{1.1}$ expressed by the continuous funotion of the Bouguer anomalies, without any Fourier analysis of these anomalies.

In this oontext, the terms in the equation (92) are differentiated in the vertioal direotion, the relation (100) is introduced, and the resulting power series development is truncated behind the term linear in $\alpha$ ,

$$\left[\frac{\partial}{\partial h} \Delta g_{Bouguer}\right]_{z = Z} = \left[\frac{\partial}{\partial h} \Delta g_{Bouguer}\right]_{z = 0} (1 - \alpha). \quad (117)$$

Further, the relation (92) leads to

$$\left[\Delta g_{Bouguer}\right]_{z = Z} = \left[\Delta g_{Bouguer}\right]_{z = 0} (1 - \alpha), \quad (118)$$

and

$$\left[\frac{\partial}{\partial h} \Delta g_{Bouguer}\right]_{z = Z} = \left[\Delta g_{Bouguer}\right]_{z = Z} \left(- \pi \sqrt{2} \frac{1}{D}\right). \quad (119)$$

The relations (109) and (119) are combined. They give the equation (121), for

$$|\alpha| \ll 1 \quad ; \quad (120)$$

$$C_{1.1} = - (h_Q - h_P) \left[\frac{\partial}{\partial h} \Delta g_{Bouguer}\right]_{z = Z} . \quad (121)$$

A look on (117) and (118) reveals the following situation. The height dependence of the Bouguer anomalies and that of the vertical

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gradients of them is small, if $\alpha$ is small, (120). If the height dependence of these values is omitted, a relative error of the order of $\alpha$ is the consequence, only. Along these lines, the relation (121) gets the following form

$$C_{1.1} = -(h_Q - h_P) \left[ \frac{\partial}{\partial h} \Delta g_{Bouguer} \right]_{z=0} . \tag{122}$$

The above vertical gradient of the Bouguer anomalies, (122), can be computed in terms of the Bouguer anomalies which are distributed along the sphere with $r_P$ as radius. Hence, (13),

$$C_{1.1} = -(h_Q - h_P) \frac{R^2}{2\pi} \iint_\omega \frac{(\Delta g_{Bouguer})_{Y^{**}} - (\Delta g_{Bouguer})_{Q^{**}}}{e_o^3} \, d\omega_{Y^{**}} . \tag{123}$$

Regarding (118) and (120), it is allowed to replace the Bouguer anomalies along the sphere $\varkappa_P$ by the corresponding values measured at the surface of the Earth $\sigma$, Fig. 1, (z = Z). Thus, in (123), the following transitions are allowed

$$\left[ \Delta g_{Bouguer} \right]_{Q^{**}} \longrightarrow \left[ \Delta g_{Bouguer} \right]_{Q} , \tag{123a}$$

$$\left( \Delta g_{Bouguer} \right)_{Y^{**}} \longrightarrow \left[ \Delta g_{Bouguer} \right]_{Y} , \tag{123b}$$

and, finally,

$$C_{1.1} \cong -(h_Q - h_P) \frac{R^2}{2\pi} \iint_\omega \frac{(\Delta g_{Bouguer})_Y - (\Delta g_{Bouguer})_Q}{e_o^3} \, d\omega_Y . \tag{123c}$$

As to the further procedures for the computation of $C_{1.1}$, the integral equation for the downwards continuation of the gravity anomalies derived by Bjerhammar from the Poisson integral is of interest here, [4]. If the rugged free-air anomalies are replaced by the smoothed Bouguer anomalies, and if the kernel function remains unchanged, this integral equation of [4] turns

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into a shape suitable for the here discussed problem, i. e. for the determination of $C_{1.1}$, (see (136)).

Thanks to the advantage that $C_{1.1}$ is small and that the Bouguer anomalies are smoothed, the numerical computation of $C_{1.1}$ in terms of the Bouguer anomalies is a rather uncomplicated and stable procedure, relative easy to handle.

Finally, the evaluation of the amount of the $C_{1.2}$ term is a problem needful to be discussed, (78) (79), (70) to (73). The deductions follow the way that did lead to the amount of $C_{1.1}$. The substitution, (81),

$$\vartheta = \frac{\partial M_2}{\partial r} \qquad (124)$$

is introduced, (81) (70a) (78) (79); and a look on (109) gives

$$C_{1.2} = - ( \vartheta )_Q \cdot \alpha \quad . \qquad (125)$$

Considering the shape of the $\vartheta$ function represented by the table 1, the horizontal extension of the gravity anomaly has about the value

$$D = 400 \text{ km}, \qquad (126)$$

and the amount of the amplitude of the $\vartheta$-wave is about

$$\left| (\vartheta)_Q \right| = 5 \text{ mgal}. \qquad (127)$$

With

$$h_Q - h_P = 1 \text{ km} \qquad (128)$$

and with the formula (100), the following equation yields,

$$\alpha = 0.0\,111 \quad , \qquad (129)$$

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the relations (125) (127) and (129) lead to

$$\left| C_{1.2} \right| = 0.06 \text{ mgal} \quad . \tag{130}$$

As is evidenced by (130), the amount of $C_{1.2}$ can be neglected without any hesitation. In order to avoid misunderstandings: The $\vartheta$ values are not of interest here directly, but the vertical gradients of these values are in the fore, since the vertical gradients determine the $C_{1.2}$ term.

Consequently, in the here discussed geodetic applications, the following relations can be taken to be right, (70) (76),

$$\Delta g_{Bouguer} \cong - \frac{\partial M}{\partial r} \quad . \tag{131}$$

$$C_{1.1} \cong C_1 \quad , \tag{132}$$

$$C_{1.2} \cong 0 \quad . \tag{133}$$

And, with (77),

$$C_1 = \left( \Delta g_{Bouguer} \right)_{Q^{**}} - \left( \Delta g_{Bouguer} \right)_Q \quad . \tag{134}$$

Consequently, (103a) (123) (123c) and Fig. 1,

$$C_1 = - (h_Q - h_P) \frac{R^2}{2 \pi} \iint\limits_{\omega} \frac{\left( \Delta g_{Bouguer} \right)_Y - \left( \Delta g_{Bouguer} \right)_Q}{e_o^3} \, d\omega_Y \quad . \tag{135}$$

Further on, certain other procedures exist for the computation of $C_1$, being defined according to (134). These procedures are free of spherical approximations for the surface of the Earth, and they are free of the condition (103a) for $\alpha$ . The formula of Poisson gives rise to a regional development of the form (136), (see [4], equation (2)),

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$$(\Delta g_{Bouguer})_Q = r_P^2 \frac{r_Q^2 - r_P^2}{4\pi r_Q} \iint\limits_{\omega} \frac{(\Delta g_{Bouguer})_{Y^{**}}}{e^3_{Q.Y^{**}}} d\omega_{Y^{**}} \quad (136)$$

with, Fig. 1,

$$r_P = R + h_P \quad , \quad (137)$$

$$r_Q = R + h_Q \quad . \quad (138)$$

$e_{Q.Y^{**}}$ is the oblique distance between the point Q on the one hand and the point $Y^{**}$ on the other hand. Q is fixed at the surface of the Earth within the course of the integration according to (136). $Y^{**}$ is variable over the sphere $\partial\mathcal{H}_P$ within the course of the integration. Thus,

$$e^2_{Q.Y^{**}} = r_P^2 + r_Q^2 - 2\, r_P\, r_Q \cos\psi \quad . \quad (139)$$

The angle $\psi$ is here the spherical distance between the two points Q and $Y^{**}$.

The inversion of the integral equation of the first kind, (136), determines the Bouguer anomalies $(\Delta g_{Bouguer})_{Y^{**}}$ along the sphere with the radius $r_P$ in terms of the measured Bouguer anomalies $(\Delta g_{Bouguer})_Q$ on the surface of the Earth $\sigma$. Thus, the inversion of (136) leads to the difference amounts : $(\Delta g_{Bouguer})_{Q^{**}}$ minus $(\Delta g_{Bouguer})_Q$. Thereafter, it is an easy step to reach $C_1$, (134); since this anomaly difference is equal to $C_1$.

The integral equation (136) can be brought into the form of an operator equation,

$$(\Delta g_{Bouguer})_Q = \Lambda \cdot (\Delta g_{Bouguer})_{Y^{**}} \quad . \quad (140)$$

It has the following inversion,

$$(\Delta g_{Bouguer})_{Y^{**}} = \Lambda^{-1} \cdot (\Delta g_{Bouguer})_Q \quad . \quad (141)$$

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The left hand side of (141) includes also $(\Delta g_{Bouguer})_{Q^{**}}$ . The corresponding anomaly for the point Q is known from the measurements. Thus, the $C_1$ value can be determined as the difference of these two values, (134).

The introduction of a set of gravitating point masses $m_i$ situated in the interior of the globe $\text{æ}$ constructs another procedure for the computation of $C_1$, /1/. Along these lines, the potential M is expressed by a regional development of the form

$$M_Q = f \sum_i \frac{1}{e_{Q.i}} m_i \; , \qquad (142)$$

for the test points Q at the surface $\sigma$ . The point masses

$$m_i \; , \quad (i = 1, 2, ..., J) \; , \qquad (143)$$

have a limited number J. They are understood as a set of point masses, a set of bounded regional extension. For instance, they cover an area of 100 km or 200 km square. In the central part of this area, the potential M will be approximated especially good by (142).

Now, the introduction of vector and matrix symbols is recommended. For a set of discrete surface points Q of the total number J plotted in the considered area of regional extension in a convenient distribution, the M values are well-defined by (142). They figure now as the elements of the vector $\underline{\underline{u}}$, this fact is described by

$$\underline{\underline{u}} = \{ M_Q \} \; . \qquad (144)$$

In a similar way, the gravitating point masses $m_i$, (143), figure as the elements of the vector $\underline{\underline{v}}$,

$$\underline{\underline{v}} = \{ m_i \} \; . \qquad (145)$$

The equation (142) follows to have this matrix shape,

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$$\underline{\underline{u}} = \underline{\underline{X}}\ \underline{\underline{v}} \quad . \tag{146}$$

The matrix $\underline{\underline{X}}$ represents the kernel function of (142), it is here understood that it is a quadratic and non-singular matrix. Further on, it is convenient to introduce the radial derivatives of the M potential (in the selected surface points Q) by the form of a vector. It will be demonstrated by the form (147),

$$\underline{\underline{w}} = \left\{ \left( \frac{\partial M}{\partial r} \right)_Q \right\} \quad . \tag{147}$$

At the surface $\sigma$ , the radial derivatives of the kernel function $\underline{\underline{X}}$ are as follows,

$$\underline{\underline{Y}} = \left\{ \left( \frac{\partial}{\partial r} \underline{\underline{X}} \right)_Q \right\} \quad . \tag{148}$$

At the sphere $\mathcal{H}_P$, the relation (148a) follows similarly,

$$\underline{\underline{Y}}^{**} = \left\{ \left( \frac{\partial}{\partial r} \underline{\underline{X}} \right)_{Q^{**}} \right\} \quad . \tag{148a}$$

Consequently,

$$\underline{\underline{w}} = \underline{\underline{Y}}\ \underline{\underline{v}} \tag{149}$$

The relations (131) and (134) yield

$$C_1 = \left( \frac{\partial M}{\partial r} \right)_Q - \left( \frac{\partial M}{\partial r} \right)_{Q^{**}} \quad . \tag{150}$$

The vector, (147),

$$\underline{\underline{w}}^{**} = \left\{ \left( \frac{\partial M}{\partial r} \right)_{Q^{**}} \right\} \tag{151}$$

gives, (149),

$$\underline{\underline{w}}^{**} = \underline{\underline{Y}}^{**}\ \underline{\underline{v}} \quad . \tag{152}$$

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Hence, (149) (150) (152),

$$\underline{\underline{c}}_1 = \underline{\underline{w}} - \underline{\underline{w}}^{**} = (\underline{\underline{E}} - \underline{\underline{Y}}^{**} \underline{\underline{Y}}^{-1})\, \underline{\underline{w}} \,, \qquad (153)$$

with,

$$\underline{\underline{c}}_1 = \left\{ C_1 \right\} . \qquad (154)$$

The elements of the vector $\underline{\underline{c}}_1$ are the here required values, $C_1$, (153) (154). Especially, that element of $\underline{\underline{c}}_1$ that is situated in the oentral part of the considered regional area is of dominating interest, since it will result in a relative high preoision, and since it will be equal to the $C_1$ term to be determined.

A similar development about gravitating mass points $m_i$ for a regional representation of the M potential (which is in olose relation to the Bouguer anomalies) was discussed earlier in another publication, /1/. The problem treated in that publication /1/ was in a very close relationship to the here investigated question of the determination of the residual term $C_1$ of the geodetic boundary value problem.

In that earlier publication /1/, the Green identity adapted to the surface of the Earth was the fundamental starting point. This identity was applied to the perturbation potential T. The thus obtained relation was rearranged for routine geodetio applications. These rearrangements did not transfer the physical values from $\sigma$ downwards to $\varkappa_P$, as here in case of the Bjerhammar sphere; but, the geometrioal values did undergo a transfer from $\varkappa_P$ upwards to the surface $\sigma$, replacing the square of the horizontal distances $e_o^2$ by the square of the oblique distances $e_o^2 + Z^2$, for instance. The final result of these rearrangements of the identity of Green was the Stokes integral supplemented by a olosed residual term - which is in the main identical with the here obtained residual term represented by C, $C_1$, $\Xi_2$ in (54) -. In the publication /1/, one of the discussed problems was the determination of the amount of

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$$\frac{\partial \mu_1}{\partial \bar{x}} + \frac{\partial \mu_2}{\partial \bar{y}} \quad , \qquad (155)$$

by a regional representation of the M potential in terms of a set of gravitating mass points. $\mu_1$ and $\mu_2$ are the surface values of the deflections of the vertical caused by the M potential. Hence, $\mu_1$ and $\mu_2$ are two - dimensional functions. The derivations in (155) refer to the $\mu_1$ and $\mu_2$ functions along the oblique surface of the Earth. Further, [1],

$$d\bar{x} = R\, d\varphi \quad , \qquad (155a)$$

$$d\bar{y} = R \cos\varphi \;\; d\lambda \quad . \qquad (155b)$$

The step from (155) to the $C_1$ term is executed by the following relation, (see equation (67) in [1]),

$$C_1 = G\,(h_Q - h_P) \left[ \frac{\partial \mu_1}{\partial \bar{x}} + \frac{\partial \mu_2}{\partial \bar{y}} \right] . \qquad (156)$$

The functions $\mu_1$ and $\mu_2$ are equal to the topographically reduced plumb-line deflections. In [1], the amount of (155) was expressed by the Bouguer anomalies ,( see equation (1o3) of [ 1 ] ).

A short discussion about the spatial distribution of the point masses $m_i$ in the interior of the Earth seems to be recommended. In the here executed developments, from (143) to (154), the places of the point masses $m_i$ have to observe the restriction that they have to be situated only within the globe $\varkappa$ with the radius R; it is on the strength of the theorem of Keldysh-Lavrentiev which postulates that the space $\Phi_1$ (between $\varkappa$ and $\sigma$ ) has to be free of masses. As opposed to this situation, the developments about the identity of Green have other presuppositions, [1]. In [1], the theorem of Keldysh-Lavrentiev is not used. Thus, the lengthes of the geocentric placement vectors of the point masses $m_i$ are not restricted to be smaller than the radius of the globe, R. In the developments described in the publication [1], the point masses $m_i$ are allowed

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to be situated also in the space $\bar{\Phi}_i$ extended between $\sigma$ and $\partial e$ .

Therefore, the methods using the identity of Green, [1], allow a greater flexibility in the choice of the positions of the point masses $m_i$, as opposed to the methods which are based on the theorem of Keldysh-Lavrentiev. The theorem of Keldysh-Lavrentiev has the constraint $r < R$ for the radii of the point masses. The theorem of Green has the constraint $r < r_\sigma$ , ($r_\sigma = R + h$), for the geocentric radii of these point masses.

Finally, it is to be stressed again, that the free-air anomalies give the $C_{1.1}$ term along the oceans, as the Bouguer anomalies do along the continents, (134). In good approximation, $C_{1.1}$ can be replaced by $C_1$ .

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## 7. A comparative survey of the free and of the fixed boundary value problem.

The solution of the boundary value problem of Molodenskij is given by the relation (1) and by the supplementary relations (7) (11) (13). In these equations, the h values figure as the heights which the surface points have above the ellipsoid or above the globe, Fig. 1. The succeeding relation, (157), is self-explanatory, (9),

$$r_P = R + h_P = R + \overset{*}{h}_P + \zeta_P \quad . \tag{157}$$

The height anomaly $\zeta_P$ is the unknown quantity of the problem which can be determined by the boundary values, (5) (6). Principally, the equation (1) has the character of the solution of the free boundary value problem, because the boundary values are distributed along the real surface of the Earth $\sigma$ ( they are not placed on the telluroid), and it must be added that the shape of $\sigma$ is beforehand unknown. It comes to be known by the computations according to (1) and (6), which yield $\zeta$ . After the height anomalies $\zeta$ are known by (1) and (6), the shape of the boundary surface $\sigma$ can be determined by the geocentric radius of it,

$$R + \overset{*}{h}_P + \zeta_P \quad . \tag{158}$$

But, even the $\zeta$ values that figure in (158) are the unknown quantities which are to be determined in the course of the solution of the boundary value problem. Thus, principally, the relations (1) and (6) are not an explicit solution for $\zeta$ . $\zeta$ appears not only on the left hand side of (1), $\zeta$ figures also on the right hand side of (1), but in a more indirect or implicit manner. Thus, the shape of the boundary surface belongs to the unknowns of the problem. This situation is typical for a free boundary value problem.

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On the strength of this fact, and to begin with the comparative deliberations - comparing the free boundary value problem with the fixed one -, it is useful to subject the free boundary value problem to a modification, reducing it and putting it into the class of the fixed boundary value problem. After this transition to the fixed problem, the boundary values are placed on the telluroid, instead of the surface $\sigma$ , Fig. 1. Therefore, the radius is not given by (158), but, it has now the relation

$$r_P = R + h_P^* \qquad (159)$$

This modification of the $r_P$ values, (158) (159), influences the coefficient before the integral on the right hand side of the equation (1). The neglection of $\zeta_P$ in the $r_P$ value, ($r_P \rightarrow R + h_P^*$ ), leads to a relative error of the amount of

$$\frac{\zeta}{R} \ll 1 \qquad (160)$$

in the T value on the left hand side of (1). The height anomaly derives with the same relative error, ( $\zeta/R$, (160); if $r_P \rightarrow R + h_P^*$ ); this fact is evidenced by the equation (6). As an example, the quotient (160) is computed for the parameters $\zeta = 0.1$ km and R = 6 370 km. These parameters lead to the fact that the transition from the radius of the surface of the Earth $\sigma$ to the radius of the telluroid, (157) (158)(159) , has an impact of not more than

$$\frac{\zeta^2}{R} = 0.2 \text{ cm} \qquad (160a)$$

on the $\zeta$ value obtained from (1) and (6).

This amount is absolute unimportant.

The very small amount of (160a) can be included in the solution ((1) (6)) easily, by a succeeding iteration step, it is self-explanatory. This iteration procedure is always convergent, because the amount of (160) is by far smaller than the unity, in all cases.

Further on, considering the solutions of the boundary value problems by the integrals (1) and (54) which solve the Stokes's problem and that of Hotine in a sufficient approximation, it is

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self-explanatory, the transition from the surface $\sigma$ to the telluroid, (157) (158) (159),

$$h = h^* + \zeta \longrightarrow h^* , \qquad (161)$$

has an impact on the small supplementary terms C, $C_1$, $\Xi_1$, $\Xi_2$; (1) (54). Obviously, this impact can be neglected in all cases.

Summarizing, the transition from the surface $\sigma$ to the fixed and well-known telluroid leads to a well-defined value for the height anomaly (with a very small relative error), along the lines of the fixed boundary value problem, (1) (6).

In order to have a more rigorous mathematical base, it will be convenient to withdraw from the intention to substitute along the lines : Surface $\sigma$ $\longrightarrow$ telluroid. In this case, it will be necessary to go back from the telluroid to the real surface of the Earth. It will be possible to bring this intention to a practical realization (i. e. the introduction of the surface $\sigma$ as the definite and free boundary surface) by a simple iteration procedure appended to the solution of the fixed boundary value problem. This iteration procedure consists in the application of (157) instead of (159) for the radius of the boundary surface, introducing a first approximation value of $\zeta$ . Further iterative approximation steps may follow. The supplementary numerical amounts which yield from this iteration procedure are negligible.

Therefore, the relations (1) and (6) compute in our problem, preferring (157) or (159), the solution for the free or for the fixed boundary value problem of the geodesy. The differences between these two problems are unimportant, as far as the impact on the resulting height anomaly is concerned.

After this above discussion of the aspects of the free and fixed boundary value problem of the Stokes type, the corresponding situation in case of the Hotine type of the boundary value problem is now put into the fore, (54). Here, the gravity disturbances figure as the boundary values on the surface $\sigma$ , (16). For the computation of the standard gravity at the surface of the Earth, it is necessary that the shape of the Earth is known in advance. The radii $r_P$ and $r_Q$ of the surface points P and Q must be known in advance. Along the oceans, the satellite altimetry offers convenient methods for the determination of $r_P$ and $r_Q$. The knowledge of the shape of the boundary surface is essential for the fixed boundary

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value problem, in our applications. Thus, the Hotine type comes near the fixed version(not the free version), in any case in our applications.

Coming to a final summation, the equation (1) represents the solution of the free and of the fixed geodetic boundary value problem .It has the free-air anomalies as boundary values along the surface of the Earth. However, in our applications, the equation (54) is connected with the fixed boundary value problem which has gravity disturbances as boundary values along the Earth's surface. The corresponding free version of (54) is not actual in the geodetic applications.

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## 8. Conclusion.

The formula (54) is a solution of the here considered boundary value problem of the Hotine type. (54) meets the requirements of the theory and of the numerical reckoning. The gravity disturbances $\delta g$ serve as the boundary values placed at the surface of the Earth which is shaped by the topography. Along these lines, the amount of the perturbation potential T at the surface of the Earth is expressed in terms of the gravity disturbances. The Hotine function appears as the kernel function. In the integral of (54), the gravity disturbances must be supplemented by the plane topographic reduction of the gravity, C, and, further on, by the $C_1$ term which is rather small and smoothed, and which is often negligible. In (54), the third supplementary term $\Xi_2$ is also trifling in most cases. The solution, (54), has no series development of bad or dubious convergence.

A similar formula is valid for the solution of the boundary value problem of Molodenskij which uses free-air anomalies as boundary values, (1) (2); cf. [1] [2] [3]. Also in this case, the supplementary terms C and $C_1$ must be added to the boundary values, (1), i. e. the free-air anomalies. But, the Stokes function serves in (1) as the kernel function, instead of the Hotine function of (54). Further, in $\Xi_2$, (54a), the second term of the expression for $\Xi_1$ does not appear, (2).

Several different ways lead to the computation of the amount of $C_1$. The relations (135) or (153)(154) are recommended, if a map of Bouguer anomalies is at disposal. The formula (156) is useful, if topographically reduced plumb-line deflections are at hand.

A discussion of the parameters which enter into the three terms $\delta g$, C and $C_1$ is of interest.

The term $\delta g$ depends only on the measurements developed in the physical geodesy and in the satellite geodesy. The gravity measurements g at the surface are here effective. Further on, the precise height of the surface above the mean ellipsoid of the Earth is needed, since this height is required for the computation of the standard gravity $\gamma$ at the surface of the Earth, Fig. 1.

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The C term figures as a gravity value, since it is the plane topographical reduction of the gravity. But, the parameters that dominate the computation of C are the rough height differences, a precision of some meters suffices for them. These height differences are free of the sophistications the physical geodesy has in store for the precise levellings.

On the other hand, the small term $C_1$ depends only on the geological density anomalies in the upper parts of the Earth : This are the amounts by which the density of the geological masses differs from the standard density $\rho = 2.65[\ g\ cm^{-3}\ ]$; and, further on, the compensation masses of the isostatic mountain roots are effective. The $C_1$ term is proportional to the second vertical derivative of the potential produced by these density anomalies. These density anomalies are close to the Bouguer anomalies, if continental areas are considered, - they are close to the free-air anomalies, if oceanic areas are treated - .

Summarizing, the three terms $\delta g$, C and $C_1$ have different sources and different characters. $\delta g$ can be taken as a physical value, C as a geometrical and $C_1$ as a geological one.

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E. A proof of the convergence of the spherical-harmonics series development of a potential exterior of a regular surface by the completeness of the system of the base functions at the surface.

Contents



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Summary

The values which the regular function of a spatial potential takes at the boundary surface being a regular surface can be represented by a uniform convergent spherical harmonics series development. The vector formed by the spherical harmonics can be multiplied by an infinite orthonormal matrix. The product is a new vector of an infinite number of orthonormal functions. The surface values of the potential are expressed in terms of these new functions. Even these new base functions are replaced by the linear independent surface functions which the solid spherical harmonics take along the boundary surface; it happens by the inverse Schmidt orthogonalization procedure. These linear independent functions construct a complete system of base functions. Along these lines, a convergent series development is obtained for the surface values of the potential. A theorem of Abel leads to the uniform convergence of the series in whole the exterior space. Picone's theorem corroborates the result. A proof of the completeness and a short proof of the convergence is added.

Zusammenfassung

Die Werte, die die reguläre Funktion eines räumlichen Potentials an einer Randfläche annimmt, können bekanntlich durch eine konvergente Kugelfunktionsentwicklung dargestellt werden. Der Vektor der Kugelfunktionen wird mit einer unendlichen orthogonalen Matrix multipliziert. Man erhält ein neues unendliches System von orthonormalen Funktionen. Die Werte des Potentials an der Randfläche werden durch diese Funktionen dargestellt. Schliesslich werden die wohldefinierten Oberflächenfunktionen eingeführt, die die räumlichen Kugelfunktionen an der Randfläche annehmen. Man erhält ein linear unabhängiges und vollständiges System von Basisfunktionen. Dabei wird das inverse Schmidt-sche Orthogonalisierungsverfahren herangezogen. Es wird eine konvergente Reihenentwicklung für die Werte des Potentials an der Randfläche erhalten. Ein Lehrsatz von Abel führt zu der Tatsache, dass die räumliche Kugelfunktionsentwicklung für ein Potential im Aussenraum der Randfläche gleichmässig konvergent ist. Dieses Ergebnis wird durch den Satz von Picone bestätigt. Ein Beweis für die Vollständigkeit und ein kurzer Beweis für die Konvergenz schliessen sich an.

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Резюме

Как известно, значения, которые пространственный потенциал принимает на граничной поверхности, могут быть представлены посредством сходящегося разложения сферической функции. Вектор сферических функций перемножается с бесконечной ортогональной матрицей. Получают новую бесконечную систему ортонормальных функций. Значения потенциала на граничной верхности представляются посредством этих функций. Наконец, вводятся вполне определенные поверхностные функции, которые принимают пространственные сферические функции на граничной по- поверхности. Получают линейную независимую систему базисных функций. При этом привлекается инверсный метод ортогонализации Шмидта. Получают сходящееся разложение в ряд для значений потенциала на граничной поверхности. Теорема Абеля приводит к тому факту, что пространственное разложение сферической функции для потенциала во внешнем пространстве граничной поверхности является равномерно сходящимся.

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## 1. Introduction

A regular harmonic function W is introduced in the three-dimensional space of the orthogonal Cartesian co-ordinates x, y, z,

$$W = W(x, y, z) \quad , \tag{1}$$

$$\Delta W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0 \quad . \tag{2}$$

The relation (2) is the well-known Laplace differential equation, [9] [12].

The gravitating sources are situated within a closed, regular and star-shaped surface D. The relation (2) is valid in the exterior of the surface D. W is a continuous and regular function in the exterior of D and on the surface D. In the exterior of D, W has continuous derivatives of the first and higher order, [9] [12].

Now, the well-known Brillouin sphere is introduced. It is a geocentric sphere that encloses the surface D and, thus, all the gravitating sources. It has the radius $\bar{R}$, (see Fig. 1). In the exterior of the Brillouin sphere, the potential W has the following uniform convergent spatial spherical harmonics series development, it is well-explained in the literature, [8] [9].

$$W = \frac{fM}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{1}{r}\right)^n \sum_{m=0}^{n} P_{n.m}(\sin\varphi)\left\{ W_{n.m.1}\cos m\lambda + W_{n.m.2}\sin m\lambda \right\}\right] \quad , r \geqslant \bar{R} \quad . \tag{3}$$

The center of the spatial polar co-ordinate system r, $\varphi$, $\lambda$ is identical with the gravity center of the Earth. r is the geocentric radius, $\varphi$ and $\lambda$ are the geocentric latitude and longitude. f is the gravitational constant, M is the mass of the Earth, $W_{n.m.1}$ and $W_{n.m.2}$ are the Stokes constants, $P_{n.m}$ are the associated spherical harmonics.

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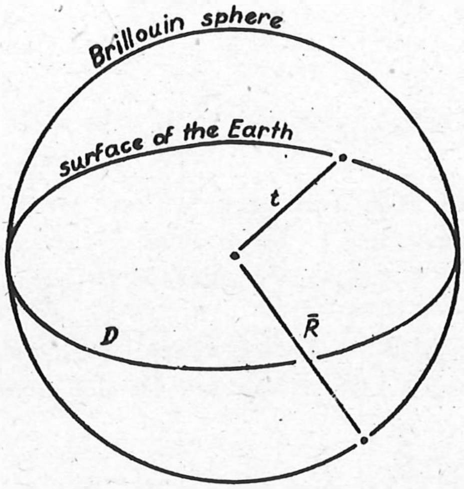


Fig. 1: The Brillouin sphere with the radius $\bar{R}$. The surface of the Earth D and its geocentric radius t.

The series development (3) is generally accepted to be uniform convergent, if $r \geqslant \bar{R}$, [8] [9] [12]. Therefore, the following theorem is valid:

Theorem 1:

For any positive number $\varepsilon_1$, however small,

$$\varepsilon_1 > 0 , \qquad (4)$$

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there exists an integer H, sufficient great, such that

$$\left| \frac{fM}{r} \sum_{n=H+1}^{\infty} \left(\frac{1}{r}\right)^{n} \sum_{m=0}^{n} P_{n.m}(\sin\varphi) \left\{ W_{n.m.1} \cos m\lambda + W_{n.m.2} \sin m\lambda \right\} \right| < \varepsilon_1 ; r \geqslant \bar{R} . \qquad (5)$$

The problem, to be discussed here, is the question whether the validity of the convergence of the series (3) can be extended to whole the exterior space of D and to the surface D, also.

An abbreviation of the following shape, (6), is recommended for the development (3) . It is to be applied in the subsequent deductions.

$$W = \sum_{n=1}^{\infty} W_n \left(\frac{1}{r}\right)^{n} u_n(\varphi, \lambda) ; r \geqslant \bar{R} . \qquad (6)$$

The meaning of $W_n$ and $u_n$ is self-explanatory, as a comparison of (3) and (6) does show

The transition from the manner of writing (3) to the manner of representation (6) can be understood in the following way : Instead of writing down all the zonal, tesseral and sectorial spherical harmonics, only the zonal harmonics are taken along and written down, since the tesseral and sectorial harmonics of the n-th degree transform in the same way as the zonal harmonics of the degree n . This abbreviation is a great relief in the writing down of the mathematical developments. It is an often used procedure, a habit usual since a long time.

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## 2. The Schmidt orthogonalization process.

The expressions

$$\left(\frac{1}{r}\right)^n u_n(\varphi,\lambda) \quad , \tag{7}$$

which appear in the formula (6), are now of special interest. The functions (7) are now specialized for the test points situated on the surface D. Here, they take the following expression,

$$\left[\left(\frac{1}{r}\right)^n\right]_D u_n(\varphi,\lambda) \quad . \tag{8}$$

The geocentric radius of the surface D is denominated by t, (see Fig. 1),

$$t = t(\varphi,\lambda) = r_D \; . \tag{9}$$

Thus, the relation (8) turns to

$$\left(\frac{1}{t}\right)^n u_n(\varphi,\lambda) = \left(\frac{1}{t(\varphi,\lambda)}\right)^n u_n(\varphi,\lambda) =$$

$$= v_n(\varphi,\lambda) \quad , \tag{10}$$

$$n = 1, 2, \ldots \quad . \tag{10 a}$$

Or

$$v_n(\varphi,\lambda) = \left(\frac{1}{t(\varphi,\lambda)}\right)^n u_n(\varphi,\lambda) =$$

$$= \left[\left(\frac{1}{r}\right)^n u_n(\varphi,\lambda)\right]_D \tag{11}$$

The spherical harmonics $u_n(\varphi,\lambda)$, ($n = 1, 2, \ldots$), establish an infinite set of orthonormal functions. This set is complete for all the regular functions on the unit sphere. On the sphere, the development of

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a regular function in terms of the $u_n$ functions is a uniform convergent series, it is well-known, [9] [11].

The functions $u_n(\varphi, \lambda)$ are orthonormal,

$$\iint_F u_n(\varphi, \lambda)\, u_m(\varphi, \lambda)\, dF = \delta_{n.m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}, \qquad (12)$$

$$n, m = 1, 2, \ldots \qquad (13)$$

F is the surface of the unit sphere,

$$dF = \cos\varphi \; d\varphi \; d\lambda \qquad (14)$$

The functions $v_n(\varphi, \lambda)$ are linear independent, (11), [1] [3] [6] [10] [13] [14]. Thus, the relation

$$\sum_{n=K}^{L} U_n v_n(\varphi, \lambda) = 0 \qquad (15)$$

cannot be valid for any limited integer K and L, unless the constant coefficients $U_n$ are equal to zero,

$$U_k = U_{K+1} = \ldots = U_L = 0 ; \qquad (16)$$

$$L \geqslant K , \qquad (17)$$

$$K = 1, 2, \ldots \qquad (18)$$

The proof of (15) and (16) is uncomplicated. In the exterior of D, the spatial potential U is introduced by the following sum,

$$U = \sum_{n=K}^{L} U_n \left(\frac{1}{r}\right)^n u_n(\varphi, \lambda) . \qquad (19)$$

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U is represented by a sum and not by an infinite series development. Therefore, the convergence problems are not involved. The expression (19) is harmonic and regular in the exterior of D. On the boundary surface D, the relation (19) turns to

$$U_D = \sum_{n=K}^{L} U_n \left(\frac{1}{t}\right)^n u_n(\varphi, \lambda) \quad , \tag{20}$$

or, (11),

$$U_D = \sum_{n=K}^{L} U_n v_n(\varphi, \lambda) \quad . \tag{21}$$

If a potential is zero along the boundary surface, it is zero also in whole the exterior space of this surface. This fact is well-proved by the Dirichlet boundary value problem, /12/. Hence, the following condition, (22), for the boundary values of the potential U, (20) (21),

$$U_D = 0 \quad , \tag{22}$$

leads necessarily to the fact that

$$U = 0 \quad , \tag{23}$$

in whole the exterior space of D. Consequently, the relation (23) is valid also along the surface of the Brillouin sphere with the radius $\bar{R}$,

$$\sum_{n=K}^{L} U_n \left(\frac{1}{\bar{R}}\right)^n u_n(\varphi, \lambda) = 0 \quad . \tag{24}$$

The multiplication of (24) with $u_n(\varphi, \lambda)$ and the application of (12) leads to

$$U_K = U_{K+1} = \ldots = U_L = 0. \tag{25}$$

(25) corroborates the linear independence of the $v_n$ functions, (15) (16).

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Since the functions $v_n$ are linear independent - but not necessarily orthogonal -, it is possible to find a system of orthonormal functions

$$w_n = w_n\ (\varphi, \lambda) \tag{26}$$

by the functions $v_n$. This aim is reached by the Schmidt orthonormalization procedure, [10] [14] [16], a way that is always possible to go, because of (15) and (16). A linear system of the following shape is obtained,

$$\begin{aligned} w_1 &= b_{1.1}\ v_1 \\ w_2 &= b_{2.1}\ v_1 + b_{2.2}\ v_2 \\ \dots & \quad \dots \quad \dots \\ w_L &= b_{L.1}\ v_1 + b_{L.2} v_2 + \dots + b_{L.L}\ v_L\ ; \end{aligned} \tag{27}$$

or, in the form of a matrix relation,

$$\underline{\underline{w}}_L = \underline{\underline{B}}_L\ \underline{\underline{v}}_L \quad . \tag{28}$$

The elements of the main diagonal of (27) (28) (33) are positive, [10],

$$b_{i.i} > 0 \ . \tag{28a}$$

$$\iint\limits_F w_n\ w_m\ dF = \delta_{n.m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \quad , \tag{29}$$

$$n, m = 1, 2, \dots, L. \tag{30}$$

The number of the relations of (27) is equal to L, (see (15) and (16)). But, there is no difficulty in continuing the above process, (27), for a value of L getting greater and greater.

There is no upper bound for the amount of the integer L. The $b_{i.k}$ values of (27) are the constant coefficients , free of φ and λ.

The column vectors and the matrix of (28) have the following shape,

$$\underline{\underline{w}}_L = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_L \end{pmatrix} \quad , \tag{31}$$

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$$\underline{\underline{v}}_L = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_L \end{pmatrix} , \qquad (32)$$

$$\underline{\underline{B}}_L = \begin{pmatrix} b_{1.1} & b_{1.2} & \dots & b_{1.L} \\ b_{2.1} & b_{2.2} & \dots & b_{2.L} \\ \dots & \dots & \dots & \dots \\ b_{L.1} & b_{L.2} & \dots & b_{L.L} \end{pmatrix} = \left\{ b_{i.k} \right\} . \qquad (33)$$

The dimension of $\underline{\underline{w}}_L$ and $\underline{\underline{v}}_L$ is equal to L. $\underline{\underline{B}}_L$ is a L x L matrix. In (33), the first suffix specifies the row, the second suffix k specifies the column. Since the functions $v_n$ are linear independent, the Gram determinants which governe the transformation by (27) and (28) are non-singular, [6] [14] [16].

The mapping (27) and (28) is unique, the same is valid for the inversion of (27) and (28). The theory of the Schmidt orthogonalization process shows that

$$\det \underline{\underline{B}}_L \neq 0 . \qquad (34)$$

(34) is right, because the $v_n$ functions are linear independent, (15) (16). Thus, the inversion of (28) is possible,

$$\underline{\underline{v}}_L = \underline{\underline{B}}_L^{-1} \underline{\underline{w}}_L . \qquad (35)$$

$\underline{\underline{B}}_L$ is a triangular matrix, (subdiagonal matrix), (see (27)). The coefficients $b_{i.k}$ of (27) and (33) have limited amounts, because the Gram determinants are non-singular.

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## 3. The infinite orthonormal matrix.

The theory of the infinite matrices develops in the vicinity of the theory of the finite matrices. But, these two theories have also certain differences, [7] [17].

For instance, the product of two infinite matrioes is well-explained only if certain convergenoe properties are valid. The product of the infinite matrix

$$\underline{\underline{X}} = \{ x_{i.k} \} \tag{36}$$

and the infinite matrix

$$\underline{\underline{Y}} = \{ y_{i.k} \} \quad , \tag{37}$$

$$(i, k = 1, 2, \ldots) \quad , \tag{38}$$

is constructed by the infinite matrix

$$\underline{\underline{XY}} = \underline{\underline{Z}} = \{ z_{i.k} \} \quad . \tag{39}$$

In (36) (37) (39), the first suffix i specifies the rows and the second suffix k the columns. The elements of the matrix of (39) are explained by

$$z_{i.k} = \sum_{j=1}^{\infty} x_{i.j}\, y_{j.k} \quad . \tag{40}$$

The product of these two matrices which are considered here is well-defined only if the right hand side of (40) is a convergent series development.

In a consideration from a more universal standpoint, the theory of the finite matrices belongs to the discipline of the algebra, but the infinite matrices are more in the vicinity of the field of the functional analysis.

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Now, the infinite orthonormal matrices $\underline{\underline{A}}$ are in the fore. These matrices are well-defined by the following relation [7].

$$\underline{\underline{A}}\ \underline{\underline{A}}^T = \underline{\underline{A}}^T\ \underline{\underline{A}} = \underline{\underline{E}}\ . \tag{41}$$

The superscript T denominates the transposition. $\underline{\underline{E}}$ is the infinite unit matrix,

$$\det \underline{\underline{E}} = 1. \tag{42}$$

But, in the theory of the infinite matrices, the value of the determinant is lost. The relations (41) and (42) give

$$\det \underline{\underline{A}} = 1, \tag{43}$$

(43) follows from

$$\det (\underline{\underline{A}}\ \underline{\underline{A}}^T) = (\det \underline{\underline{A}})\ (\det \underline{\underline{A}}^T) = (\det \underline{\underline{A}})^2 = \det \underline{\underline{E}} = 1. \tag{44}$$

$\underline{\underline{A}}$ is a twofold table,

$$\underline{\underline{A}} = \left\{ a_{i.k} \right\} , \tag{45}$$

$$i,\ k = 1,\ 2,\ \ldots\ . \tag{46}$$

$$\underline{\underline{A}} = \begin{pmatrix} a_{1.1} & a_{1.2} & a_{1.3} & \cdots \\ a_{2.1} & a_{2.2} & a_{2.3} & \cdots \\ a_{3.1} & a_{3.2} & a_{3.3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} , \tag{47}$$

$$\underline{\underline{A}}^T = \begin{pmatrix} a_{1.1} & a_{2.1} & a_{3.1} & \cdots \\ a_{1.2} & a_{2.2} & a_{3.2} & \cdots \\ a_{1.3} & a_{2.3} & a_{3.3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} , \tag{48}$$

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$$\underline{\underline{A}}^T = \left\{ \bar{a}_{i.k} \right\} \quad , \tag{49}$$

$$\bar{a}_{i.k} = a_{k.i} \quad . \tag{50}$$

The basing definition of the infinite orthonormal matrix, (41), leads to

$$\sum_{j=1}^{\infty} a_{i.j}\, a_{k.j} = \delta_{i.k} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \quad , \tag{51}$$

$$\sum_{j=1}^{\infty} a_{j.i}\, a_{j.k} = \delta_{i.k} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \quad . \tag{52}$$

$\delta_{i.k}$ is the Kronecker symbol.

Introducing the infinite set of the spherical harmonics $u_n(\varphi, \lambda)$ into the theory of the infinite orthonormal matrices $\underline{\underline{A}}$, these harmonics construct the following infinite - dimensional column vector,

$$\underline{\underline{u}} = \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \\ \cdots \end{pmatrix} \quad . \tag{53}$$

The vector $\underline{\underline{u}}$ can be transformed by the multiplication with $\underline{\underline{A}}$. The elements of $\underline{\underline{u}}$ are the orthonormal bases $u_n(\varphi, \lambda)$. The system of the $u_n$ functions is well-known to be complete and closed in the space of the regular functions on the sphere, (12).

The multiplication of $\underline{\underline{u}}$ and $\underline{\underline{A}}$ gives the vector $\underline{\underline{e}}$,

$$\underline{\underline{e}} = \underline{\underline{A}}\, \underline{\underline{u}} \quad , \tag{54}$$

and, because $\underline{\underline{A}}^T$ is the inverse of $\underline{\underline{A}}$, (41),

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$$\underline{\underline{u}} = \underline{\underline{A}}^T \underline{\underline{e}} \quad , \tag{55}$$

$$\underline{\underline{e}} = \begin{pmatrix} e_1 \\ e_2 \\ \dots \\ e_n \\ \dots \end{pmatrix} \quad , \tag{56}$$

$$e_n = e_n ( \varphi , \lambda ) . \tag{57}$$

The relation (54) leads to

$$e_n = e_n ( \varphi , \lambda ) = \sum_{i=1}^{\infty} a_{n.i} \, u_i ( \varphi , \lambda ) \quad . \tag{58}$$

The equations (12) (51) (52) (58) result in the following orthogonality relations,

$$\iint_F e_n \, e_m \, dF = \iint_F u_n \, u_m \, dF = \sum_{i=1}^{\infty} a_{n.i} \, a_{m.i} =$$

$$= \delta_{n.m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \quad . \tag{59}$$

The relations (54) (55) describe a one - to - one mapping, relating $\underline{\underline{e}}$ and $\underline{\underline{u}}$.

As to the completeness of the mappings by $\underline{\underline{A}}$ and $\underline{\underline{A}}^T$, the constraint

$$\underline{\underline{u}} = 0 \tag{60}$$

has the consequence, (54),

$$\underline{\underline{e}} = 0 \quad . \tag{61}$$

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Thus, (54) allows to find the unique vector $\underline{\underline{e}}$ by $\underline{\underline{u}}$ and $\underline{\underline{A}}$. And, vice versa, the constraint

$$\underline{\underline{e}} = 0 \tag{62}$$

leads to, (55),

$$\underline{\underline{u}} = 0 \quad . \tag{63}$$

(62) and (63) show that (55) allows to determine the unique vector $\underline{\underline{u}}$ from the vector $\underline{\underline{e}}$ and the matrix $\underline{\underline{A}}^T$. The mappings of $\underline{\underline{u}}$ on $\underline{\underline{e}}$, and that of $\underline{\underline{e}}$ on $\underline{\underline{u}}$, are unique procedures.

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## 4. The representation of the boundary values of the potential by orthonormal base functions.

Now, on the regular surface D, a regular function

$$f = f\,(\varphi, \lambda) \tag{64}$$

is considered. It is well-known that f has the following convergent series development in spherical harmonics,

$$f = \sum_{n=1}^{\infty} f_n\, u_n\,(\varphi, \lambda) \quad . \tag{65}$$

$f_n$ are the constant coefficients, (Stokes constants). Therefore, the succeeding sentence is valid, it can be found in the textbooks, [8] [9] [12].

Theorem 2:

For any positive number $\varepsilon_2$, however small,

$$\varepsilon_2 > 0 \ , \tag{66}$$

there exists an integer M, sufficient great, such that

$$\left| \sum_{n=M+1}^{\infty} f_n\, u_n\,(\varphi, \lambda) \right| < \varepsilon_2 \ . \tag{67}$$

The Parseval relation gives for the norm of f, (12) (65), [6] [11], (completeness relation),

$$\| f \|^2 = \iint_F f^2\, dF = \sum_{n=1}^{\infty} f_n^2 \quad . \tag{68}$$

The equation (65) can be brought into the form of a scalar product of two vectors, (53),

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$$f = \underline{\underline{f}}^T \underline{\underline{u}} \quad , \tag{69}$$

with

$$\underline{\underline{f}} = \begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \\ \cdots \end{pmatrix} \quad . \tag{70}$$

In (69), the vector $\underline{\underline{e}}$ is to be introduced as a substitute for the vector $\underline{\underline{u}}$, (55) (56). Consequently,

$$f = \underline{\underline{f}}^T \underline{\underline{A}}^T \underline{\underline{e}} \quad . \tag{71}$$

With (68) and (70), the norm of f takes on the following form

$$\| f \|^2 = \underline{\underline{f}}^T \underline{\underline{f}} \quad . \tag{72}$$

In a rigorous consideration, (71) has to assume the form

$$f = \underline{\underline{f}}^T (\underline{\underline{A}}^T \underline{\underline{e}}) \quad . \tag{73}$$

Associating $\underline{\underline{A}}^T$ with $\underline{\underline{e}}$, as it is shown in (73), the right hand side of (73) is convergent, (69). But, associating the two terms $\underline{\underline{f}}^T$ and $\underline{\underline{A}}^T$ in (73),

$$f = (\underline{\underline{f}}^T \underline{\underline{A}}^T) \underline{\underline{e}} = \underline{\underline{g}}^T \underline{\underline{e}} \quad , \tag{74}$$

it remains as an open question whether the right hand side of (74) continues to be convergent.

$$\underline{\underline{g}}^T = \underline{\underline{f}}^T \underline{\underline{A}}^T \quad . \tag{75}$$

Hence,

$$f = \underline{\underline{g}}^T \underline{\underline{e}} \quad . \tag{76}$$

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$\underline{g}$ is an infinite - dimensional column vector,

$$\underline{g} = \begin{pmatrix} g_1 \\ g_2 \\ \cdots \\ g_n \\ \cdots \end{pmatrix} . \qquad (77)$$

The elements of (77) have constant values. (76) is equivalent to

$$f = \sum_{n=1}^{\infty} g_n \, e_n ( \varphi , \lambda ) . \qquad (78)$$

The individual elements $g_1$, $g_2$, ... of (77) have limited amounts and convergent developments, if derives from (75). This fact is obviously right, since the Schwarz inequality can be applied to (75), and especially to the residual term of (75),

$$g_n = f_1 \, a_{n.1} + f_2 \, a_{n.2} + \cdots = \sum_{i=1}^{\infty} f_i \, a_{n.i} . \qquad (78a)$$

The residual term of (78a) has the relation

$$\left| \sum_{j=M+1}^{\infty} f_j \, a_{n.j} \right| \leqslant \left| \sqrt{\sum_{j=M+1}^{\infty} f_j^2} \cdot \sqrt{\sum_{j=M+1}^{\infty} a_{n.j}^2} \right| . \qquad (78b)$$

Because of (68), and because of (51) - for $i = k$ -, and because all the series developments of positive terms and of limited amount are always uniform convergent, it follows that (68) and (51) are uniform convergent series development, /13/. Thus, each of the two factors which construct the product on the right hand side of (78b) tends to zero, if M tends to infinity. Consequently, the left hand side of (78b) tends to zero also, if M tends to infinity.

The convergence of (65) is known to be proved, but the convergence of (78) is a problem to be discussed in the following paragraph.

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The relations (41) (52) (58) (59) (68) (74) (75) and (78) lead to the following equations,

$$\sum_{n=1}^{\infty} g_n^2 = \underline{\underline{g}}^T \underline{\underline{g}} = (\underline{\underline{f}}^T \underline{\underline{A}}^T)\,(\underline{\underline{f}}^T \underline{\underline{A}}^T)^T =$$

$$= (\underline{\underline{f}}^T \underline{\underline{A}}^T)\,(\underline{\underline{A}}\,\underline{\underline{f}}) = \underline{\underline{f}}^T(\underline{\underline{A}}^T \underline{\underline{A}})\,\underline{\underline{f}} =$$

$$= \underline{\underline{f}}^T \underline{\underline{f}} = \sum_{n=1}^{\infty} f_n^2 = \| f \|^2 \quad . \tag{79}$$

Consequently,

$$\sum_{n=1}^{\infty} g_n^2 = \sum_{n=1}^{\infty} f_n^2 \quad . \tag{80}$$

The surface function $f = f(\varphi, \lambda)$ represents here the amounts of the potential W for testpoints situated on the surface D, (1). For the surface of the Earth, it is well-known that the potential is a regular function along of it,

$$f = W_D \quad . \tag{81}$$

The relations (65) and (78) are the series developments for f in terms of the base functions $u_n(\varphi, \lambda)$ and $e_n(\varphi, \lambda)$. The uniform convergence of (65) is well-known, [9] [12]. The uniform convergence of (78) is now intended to be proved, [1] [2] [3] [4] [5].

As to the meaning of the essential mathematical property to be a regular function, such a function is unique and continuous, the first derivatives of it are continuous functions. These conditions are observed by the potential values of the gravitating body of the Earth, (81).

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## 5. The convergence in the mean and the uniform convergence of a series development in terms of orthonormal base functions.

The equations (79) and (80) construct the Parseval completeness relation for the representation of the function f in terms of the orthonormal base functions $u_n(\varphi, \lambda)$ and $e_n(\varphi, \lambda)$, (65) (78). These relations, (79) (80), include the convergence in the mean of the series developments (65) and (78) for the boundary values $f = f(\varphi, \lambda)$ at the regular surface D, (see (68)).

The convergence in the mean has the following relation: The function

$$f_M(\varphi, \lambda) = \sum_{n=1}^{M} f_n u_n(\varphi, \lambda) \tag{82}$$

converges in the mean to the function f, if

$$\lim_{M \to \infty} \iint_F \left[ f_M(\varphi, \lambda) - f(\varphi, \lambda) \right]^2 dF = 0 , \tag{83}$$

(cf. (64) (65)). A necessary and sufficient condition for the convergence in the mean is the following theorem 3.

Theorem 3:

The development $f_M(\varphi, \lambda)$, (82), converges in the mean, if, for a given number $\varepsilon_3$, there exists an integer

$$M' > 0 , \tag{83a}$$

such that for

$$\varepsilon_3 > 0 , \tag{84}$$

the inequation (85) is valid,

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$$\iint_F (f_{M_1}(\varphi, \lambda) - f_{M_2}(\varphi, \lambda))^2 \, dF < \varepsilon_3 , \qquad (85)$$

for all the integers

$$M_1, M_2 \quad > \quad M \qquad (86)$$

Thus, the very problem which is now to be investigated demands to show that (79) leads to the uniform convergence of (78). The formula (79) is by itself already the proof that the series (78) converges in the mean to the function f ( $\varphi$ , $\lambda$ ). The convergence in the mean has the uniform convergence of (78) as consequence, as far as a regular function f is considered, it is well-known from the textbooks, [10] [13] [15]:

A complete system of orthonormal functions has the same convergence property as the well-established Fourier series development. For regular functions, the representation by a Fourier series development is uniform convergent. Thus, the representation of the regular function f by the orthonormal and complete systems of the $u_n$ ( $\varphi$ , $\lambda$ ) or $e_n$ ( $\varphi$ , $\lambda$ ) base functions leads to a uniform convergent series development, (about the completeness, see paragraph 6.9.). The constants of this development are the well-defined Fourier-type coefficients, (see the theorems of Dirichlet-Jordan and Dini-Lipschitz), [13] [15].

A detailed description of the process which leads from the convergence in the mean to the uniform convergence is now intended to be given.

The left and the right hand side of (80) is a series development of constant positive terms. The amount of it is equal to the square of the norm of the function f. This amount is limited, since f has limited amounts. In the theory of such series developments of constant positive terms. the following sentence is proved to be valid, [13].

<u>Theorem 4:</u>

The necessary and sufficient condition for the convergence of an infinite series development that consists of positive terms only is the fulfillment of the demand that the partial sums of this series have limited values, [13].

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The partial sums of (80) are, (left hand side of (80)),

$$g_1^2 ,$$

$$g_1^2 + g_2^2 ,$$

$$g_1^2 + g_2^2 + g_3^2 , \tag{87}$$

$$g_1^2 + g_2^2 + g_3^2 + g_4^2 ,$$

$$\ldots \quad \ldots \quad \ldots$$

With regard to the representation of the norm, (79), all the partial sums of (87) have necessarily limited amounts, irrespective of the number of the terms they consist of. These amounts have to be smaller than the square of the norm of f. Since this norm of f has a limited amount - the function f is limited -, the partial sums (87) have limited amounts. Therefore, the above theorem 4 shows that the left hand side of (80) is a uniform convergent series development.

Thus, the following theorem is valid.

Theorem 5:

For any positive number $\varepsilon_4$, arbitrary small

$$\varepsilon_4 > 0, \tag{88}$$

there exists an integer N, sufficient great, such that, (79),

$$\sum_{n=N+1}^{\infty} g_n^2 < \varepsilon_4 . \tag{89}$$

The uniform convergence of (78) is a property that derives as a corollary of the theorem 5, (88) (89). As to the definition of the uniform convergence of a series development, the subsequent theorem 6 is an application of this definition to the here discussed series development, (78).

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Theorem 6:

For any positive number $\varepsilon_5$ which can be chosen arbitrary small,

$$\varepsilon_5 > 0 \quad , \tag{90}$$

there exists an integer Q, sufficient great, such that

$$\left| \sum_{n=Q+1}^{\infty} g_n e_n(\varphi, \lambda) \right| < \varepsilon_5 . \tag{91}$$

As to the proof of the theorem 6, the formula (54) is the matrix shape of the development of the functions $e_n(\varphi, \lambda)$ in terms of the spherical harmonics. The theory of the spherical harmonics shows that every regular function given along the sphere can be represented completely by a spherical harmonics series development, [9]. In the following derivations, the functions

$$e_1, e_2, e_3, \dots, e_Q \tag{92}$$

are understood to be in the space of the regular functions. Q is a sufficient great integer, (see (90) (91)). There is no upper bound for Q.

Thus, the sum

$$\sum_{n=1}^{Q} g_n e_n(\varphi, \lambda) \tag{93}$$

is a regular function. (93) is a truncation of (78). The constant coefficients

$$g_1, g_2, g_3, \dots, g_Q \tag{94}$$

of (78) and (93) have limited amounts, (see (78a), (78b)). It follows, - computing them by Fourier type integrals -,

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$$g_n = \iint_F f\, e_n(\varphi, \lambda)\, dF \quad , \qquad (95)$$

as a look on the orthogonality relation (59), and on (78), does show. Because the functions f and $e_n$ are introduced as regular and limited functions, (92), the coefficients $g_n$ follow to be limited also, (95), (78a) (78b),

Therefore, the difference expression $p_Q$, (78),

$$p_Q(\varphi, \lambda) = f(\varphi, \lambda) - \sum_{n=1}^{Q} g_n\, e_n(\varphi, \lambda) \quad , \qquad (96)$$

is necessarily a regular function. Consequently, the function $p_Q$ can be developed in a spherical harmonics series development, the uniform convergence of it is secured, [9] [12] ,

$$p_Q(\varphi, \lambda) = \sum_{n=1}^{\infty} p_{Q.n}\, u_n(\varphi, \lambda) \; . \qquad (97)$$

Since $p_Q$ is a limited, continuous and regular function, the uniform convergence of (97) can be taken for granted, (96). The development (97) allows the separation of the arbitrary small residual term $\varepsilon_6$,

$$p_Q = \sum_{n=1}^{J} p_{Q.n}\, u_n(\varphi, \lambda) + \varepsilon_6 \; . \qquad (98)$$

<u>Theorem 7:</u>

For any positive number $|\varepsilon_{6.0}|$ which is chosen arbitrary small,

$$|\varepsilon_{6.0}| > 0 \quad , \qquad (99)$$

there exists an integer J, sufficient great, such that

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$$\left| \sum_{n=J+1}^{\infty} p_{Q.n} \, u_n(\varphi, \lambda) \right| = \left| \varepsilon_6 \right| < \left| \varepsilon_{6.0} \right| \qquad (100)$$

The following lines are in the focus of interest, in this context: For a given number $\varepsilon_{6.0}$, the integer J has the property that it has a certain upper value which is not needed to be exceeded, $J = J(\varepsilon_{6.0})$, as long as the inequation (100) has to be fulfilled, whatever the amount of Q may be, (96) (98). Therefore, to be more precise, the theorem 7 can be supplemented by the following theorem 8, (96) (98) (99) (100).

Theorem 8:

For all the regular functions $p_Q$, (96) (98), - whatever the amount of the integer parameter Q may be -, a positive number $|\varepsilon_7|$ can be chosen having an arbitrary small amount - independent of Q -

$$|\varepsilon_7| > 0 , \qquad (101)$$

in such a way, that there exists a fixed upper bound $[J]$ for the integer J, (98),

$$J \leqslant [J] , \qquad (102)$$

fulfilling the relation

$$p_Q = \sum_{n=1}^{[J]} p_{Q.n} \, u_n(\varphi, \lambda) + \varepsilon_8(Q) , \qquad (103)$$

with

$$|\varepsilon_8(Q)| \leqslant |\varepsilon_7| . \qquad (104)$$

The relations (102) (103) (104) are valid for all the amounts of the positive integer parameter Q. The values of $[J]$ and $\varepsilon_7$ can be considered as being independent of the parameters Q, $\varphi$, $\lambda$.

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The inequality of Schwarz leads to a relation for the truncated series development (97) which is identical with the first term on the right hand side of (103),

$$\left| \sum_{n=1}^{[J]} p_{Q.n}\, u_n\,(\varphi,\lambda) \right| \leqslant \left| \sqrt{\sum_{n=1}^{[J]} p^2_{Q.n}} \right| \cdot \left| \sqrt{\sum_{n=1}^{[J]} u_r^2\,(\varphi,\lambda)} \right| . \quad (105)$$

The orthonormal spherical harmonics $u_n\,(\varphi\,,\lambda\,)$,

$$u_1,\ u_2,\ u_3,\ \ldots\ ,\ u_{[J]}\ , \qquad (106)$$

are well-known to be continuous and limited functions, [9] [12]. Since both the functions $u_n\,(\varphi\,,\lambda\,)$, $(1 \leqslant n \leqslant [J])$, and the integer $[J]$ have limited amounts, (102) (103) (104), the second term on the right hand side of (105) has consequently also a limited amount. Thus,

$$\left| \sqrt{\sum_{n=1}^{[J]} u_n^2\,(\varphi\,,\lambda\,)} \right| < S\ . \qquad (107)$$

S is a positive and limited amount.

Further, with (59) (78) (91) (96) (97),

$$\| p_Q \|^2 = \sum_{n=1}^{\infty} p^2_{Q.n} = \left\| f - \sum_{n=1}^{Q} g_n\, e_n\,(\varphi\,,\lambda\,) \right\|^2 =$$

$$= \left\| \sum_{n=Q+1}^{\infty} g_n\, e_n\,(\varphi\,,\lambda\,) \right\|^2 = \sum_{n=Q+1}^{\infty} g_n^2\ . \qquad (108)$$

A view on (88) and (89), and (91) shows that the integer Q in (108) can be chosen in such a way that the last term of (108),

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$$\sum_{n=Q+1}^{\infty} g_n^2 < \varepsilon_4 , \tag{109}$$

is arbitrary small.

Theorem 9:

For any positive number $\varepsilon_9$, arbitrary small,

$$\varepsilon_9 > 0 , \tag{110}$$

there exists an integer Q, sufficient great, such that

$$\left| \sqrt{\sum_{n=1}^{\infty} p_{Q.n}^2} \right| < \varepsilon_9 . \tag{111}$$

The theorem 9 is the consequence of the formulas (108) and (109).

The procedure constructed by the relations (105) (107) (110) (111) reveals the validity of the subsequent theorem.

Theorem 10:

Considering the relation (96),

$$p_Q(\varphi, \lambda) = f(\varphi, \lambda) - \sum_{n=1}^{Q} g_n \, \bullet_n (\varphi, \lambda) , \tag{112}$$

which is governed by the integer Q, it is possible to choose an arbitrary small positive number $\varepsilon_{10}$,

$$\varepsilon_{10} > 0 ; \tag{113}$$

it represents the right hand side of (105). There exists a sufficient great integer Q such that the development (97),

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$$p_Q(\varphi,\lambda) = \sum_{n=1}^{\infty} p_{Q.n}\, u_n(\varphi,\lambda) \quad , \tag{114}$$

fulfills - after a truncation - the following inequation, (105),

$$\left| \sum_{n=1}^{[J]} p_{Q.n}\, u_n(\varphi,\lambda) \right| \leqslant \varepsilon_{10} \leqslant \varepsilon_9 \cdot S \; . \tag{115}$$

The relations (90) (91) (98) (103) and (115) prove the validity of the inequation (116),

$$\left| p_Q \right| \leqslant \left( \varepsilon_{10} + \left| \varepsilon_8 \right| \right) \leqslant \left( \varepsilon_9 \cdot S + \left| \varepsilon_8 \right| \right) \; . \tag{116}$$

Since the terms $\varepsilon_{10}$, $\varepsilon_9$ and $|\varepsilon_8|$ can take on arbitrary small amounts, the right hand side of (116) can be considered as a term the amount of which is arbitrary small, if the integer Q is sufficient great. Therefore, the following inequation is right, (90) (91) (96) (116),

$$\left| p_Q \right| < \varepsilon_5 \; . \tag{117}$$

This above inequation corroborates the validity of the theorem 6, (90) (91). The theorem 6 is the formulation of the condition for the uniform convergence of the here discussed series development, (78).

Thus, the inequation (117) proves the uniform convergence of the series development (78). The introduction of properly chosen amounts for the integer $\eta$ has the consequence that (117) and the theorem 6, (90) (91), are valid for an arbitrary small positive number $\varepsilon_5$. This fact includes the uniform convergence of (78).

Before this background and with regard to (96), it is permitted to write the equation (118),

$$f(\varphi,\lambda) = \sum_{n=1}^{\infty} g_n\, e_n(\varphi,\lambda) \; . \tag{118}$$

The series development (118) can be brought into the following shape,

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$$f(\varphi,\lambda) = \sum_{n=1}^{Q} g_n e_n(\varphi,\lambda) + \varepsilon_{11} , \tag{119}$$

$$|\varepsilon|_{11}| < |\varepsilon|_{11.0}| > 0 , \text{ if } Q > Q_o . \tag{120}$$

$|\varepsilon|_{11.0}|$ is arbitrary small, it computes the limited value $Q_0$,

$$Q_0 = Q_0 ( \varepsilon|_{11.0}) . \tag{121}$$

Later on, in another paragraph, the functions $e_n(\varphi,\lambda)$ will be identified with the functions $w_n(\varphi,\lambda)$ which derive from the linear independent functions $v_n(\varphi,\lambda)$, (10) (11) , (26) (27).

The linear independence is a property that is defined for a limited number of functions, (15). The definition of $p_Q(\varphi,\lambda)$ according to (96) does not conflict with the linear independence of the functions $v_n(\varphi,\lambda)$, (15), if the $e_n$ functions are substituted by the functions $w_n$, and, further on, if the functions $w_n(\varphi,\lambda)$ are substituted by the functions $v_n(\varphi,\lambda)$, (see (27) (28)), since Q has a limited amount, (119) (120). The following sentence is derived by (119), and up to (121). (The completeness of the systems $w_n(\varphi,\lambda)$ or $v_n(\varphi,\lambda)$: See paragraph 6.9).

Theorem 11:

The series development (118) is convergent, because, after the choice of a positive number, arbitrary small,

$$|\varepsilon|_{11.0}| > 0 , \tag{122}$$

an integer $Q_0 = Q_0(\varepsilon_{11.0})$ can be found such that for the integer Q,

$$Q > Q_0 , \tag{123}$$

the following relation is valid, (96),

$$\left| f(\varphi,\lambda) - \sum_{n=1}^{Q} g_n e_n(\varphi,\lambda) \right| < |\varepsilon_{11.0}| \tag{124}$$

In the second term on the left hand side of (124), the functions $e_n(\varphi,\lambda)$ can be identified with the functions $w_n(\varphi,\lambda)$, (26). After it, $w_n(\varphi,\lambda)$ can be replaced by $v_n(\varphi,\lambda)$, introducing (27) and (28). The modified form of (124) obtained along these lines considers $v_n(\varphi,\lambda)$ only for $n = 1,2,\ldots,Q$. Thus, this modified form is in

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keeping with (15), because Q represents a limited number. There is no upper bound for Q. (See also the paragraphs 6.3 and 6.4).

The introduction of the convergence criterion of Cauchy avoids likewise a conflict with the fact, that (15) refers to a limited number of terms only, (if $e_n \longrightarrow w_n \longrightarrow v_n$). In this context, the inequality (91) is replaced by

$$\left| \sum_{n=Q+1}^{Q+Q^*} g_n\, e_n\,(\varphi,\lambda) \right| < \varepsilon_{12} \quad . \tag{125}$$

<u>Theorem 12:</u>

After the choice of a positive number, arbitrary small,

$$\varepsilon_{12} > 0 \quad , \tag{126}$$

an integer $Q_{0.0} = Q_{0.0}\,(\varepsilon_{12})$ can be found such that for the integer Q,

$$Q > Q_{0.0} \quad , \tag{127}$$

and for the integer

$$Q^* \geq 1 \quad , \tag{128}$$

the above inequation (125) is right.

The validity of the sentence expressed by (125) to (128) is easily proved, as follows:
(125) and (96) give

$$\sum_{n=Q+1}^{Q+Q^*} g_n\, e_n(\varphi,\lambda) = p_Q(\varphi,\lambda) - p_{Q+Q^*}(\varphi,\lambda) = p_{Q.Q^*}(\varphi,\lambda) \; . \tag{129}$$

$p_Q\,(\varphi,\lambda)$ and $p_{Q+Q^*}\,(\varphi,\lambda)$ are continuous functions, (96). Therefore, $p_{Q.Q^*}\,(\varphi,\lambda)$ is a continuous function also. It has the following uniform convergent series development in terms of spherical harmonics, (97),

$$p_{Q.Q^*}\,(\varphi,\lambda) = \sum_{n=1}^{\infty} p_{Q.Q^*.n}\; u_n\,(\varphi,\lambda) \quad . \tag{130}$$

(129) and (130) give

$$\left\| p_{Q.Q^*}\,(\varphi,\lambda) \right\|^2 = \sum_{n=1}^{\infty} p^2_{Q.Q^*.n} = \sum_{i=Q+1}^{Q+Q^*} g^2_i . \tag{130a}$$

In analogy to (98), the relation (130) is transformed to

$$p_{Q.Q^*}\,(\varphi,\lambda) = \sum_{n=1}^{J^*} p_{Q.Q^*.n}\; u_n\,(\varphi,\lambda) + \varepsilon^*_6 \; . \tag{131}$$

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$\varepsilon^{*}_{6}$ is an arbitrary small residuum, as $\varepsilon_6$, (98). $\varepsilon^{*}_{6}$ fulfills a theorem analogous to the Theorem 7 for $\varepsilon_6$. (103) and (131) lead to

$$p_{Q.Q^{*}}(\varphi,\lambda) = \sum_{n=1}^{[J^{*}]} p_{Q.Q^{*}.n}\, u_n(\varphi,\lambda) + \varepsilon^{*}_{8}(Q,Q^{*}) \;. \tag{132}$$

$[J^{*}]$ is a fixed upper bound for $J^{*}$, which derives from the constraint that the function $|\varepsilon^{*}_{8}(Q,Q^{*})|$ does not surmount the amount (which is independent of $Q,Q^{*}$) of a certain upper bound $|\varepsilon^{*}_{7}|$ , for all values of $Q \geq 1$ and $Q^{*} \geq 1$, (96)(128)(129), (see Theorem 8). With (105), the inequality of Schwarz gives, (132),

$$\left|\sum_{n=1}^{[J^{*}]} p_{Q.Q^{*}.n}\, u_n(\varphi,\lambda)\right| \leq \left|\sqrt{\sum_{n=1}^{[J^{*}]} p^2_{Q.Q^{*}.n}}\right| \left|\sqrt{\sum_{n=1}^{[J^{*}]} u_n^2(\varphi,\lambda)}\right| \;. \tag{133}$$

The functions $u_n(\varphi,\lambda)$ and the fixed integer $[J^{*}]$ have limited upper bounds. Thus,

$$\left|\sqrt{\sum_{n=1}^{[J^{*}]} u_n^2(\varphi,\lambda)}\right| < S^{*} \;. \tag{134}$$

$S^{*}$ is a positive and limited amount. After the choice of a positive number $\varepsilon_{12}$,

$$\varepsilon_{12} > 0 \;, \tag{135}$$

arbitrary small, an integer $Q_{0.0}(\varepsilon_{12})$ can be found such that, considering (109) (110)(111)(125)(127)(128)(129)(130a)(132),

$$\left|p_{Q.Q^{*}}(\varphi,\lambda)\right| \leq \left(\varepsilon_{14} + \left|\varepsilon^{*}_{8}(Q,Q^{*})\right|\right) = \varepsilon_{12} \;, \tag{136}$$

with

$$\left|\sum_{n=1}^{[J^{*}]} p_{Q.Q^{*}.n}\, u_n(\varphi,\lambda)\right| = \varepsilon_{14} \leq \varepsilon_{13} S^{*} < \varepsilon_4 S^{*} \;, \tag{136a}$$

and with

$$\left|\sqrt{\sum_{n=1}^{[J^{*}]} p^2_{Q.Q^{*}.n}}\right| = \varepsilon_{13} \leq \left|\sqrt{\sum_{i=Q+1}^{Q+Q^{*}} g_i^2}\right| \leq \left|\sqrt{\sum_{i=Q+1}^{\infty} g_i^2}\right| < \varepsilon_4 \;. \tag{136b}$$

$\varepsilon_{13}$, $\varepsilon_{14}$ are positive amounts, arbitrary small; $\varepsilon_{13} < \varepsilon_4$, $\varepsilon_{14} < \varepsilon_4 S^{*}$, (109)(134). After the choice of $\varepsilon_{12}$, $\varepsilon^{*}_{7}$ can be chosen such that $|\varepsilon^{*}_{8}(Q,Q^{*})| \leq |\varepsilon^{*}_{7}| < \frac{1}{2}\varepsilon_{12}$, for all values of Q and $Q^{*}$,(136). $Q_{0.0}$ can be chosen such that $\varepsilon_{14} < \frac{1}{2}\varepsilon_{12}$, by a sufficient far extension of Q in (96),

$$\sum_{n=1}^{Q} g_n e_n(\varphi,\lambda) \;, \tag{136c}$$

before the background of the convergence in the mean, (79)(80)(88)(89). Consequently, (125) is right. Thus, the uniform convergence of (118) is corroborated. (136) does not conflict with (15), because Q and $Q^{*}$ have certain values, according to (127)(128); they do not go to infinity, a property demanded by the

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definition of the linear independence of the $v_n$ functions. These $v_n$ functions lead to the $w_n$ functions identified later on with the $e_n$ functions of (136c).

Now, the often discussed counterexample about the convergence of the considered series should be mentioned, (207a)(265): A point mass is introduced, it has the distance b to the gravity center of a spherical body with the radius R, b > R. For the computation of the potential T of this point mass, in the exterior of the surface of a rotation ellipsoid enclosing all the masses, the straight distance to the point mass is developed in spherical harmonics converging only if a > b (a: Radius of the test point). The exterior potential T can be expressed by the masses $m_i$ . There exists an infinity of different systems of mass distributions $m_i$ (i = 1,2,3,...), each of them gives the same exterior potential field T,

$$T = \alpha(m_i), \qquad (i = 1,2,3,...). \qquad (136d)$$

The expression (136d) for a special parameter i has not a unique inversion. The mass distribution in the interior can not be determined in a unique way by the exterior T values - a well-known fact -. One mass distribution of the infinity of mass distributions $m_i$, generating the one exterior T potential, can be in keeping with the convergence. For instance, the system of the mass distribution within the Bjerhammar sphere gives rise to a convergent series, sure. Thus, the above counterexample is not convincing.

A potential of masses within an ellipsoidal boundary surface can be developed in Lamé functions which give a well-known convergent series development in the exterior. In case of a rotation ellipsoid, the Lamé functions degenerate to spherical harmonics and absolute convergent series. Also in this case, the convergence is never in question.

The exterior potential and the gravitating masses have not a one-to-one mapping, this fact is a clear handicup in the here discussed problem. But the exterior potential and the boundary values of it have the preference to be connected by a one-to-one mapping, as it is proved within the scope of the Dirichlet boundary value problem. Therefore, it is not convenient to consider the convergence of the here discussed series development in terms of the gravitating masses. But, certainly, if these considerations are carried out in terms of the boundary values of the potential, the questions about the convergence of this series development find a clear, positive, and satisfactory answer.

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## 6. The convergence of the spherical harmonics series development of a potential in the exterior space of a regular surface.

### 6.1. The interrelations between the different systems of base functions.

The Schmidt orthogonalization process conducts from a system of linear independent functions, as $v_i$, to a set of orthonormalized functions, as $w_i$, (see (27) (28)). The functions

$$w_1 ( \varphi , \lambda ), w_2 ( \varphi , \lambda ), w_3 ( \varphi , \lambda ), \ldots, w_L ( \varphi , \lambda ) \quad (137)$$

construct the vector $\underline{\underline{w}}_L$, they depend on the linear independent and regular functions $v_i ( \varphi , \lambda )$ by the linear systems (27) and (28). The $v_i$ functions are linear independent, (15) (16). Therefore, the Gram determinants that appear in the course of the orthogonalization process are necessarily non-singular, it can be taken from the textbooks, [6] [10] [14]. Thus, the constant coefficients $b_{i.k}$ of (27) and (28) follow to have finite amounts, and the functions $w_i$, $(1 \leqslant i \leqslant L)$, yield as continuous, regular and orthonormal functions. Consequently, the functions $w_i$ have convergent series developments in terms of the spherical harmonics, [9].

It is easily shown that the function $w_i$, (26) (27) (137), have the same properties as the functions $e_i$, (56) (119). The functions $w_i$ are orthonormal, as $e_i$. The functions $w_i$ are regular, as $e_i$.

There is a one-to-one mapping between the elements $w_i$ and $v_i$, $(w_i \leftrightarrow v_i)$, (see (27)). There is also a one-to-one mapping between the elements $v_i$ and $u_i$, $(v_i \leftrightarrow u_i)$, (see (10)). Consequently, there follows also a one-to-one mapping between the elements $w_i$ and $u_i$, $(w_i \leftrightarrow u_i)$. Such a one-to-one mapping exists also between the elements $e_i$ and $u_i$, $(e_i \leftrightarrow u_i)$, (see (54) (55)). There is no upper bound for i, sure.

Hence, there exists a one-to-one mapping also between the elements $w_i$ and $e_i$, $(w_i \leftrightarrow e_i)$.

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The functions $v_i(\varphi, \lambda)$ are linear independent, they have the representation according to (11). The functions $w_i(\varphi, \lambda)$ ensue from the functions $v_i(\varphi, \lambda)$ by means of the Schmidt orthogonalization process, (35). The functions $u_i(\varphi, \lambda)$ are the orthonomalized spherical harmonics. The detailed mutual dependences are expressed by the following lines, (10),

$$v_n(\varphi, \lambda) = \sum_{i=1}^{\infty} b'_{n.i}\, u_i(\varphi, \lambda) \quad , \tag{138}$$

and, (27),

$$w_n(\varphi, \lambda) = \sum_{i=1}^{n} b_{n.i}\, v_i(\varphi, \lambda) \quad , \tag{139}$$

$$n = 1, 2, \ldots \quad . \tag{140}$$

With the infinite column vector,

$$\underline{\underline{u}} = \begin{pmatrix} u_1(\varphi, \lambda) \\ u_2(\varphi, \lambda) \\ u_3(\varphi, \lambda) \\ \ldots \end{pmatrix} \quad , \tag{141}$$

of the spherical harmonics, and with the representation of $\underline{\underline{v}}_L$ and $\underline{\underline{w}}_L$ according to (32) and (31), the subsequent matrix relations are obtained, (28) (35), (138),

$$\underline{\underline{v}}_L = \underline{\underline{B}}'_L\, \underline{\underline{u}} \quad , \tag{142}$$

$$\underline{\underline{w}}_L = \underline{\underline{B}}_L\, \underline{\underline{v}}_L \quad , \tag{143}$$

$$\underline{\underline{w}}_L = \underline{\underline{C}}_L\, \underline{\underline{u}} \quad ; \tag{144}$$

here is

$$\underline{\underline{C}}_L = \underline{\underline{B}}_L\, \underline{\underline{B}}'_L \quad . \tag{145}$$

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$\underline{\underline{B}}_L$ is a lower triangular matrix which results from the orthonormalization process. Therefore, it is a non-singular matrix, - eo ipso -. Thus, the inversion of (143) is possible, (35),

$$\underline{\underline{V}}_L = (\underline{\underline{B}}_L)^{-1}\, \underline{\underline{W}}_L \quad . \tag{146}$$

$(\underline{\underline{B}}_L)^{-1}$ is a lower triangular matrix, too. $\underline{\underline{B}}_L$ and $(\underline{\underline{B}}_L)^{-1}$ are non-singular, square, and $L \cdot L$ dimensional matrices,

$$\det \underline{\underline{B}}_L \neq 0 \quad . \tag{147}$$

But the matrix $\underline{\underline{B}}'_L$ is not square , it has the elements $b'_{i.k}$, $(i = 1, 2, \ldots, L)$ $(k = 1, 2, \ldots)$ . i has a finite sequence, but k does go to infinity.

$$\underline{\underline{B}}'_L = \begin{pmatrix} b'_{1.1} & b'_{1.2} & b'_{1.3} & \cdots \\ b'_{2.1} & b'_{2.2} & b'_{2.3} & \cdots \\ b'_{3.1} & b'_{3.2} & b'_{3.3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b'_{L.1} & b'_{L.2} & b'_{L.3} & \cdots \end{pmatrix} , \tag{148}$$

or,

$$\underline{\underline{B}}'_L = \left\{ b'_{i.k} \right\} , \tag{149}$$

with (9) (10), (12),

$$b'_{i.k} = \iint_F \left(\frac{1}{t}\right)^i u_i(\varphi, \lambda)\, u_k(\varphi, \lambda)\, dF \; . \tag{150}$$

For $i = k$, the relation (150) leads to, $(t > 0)$,

$$b'_{n.n} > 0 \quad . \tag{151}$$

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## 6.2. The space of the base functions.

As to the system of the functions $v_n$ ( $\varphi$ , $\lambda$ ) and that of the functions $u_n$ ( $\varphi$ , $\lambda$ ), each of them consists of an infinity of functions depending on $\varphi$ and $\lambda$ . They cover the sphere, (11). The first system consists of linear independent functions, the second system is orthonormal. But, this is not a complete description of the properties of the two systems. Beyond it, $v_n$ derives from $u_n$ - and only from the one single function $u_n$ selected from all the other functions $u_i$ (i = 1, 2, 3, ...), - by the well-defined and unique relation (11) which multiplies the function $u_n$ with $(1/t)^n$. This relation (11) has a unique inversion,

$$u_n ( \varphi , \lambda ) = (t ( \varphi , \lambda ))^n v_n ( \varphi , \lambda ) . \tag{152}$$

A single member $v_n$ of the system of the $v_n$ functions is generated only by a single member $u_n$ of the system of the $u_n$ functions. And inverse, a single member $u_n$ of the system of the $u_n$ functions is generated only by a single member $v_n$ of the system of the $v_n$ functions. The interrelation (11) demonstrates that the one-to-one mapping

$$v_n \leftrightarrow u_n , \tag{153}$$

$$r : 1, 2, ..., L, \tag{154}$$

$$L \longrightarrow \infty , \tag{155}$$

is possible, sure. There is no upper bound for the integer L, sure. The system of the functions $v_n$ has - so to speak - the same number of elements as the system of the functions $u_n$. Further, on the strength of (11), the regular functions $v_n$ are situated in the space of the regular functions described by the system of the $u_n$ functions. Consequently, it is obvious that the functions $v_n$ determine and generate completely the space of the functions $u_n$ which is the space of the regular functions.

By means of (27), the $v_n$ functions describe completely the same space as the $w_n$ functions. Further, in a similar way, by means of

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(27), the $w_n$ functions describe completely the same space as the $u_n$ functions.

Thus, obviously, a regular function can be expressed completely by the $v_n$ functions or by the $u_n$ functions. On the strength of (11) and (153)(154)(155), the functions $v_n$ cannot determine only a subspace of the space of the functions $u_n$, ($n = 1, 2, \ldots, L$; $L \longrightarrow \infty$). Later on, in this chapter (paragraph 6.9.), this fact is corroborated by the detailed investigations about the completeness of the system of the $v_n$ functions, described in the paragraph 6.9.

Summarizing, the relation (11) leads from the $u_n$ system to the $v_n$ system. The orthonormalization is the step from the $v_n$ system to the $w_n$ system. The above deliberations show that the $u_n$ system can be transformed into the $w_n$ system by the multiplication with an infinite orthonormal matrix $\underline{\underline{A}}$ which has the property of (41).

## 6.3. The determinant of the orthonormal matrix.

Now, the matrix $\underline{\underline{B}}'_L$ of (142) (148) (149) is put into the fore. Truncating the rows of (148) behind the element of the suffix k = L, the following L · L dimensional square matrix is obtained,

$$\underline{\underline{B}}'_{L.L} = \begin{pmatrix} b'_{1.1} & b'_{1.2} & b'_{1.3} & \cdots & b'_{1.L} \\ b'_{2.1} & b'_{2.2} & b'_{2.3} & \cdots & b'_{2.L} \\ b'_{3.1} & b'_{3.2} & b'_{3.3} & \cdots & b'_{3.L} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b'_{L.1} & b'_{L.2} & b'_{L.3} & \cdots & b'_{L.L} \end{pmatrix} , \qquad (155a)$$

or

$$\underline{\underline{B}}'_{L.L} = \left\{ b'_{i.k} \right\} , \qquad (155b)$$

$$i, k = 1, 2, 3, \ldots, L. \qquad (155c)$$

The elements of the main diagonal of $\underline{\underline{B}}'_{L.L}$ are positive, there is no row and no column of $\underline{\underline{B}}'_{L.L}$ which consists of zero elements only, (151).

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Further, the truncated column vector $\underline{\underline{u}}_L$ is introduced, (141) (142),

$$\underline{\underline{u}}_L = \begin{bmatrix} u_1(\varphi, \lambda) \\ u_2(\varphi, \lambda) \\ u_3(\varphi, \lambda) \\ \cdots \\ u_L(\varphi, \lambda) \end{bmatrix} . \tag{155d}$$

The greater the value of L the better the product

$$\underline{\underline{B}}'_{L.L} \; \underline{\underline{u}}_L \tag{155e}$$

approximates the vector $\underline{\underline{v}}_L$, (142). The greater the value of L the more precise the row vectors of $\underline{\underline{B}}'_{L.L}$ tend to construct a L - dimensional parallelepiped the volume of which is never equal to zero. This important fact is right, because these row vectors of $\underline{\underline{B}}'_{L.L}$ tend to be linear independent if L becomes greater and greater. The greater the value of L the better the product (155e) represents the linear independent functions $v_n(\varphi, \lambda)$, (142).

The following transition behavior is valid:
If the suffix L tends to infinity, $L \longrightarrow \infty$ , the consequences are,

$$\underline{\underline{B}}'_{L.L} \longrightarrow \underline{\underline{B}}' \quad , \tag{156}$$

$$\det \underline{\underline{B}}' \neq 0 \quad , \tag{157}$$

and, (143) (147),

$$\underline{\underline{B}}_L \longrightarrow \underline{\underline{B}} \quad , \tag{157a}$$

$$\det \underline{\underline{B}} \neq 0 \quad . \tag{157b}$$

The relations (145) (147) (155e) (157) and

$$\underline{\underline{B}}_L \, \underline{\underline{B}}'_{L.L} \longrightarrow \underline{\underline{B}} \, \underline{\underline{B}}' = \underline{\underline{C}} \quad , \quad \text{for } L \longrightarrow \infty \quad , \tag{158}$$

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lead to,

$$\det \underline{\underline{C}} = \det \underline{\underline{B}} \cdot \det \underline{\underline{B}}' \neq 0 \ . \qquad (159)$$

Because of (159), the inversion of the matrix $\underline{\underline{C}}$ is possible, (144),

$$\underline{\underline{u}} = \underline{\underline{C}}^{-1} \underline{\underline{w}} \ , \qquad (160)$$

$$\underline{\underline{w}} = \underline{\underline{C}} \ \underline{\underline{u}} \ . \qquad (161)$$

The functions $w_n$ and $u_n$ are orthonormalized. Thus, $\underline{\underline{C}}$ is orthonormal, (41).

$$\underline{\underline{C}} \ \underline{\underline{C}}^T = \underline{\underline{C}} \ \underline{\underline{C}}^{-1} = \underline{\underline{E}} \ . \qquad (162)$$

Further, as to the diagonal elements of $\underline{\underline{C}}^T \underline{\underline{C}}$, these values are the diagonal elements of

$$(\underline{\underline{B}} \ \underline{\underline{B}}')^T \ (\underline{\underline{B}} \ \underline{\underline{B}}') \ , \qquad (163)$$

see (158). Because of $b'_{n.n} > 0$, (151), and $b_{n.n} > 0$, (28a), and regarding (27) (28) (28a) (152), it is easily seen that the column vectors of the product matrix $\underline{\underline{B}} \ \underline{\underline{B}}'$ have at least one component which does not vanish. Thus, no diagonal element of $\underline{\underline{C}}^T \underline{\underline{C}}$ is equal to zero.

Consequently, the transition $L \longrightarrow \infty$ and the equation (162) lead to. (159),

$$\det \underline{\underline{C}} \neq 0 \ , \qquad (164)$$

and, more precise,

$$(\det \underline{\underline{C}}) \ (\det \underline{\underline{C}}^T) = (\det \underline{\underline{C}})^2 = \det \underline{\underline{E}} = 1. \qquad (165)$$

Hence, more detailed than (159) (164),

$$\det \underline{\underline{C}} = 1 \ . \qquad (166)$$

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The relation (166) corroborates the fact that the matrix $\underline{\underline{C}}$ is non-singular. $\underline{\underline{C}}$ can be inverted, (160) (161).

The subsequent relations are self-explanatory,

$$\underline{\underline{C}}^{-1} = \underline{\underline{C}}^{T} , \qquad (167)$$

$$\underline{\underline{C}}\,\underline{\underline{C}}^{T} = \underline{\underline{C}}^{T}\,\underline{\underline{C}} = \underline{\underline{E}} . \qquad (168)$$

(168) gives the definition of an infinite orthonormal matrix, [7], (see also (41)). Thus, the matrix $\underline{\underline{C}}$ according to (145) and (158) is an infinite orthonormal matrix.

## 6.4 The convergence property derived by the consideration of a determinant.

Before the background of the above lines, (see the equations from (138) to (168)), it is possible to identify the functions $w_i$, $(1 \leq i \leq Q)$, with the functions $e_i$, (56) (119), - (see, in paragraph 6.9, the detailed completeness proof for the functions $v_i$ and, consequently, for the functions $w_i$, too) -,

$$e_1 (\varphi , \lambda), e_2 (\varphi , \lambda), e_3 (\varphi , \lambda), \ldots, e_Q (\varphi , \lambda), \qquad (169)$$

$$(i = 1, 2, \ldots, Q).$$

Here, it is allowed to replace the integer L in (137) by the integer Q of (119).

There is no upper limit for the integer Q and for the integer L.

Further on, it is allowed to identify the matrix $\underline{\underline{C}}$ with the matrix $\underline{\underline{A}}$.

Consequently, the formula (119) can be transformed into (170),

$$f (\varphi , \lambda) = \sum_{n=1}^{Q} g_n \, w_n (\varphi , \lambda) + \varepsilon_{11} . \qquad (170)$$

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The detailed shape of the functions $w_n$, $n > Q$, is not required here in the equation (170); the share exerted by these functions is represented by the residual term $\varepsilon_{11}$ which can be neglected. The uniform convergence of the development (170) is corroborated later - desisting from using (119) to (121) - by a consideration of the system of the $w_n$ functions, (see the paragraphs 6.9. and 6.10.).

## 6.5. The separation of a limited number of linear independent functions.

Some explanatory lines about the transition from the $e_i$ functions to the $w_i$ functions, ($i = 1, 2, 3, \ldots, Q$), should be added. For instance, it is possible that the $\underline{\underline{A}}$ matrix of (54) transforms the $\underline{\underline{u}}$ vector into an $\underline{\underline{e}}$ vector of which the foremost components - Q in number - are not identical with the orthonormal functions

$$w_1(\varphi, \lambda), w_2(\varphi, \lambda), \ldots, w_Q(\varphi, \lambda) \; ; \qquad (171)$$

(See the transition from (119) to (170)). In this case, the equation (54) is multiplied on both sides with a second infinite orthonormal matrix $\underline{\underline{A}}'$,

$$\underline{\underline{w}}' = \underline{\underline{A}}' \, \underline{\underline{e}} = \underline{\underline{A}}' \, \underline{\underline{A}} \, \underline{\underline{u}} \; . \qquad (172)$$

$$\underline{\underline{w}}' = \begin{bmatrix} w_1(\varphi, \lambda) \\ w_2(\varphi, \lambda) \\ \ldots \\ w_Q(\varphi, \lambda) \\ w'_{Q+1}(\varphi, \lambda) \\ w'_{Q+2}(\varphi, \lambda) \\ \ldots \end{bmatrix} . \qquad (173)$$

As to (173), the following relation is valid,

$$w_j(\varphi, \lambda) = w'_j(\varphi, \lambda) \; ; \; \text{if } j = 1, 2, \ldots, Q. \qquad (174)$$

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The product of two infinite orthonormal matrices is again an infinite orthonormal matrix, it is obvious. Therefore, $\underline{\underline{A}}' \underline{\underline{A}}$ is again an infinite orthonormal matrix, (172). Thus, the relation (172) leads to the fact that the vector $\underline{\underline{w}}'$ represents an infinite system of orthonormal base functions $w_i^{!}$ ( $\varphi$ , $\lambda$ ), similar as the systems of the $e_i$ and $u_i$ functions. The Parseval completeness relation is also valid for the system of the $w_i^{!}$ functions, (i = 1, 2, ...), (79) (80) (172), similar as for the systems of the $u_i$ or $e_i$ functions. There is a one-to-one mapping between the elements $e_i$ and $w_i^{!}$ and between $u_i$ and $w_i^{!}$, (174). This mapping happens on the basis of (10) and (11) and (27).

The subsequent relations are self-explanatory, (45) (59),

$$\underline{\underline{A}}' = \left\{ a^{!}_{i.k} \right\} \quad ; \quad i, k = 1, 2, \ldots \quad . \tag{175}$$

The orthogonality relations for the $e_i$ functions and the equations (172) lead to

$$a^{!}_{i.k} = \iint_F w_i ( \varphi , \lambda ) \, e_k ( \varphi , \lambda ) \, d F \quad ; \tag{176}$$

for

$$i = 1, 2, \ldots, Q, \tag{177}$$

and for

$$k = 1, 2, \ldots \quad . \tag{178}$$

Because of (174), the foremost components of the vector w' , - Q in number -, are equal to the $w_i$ functions with the suffixes i = 1, 2, ..., Q. Therefore, the relations (119) and (170) show that the subsequent 3 equations are right,

$$f ( \varphi , \lambda ) = \sum_{n=1}^{Q} g'_n \, w'_n ( \varphi , \lambda ) + \varepsilon_{11} \quad , \tag{179}$$

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$$f(\varphi, \lambda) = \sum_{n=1}^{Q} g_n' w_n(\varphi, \lambda) + \varepsilon_{11} \quad , . \tag{180}$$

$$f(\varphi, \lambda) = \sum_{n=1}^{Q} g_n w_n(\varphi, \lambda) + \varepsilon_{11} \quad . \tag{181}$$

It is possible to choose a certain number, arbitrary small,

$$|\varepsilon_{11.0}| > 0 \quad , \tag{182}$$

having the property, that

$$|\varepsilon_{11}(Q)| < |\varepsilon_{11.0}| \tag{183}$$

for a sufficient great value of the integer Q.

The explicit shape of the functions $w_n'$, for the suffixes $n > Q$, is not necessary to be discussed in this context. These functions are involved by the effect they take on the residual term $\varepsilon_{11}$ of (179) (180) (181): $\varepsilon_{11}$ can be considered as an arbitrary small amount.

Within the course of the deductions of this above paragraph 6.5., the system (27) needs not to be extended to infinity. Therefore, it is possible to introduce the functions $v_i$ with the characteristic to be linear independent. Thus, the definition of this characteristic is not violated, since the above lines avoid an infinite extension of the number of the functions $v_i$.

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## 6.6. The convergence property at the surface.

The $\underline{\underline{B}}$ matrix transforms the vector $\underline{\underline{v}}$ into the vector $\underline{\underline{w}}$, (157a), (see (28) to (35)). With regard to the equation (170), the integer L has to be replaced by the integer Q. Hence, the matrix relation (28) gets the following form,

$$\underline{\underline{w}}_Q = \underline{\underline{B}}_Q \underline{\underline{v}}_Q \quad . \tag{184}$$

The formula (170) can be expressed by a scalar product,

$$f(\varphi, \lambda) = \underline{\underline{g}}_Q^T \underline{\underline{w}}_Q + \varepsilon_{11} \quad . \tag{185}$$

In (185), $\underline{\underline{g}}_Q$ is the following column vector for the constant coefficients,

$$\underline{\underline{g}}_Q = \begin{bmatrix} g_1 \\ g_2 \\ \cdots \\ g_Q \end{bmatrix} \quad . \tag{186}$$

The right hand side of (184) is a substitute for the function $\underline{\underline{w}}_Q$ in (185),

$$f(\varphi, \lambda) = \underline{\underline{g}}_Q^T \underline{\underline{B}}_Q \underline{\underline{v}}_Q + \varepsilon_{11} \quad . \tag{187}$$

The introduction of the following abbreviation is advantageous, (188),

$$\underline{\underline{q}}_Q^T = \underline{\underline{g}}_Q^T \underline{\underline{B}}_Q \quad , \tag{188}$$

The coefficients in the $\underline{\underline{v}}_Q$ system have the following column vector,

$$\underline{\underline{q}}_Q = \begin{bmatrix} q_1 \\ q_2 \\ \cdots \\ q_Q \end{bmatrix} \quad . \tag{189}$$

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Hence,

$$f(\varphi,\lambda) = \underline{q}_Q^T \ \underline{v}_Q + \varepsilon_{11} \ ; \tag{190}$$

and the corresponding expression by the components of the concerned vectors is,

$$f(\varphi,\lambda) = \sum_{n=1}^{Q} q_n v_n(\varphi,\lambda) + \varepsilon_{11} \ . \tag{191}$$

The formula (191) is a uniform convergent series development for the surface function f in terms of the linear independent functions $v_n(\varphi,\lambda)$, (see (119) (120)).

It is a small step only, the way that leads from the series (191) for the surface values $f(\varphi,\lambda)$ to a spatial spherical harmonics series development specialized for test points situated on the boundary surface D. The relations (10) and (191) give

$$f(\varphi,\lambda) = \sum_{n=1}^{Q} q_n \left(\frac{1}{t(\varphi,\lambda)}\right)^n u_n(\varphi,\lambda) + \varepsilon_{11} . \tag{192}$$

The combination of (8) (192) (120) leads to

$$f(\varphi,\lambda) = \sum_{n=1}^{Q} q_n \left[\left(\frac{1}{r}\right)^n\right]_D u_n(\varphi,\lambda) + \varepsilon_{11} \ , \tag{193}$$

$$\varepsilon_{11} \longrightarrow 0, \text{ if } Q \longrightarrow \infty \ . \tag{194}$$

The relations (193) and (194) are equivalent to the expression (195),

$$f(\varphi,\lambda) = \sum_{n=1}^{\infty} q_n \left[\left(\frac{1}{r}\right)^n\right]_D u_n(\varphi,\lambda) . \tag{195}$$

The formula (195) describes the following fact.

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If there is given a regular function f on a regular surface D, it is possible to develop f in a uniform convergent series in terms of the solid or spatial spherical harmonics being specialized for the points of the surface D.

## 6.7. The convergence of a series development in spatial spherical harmonics.

It is a short way only that leads from the convergence of (195) along the surface D to the uniform convergence of (3) (5) (6) in the exterior space of D. The formulas (81) and (195) give,

$$W_D = \sum_{n=1}^{\infty} q_n \left[\left(\frac{1}{r}\right)^n\right]_D u_n(\varphi, \lambda) . \qquad (196)$$

For the investigation into whether the validity of the convergence of (196) can be extended into the exterior space, a series criterion of Abel is in the focus of interest:

A convergent series development

$$\sum_{n=1}^{\infty} a_n \qquad (197)$$

is given. Further, a monotone sequence of terms with limited amounts is defined,

$$\{b_n\} = b_1, b_2, b_3, \ldots . \qquad (198)$$

The theorem of Abel states that (197) and (198) lead necessarily to the uniform convergence of the subsequent series development (199), /13/,

$$\sum_{n=1}^{\infty} b_n a_n . \qquad (199)$$

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In the here discussed applications, the terms $a_n$ of (197) are replaced by, (196),

$$a_n = q_n \left[ \left( \frac{1}{r} \right)^n \right]_D u_n ( \varphi , \lambda ) , \tag{200}$$

further, instead of $b_n$, the following expressions are introduced,

$$b_n = \frac{\left( \frac{1}{r} \right)^n}{\left[ \left( \frac{1}{r} \right)^n \right]_D} = \left( \frac{r_D}{r} \right)^n . \tag{201}$$

All the points of the same geocentrical latitudes and longitudes, $\varphi$ and $\lambda$ , are situated on the same geocentrical radius vector; the radii of all these points have the lower bound $r_D = r_D ( \varphi , \lambda )$. The upper bound of these radii extends to infinity,

$$r_D \leqslant r \leqslant \infty \tag{202}$$

(202) is valid for a certain parameter couple $\varphi$ , $\lambda$ of fixed values. Hence, (202),

$$0 \leqslant \frac{r_D}{r} \leqslant 1 . \tag{203}$$

Therefore, the $b_n$ terms construct here a monotone decreasing sequence,

$$\left\{ b_n \right\} = \frac{r_D}{r} , \left( \frac{r_D}{r} \right)^2 , \left( \frac{r_D}{r} \right)^3 , \dots . \tag{204}$$

The introduction of the relations (200) and (201) into the series (199) leads to the here important statement that

$$W = \sum_{n=1}^{\infty} q_n \left( \frac{1}{r} \right)^n u_n ( \varphi , \lambda ) \tag{205}$$

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is a uniform convergent series development which is valid in the exterior space of the Earth's surface D, and on it.

The formula (205) shows that the spatial function W is harmonic, (2). W fulfills the Laplace differential equation, (2), because the members of (205) fulfill the equation (2),

$$\triangle \left[ \left(\frac{1}{r}\right)^n u_n (\varphi, \lambda) \right] = 0 \quad ; \qquad (206)$$

the solid spherical harmonics are in the brackets of (206).

As a supplementary remark, it is to be stated that the first theorem of Harnack leads also from (196) to (205) - from the convergence on the surface D to the convergence in the exterior space of D, /12/.

## 6.8. The uniqueness.

The uniform convergent series (205) valid in the exterior space of D solves the Dirichlet boundary value problem: If $W_D$, (196), represents the boundary values on D, the uniform convergent series development (205) determines the attached spatial potential W in the exterior space of D.

The solution of the Dirichlet boundary value problem is well-known to be unique, /12/. Therefore, the spatial representation (205) is necessarily a unique one. There is no other W potential which is harmonic in the exterior space of D and which observes the boundary values $W_D$.

If $W_D$ on the left hand side of (196) is given as a regular surface function, this fact leads necessarily to a unique system of the $q_n$ coefficients on the right hand side of (196), because the terms

$$\left[ \left(\frac{1}{r}\right)^n \right]_D u_n (\varphi, \lambda) \qquad (206a)$$

are linear independent surface functions. This unique system of the $q_n$

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coefficients leads necessarily to a unique spatial representation of W by (205).

The boundary values $W_D$ construct a regular two-dimensional function; the solution of the Dirichlet boundary value problem is well-known to be a regular function in the exterior space of D, [12]. Hence, it follows that the expression (205) is a regular function in the three-dimensional space exterior of the boundary surface D.

The formula (205) - being valid in the exterior space of D - implies an extension of the validity of (3) and (6). The validity of the latter formulas is restricted to test points situated only in the exterior space of the Brillouin sphere, ($r = \bar{R}$), (see Fig. 1). The $q_n$ coefficients of the series (205) can be identified with the $W_n$ coefficients of (6),

$$q_n = W_n \quad . \tag{207}$$

If the relation (207) is fulfilled, the developments (6) and (205) are identical, (for $r \geqslant \bar{R}$), member by member.

Further, both of these series are convergent for $r \geqslant \bar{R}$. As to the harmonic downwards continuation of these functions from the Brillouin sphere down to the Earth's surface D, the theorem about the harmonic continuation has the following text: There is given a harmonic function V in the three-dimensional space G. In a subspace $G_1$ of G, the V function is identical to zero. It follows that the harmonic function V is necessarily equal to zero in whole the space G, too.

In the here discussed applications, the harmonic potential V is the difference of the two harmonic expressions (1) (6) and (205). Obviously, regarding (206), this difference function is harmonic in any case and for any test point. It is harmonic in the exterior of D. Further on, (207), this difference function is equal to zero in the exterior of the Brillouin sphere ($r \geqslant \bar{R}$). Thus, according to the theorem about the harmonic continuation, it is equal to zero also in whole the exterior of D.

It follows that the introduction of the $W_n$ coefficients into (205) leads necessarily to the uniform convergent spherical harmonics series

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development of the gravitational potential valid in the exterior space of D,

$$W = \sum_{n=1}^{\infty} W_n \left(\frac{1}{r}\right)^n u_n(\varphi, \lambda) \; . \qquad (207a)$$

The identification of the analytical expression (205) with the real potential in the exterior of the Brillouin sphere includes this identification for whole the exterior of the surface of the Earth.

## 6.9. The completeness of the system of the linear independent functions.

The investigation into whether the system of the linear independent functions $v_i(\varphi, \lambda)$, (11), (195), is complete - considering the space of the regular functions f - is governed by the following condition:

If all the integrals

$$\iint_F f \cdot v_i(\varphi, \lambda)\, dF = 0 \; , \qquad (208)$$

$$i = 1, 2, 3, \ldots \; , \qquad (209)$$

are equal to zero, in this case, the regular function f is necessarily equal to zero over the whole surface of the sphere, [6] [10] [11] [14] [15] [16]. Thus, from (208) (209) the subsequent equation has to follow,

$$f = 0 \; , \; - \text{ for the whole surface } F \; . \qquad (210)$$

In case of orthonormal base functions, $w_i(\varphi, \lambda)$, the completeness conditions have a shape similar as (208) (209) (210): The system of the orthonormal functions $w_i(\varphi, \lambda)$ is complete in the space of the regular functions f if

$$\iint_F f \cdot w_i \cdot dF = 0 \; , \qquad (211)$$

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$$i = 1, 2, 3, \ldots \quad , \tag{212}$$

leads to

$$f = 0 \quad , \quad \text{- for the whole surface } F. \tag{213}$$

These relations for a system of orthonormal functions, (211) (212) (213), are more common in use in the textbooks than those relations (208) (209) (210) for a system of linear independent functions.

A look on the relations (27) shows that the fulfillment of (208) (209) (210) is a consequence of the conditions (211) (212) (213), and vice versa. The two systems of condition equations, described by the relations (208) to (210) on the one hand and (211) to (213) on the other hand, are equivalent.

In order to investigate into whether the condition relations, (208) to (210), are observed by the functions $v_i$ ( $\varphi$ , $\lambda$ ), (11) (195), the potential of a surface distribution is introduced, now. It covers the surface of the Earth D, [12].

$$T = \iint_D \frac{1}{e} m \cdot dD \quad . \tag{214}$$

m represents the surface distribution, dD is the surface element, e is the straight distance between the surface element and the spatial test point for which the potential T on the left hand side of (214) is taken.

The relation (214) is valid for test points situated in the interior space being enclosed by D, as well as for test points in the complementary space i. e. the space exterior of the body of the Earth. The relation (214) is valid also for test points situated on the surface D. Thus, the expression (214) for the potential of a surface distribution is valid for test points situated anywhere in the three-dimensional space.

The expression for 1/e in the relation (214) is developed in a uniform convergent series in terms of spherical harmonics, (see Fig. 2), [9] [12],

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$$\frac{1}{e} = \sum_{n=0}^{\infty} \frac{r^n}{(r')^{n+1}} P_n (\cos \psi) , \quad r < r' \; ; \tag{215}$$

or,

$$\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(r')^n}{r^{n+1}} P_n (\cos \psi) , \quad r' < r . \tag{216}$$

$P_n$ are the Legendre functions.

The series development (215) is valid for test points P situated in the spherical volume which is enclosed by the interior Brillouin sphere having the radius $R_{B.i}$, $r < R_{B.i}$; i. e. the greatest geocentric sphere being enclosed completely by the Earth's surface D. The validity of (215) is given also for $r = R_{B.i}$, Fig. 2. r' is equal to the geocentric radius of the Earth, Fig. 1 and Fig. 2,

$$r' = r_D = t \; . \tag{217}$$

Hence, (215) and (217) are combined to

$$\frac{1}{e} = \sum_{n=0}^{\infty} \frac{r^n}{t^{n+1}} P_n (\cos \psi), \quad r \leqslant R_{B.i} \; . \tag{218}$$

(214) and (218) give

$$T = \iint_D \sum_{n=0}^{\infty} \frac{r^n}{t^{n+1}} P_n (\cos \psi) \cdot m \cdot d D; \quad r \leqslant R_{B.i} \; . \tag{219}$$

Or,

$$T = \sum_{n=0}^{\infty} r^n \iint_D \frac{1}{t^{n+1}} P_n (\cos \psi) \cdot m \cdot d D; \; r \leqslant R_{B.i} \; . \tag{220}$$

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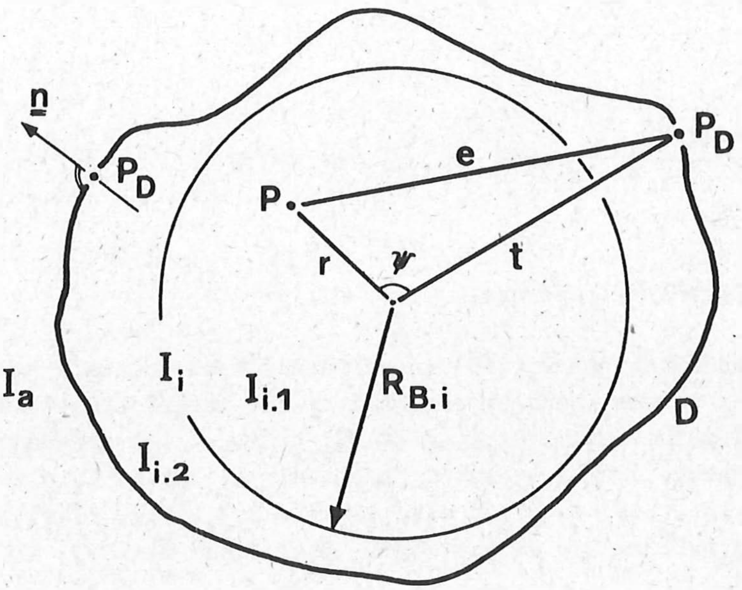


Fig. 2: The interior Brillouin sphere of the Earth with the radius $R_{B.i}$; the interior and the exterior space, $I_i$ and $I_a$, of the body of the Earth; the surface of the Earth, D; the surface point $P_D$; the surface normal vector $\underline{\underline{n}}$ directed into the exterior space $I_a$.

The addition theorem for the normalized spherical harmonics is, [8] [9],

$$P_n (\cos \psi) = \frac{1}{2n+1} \sum_{m=0}^{n} \left\{ \overline{R}_{n.m} (\varphi, \lambda) \overline{R}_{n.m} (\varphi', \lambda') + \overline{S}_{n.m} (\varphi, \lambda) \overline{S}_{n.m} (\varphi', \lambda') \right\} , \qquad (221)$$

with

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$$\bar{R}_{n.m}(\varphi, \lambda) = \bar{P}_{n.m}(\sin\varphi)\cos m\lambda \quad , \tag{222}$$

$$\bar{S}_{n.m}(\varphi, \lambda) = \bar{P}_{n.m}(\sin\varphi)\sin m\lambda \quad , \tag{223}$$

$$\iint_F \bar{R}^2_{n.m} \, dF = 4\pi \quad , \tag{224}$$

$$\iint_F \bar{S}^2_{n.m} \, dF = 4\pi \quad . \tag{225}$$

$\bar{P}_{n.m}$ are the normalized associated spherical harmonics.

The addition theorem (221) is introduced into (220). The primed latitude and longitude refer to the surface element d D which is moving during the execution of the integration process,

$$T = \sum_{n=0}^{\infty} r^n \frac{1}{2n+1} \sum_{m=0}^{n} \bar{R}_{n.m}(\varphi, \lambda) \iint_D \frac{1}{t^{n+1}} \bar{R}_{n.m}(\varphi', \lambda')\, m \cdot dD +$$

$$+ \sum_{n=0}^{\infty} r^n \frac{1}{2n+1} \sum_{m=0}^{n} \bar{S}_{n.m}(\varphi, \lambda) \iint_D \frac{1}{t^{n+1}} \bar{S}_{n.m}(\varphi', \lambda')\, m \cdot dD,$$

$$r \leq R_{B.i} \quad . \tag{226}$$

Comparing (10) and (11) with (226), and considering that the function $u_{n+1}$ is a substitute for $\bar{R}_{n.m}$ and $\bar{S}_{n.m}$, the following transition relations are valid,

$$v_{n+1} \longrightarrow \frac{1}{t^{n+1}} \bar{R}_{n.m}(\varphi', \lambda') \quad , \tag{227}$$

$$(1/t^{n+1})\, \bar{R}_{n.m}(\varphi,\lambda) \rightarrow v_{n+1}(\varphi,\lambda) = (1/t^{n+1})\, u_{n+1}(\varphi,\lambda) \quad , \tag{227a}$$

$$\bar{R}_{n.m}(\varphi,\lambda) \rightarrow u_{n+1}(\varphi,\lambda) \quad , \quad \bar{R}_{n.m}(\varphi',\lambda') \rightarrow u_{n+1}(\varphi',\lambda') \quad , \tag{227b}$$

$$(\; n = 0, 1, 2, \ldots \;) \quad , \tag{227c}$$

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and in a similar way for $\bar{S}_{n.m}(\varphi,\lambda)$ and $\bar{S}_{n.m}(\varphi',\lambda')$ , too.

Thus, for instance ( see (227) (227a) (227b) (227c) )

$$v_{n+1} \longrightarrow \frac{1}{t^{n+1}} \bar{S}_{n.m}(\varphi', \lambda') \quad . \tag{228}$$

In view of the intentions here followed up, a combination of (227) and (228) with (226) leads to the subsequent symbolic representation of T, (the zonal harmonics only are written down, as substitutes for the zonal, tesseral and sectorial harmonics/ transforming in the same way).

$$T = \sum_{n=0}^{\infty} r^n \frac{1}{2n+1} u_{n+1}(\varphi,\lambda) \iint_D v_{n+1}(\varphi',\lambda')\, m \cdot d\, D,$$

$$r \leqslant R_{B.i} \quad . \tag{229}$$

(229) is an abbreviating version of (226).

The element d D of the surface of the Earth D is now expressed by the surface element d F of the unit sphere F,

$$d\, D \cdot \cos \alpha = d\, F \cdot (R + h)^2 \quad . \tag{230}$$

The angle $\alpha$ is the slope of the terrain, R is the radius of the globe, and h is the topographical height above the globe. In case of a star-shaped Earth, the following relations are valid,

$$1 \geqslant \cos \alpha > 0 \quad , \tag{231}$$

$$R + h > 0 \quad . \tag{232}$$

Thus,

$$d\, D = (R + h)^2 \frac{1}{\cos \alpha} d\, F \, . \tag{233}$$

A function f is introduced by

$$f = m\, (R + h)^2 \frac{1}{\cos \alpha} \quad . \tag{234}$$

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f is an arbitrarily chosen regular function. (233) and (234) are combined with (229),

$$T = \sum_{n=0}^{\infty} r^n \frac{1}{2n+1} u_{n+1}(\varphi, \lambda) \iint_F v_{n+1}(\varphi', \lambda') \cdot f \cdot dF,$$

$$r \leqslant R_{B.i} \quad . \tag{235}$$

or,

$$T = \sum_{n=1}^{\infty} r^{n-1} \frac{1}{2n-1} u_n(\varphi, \lambda) \iint_F v_n(\varphi', \lambda') \cdot f \cdot dF,$$

$$r \leqslant R_{B.i} \quad . \tag{236}$$

Comparing (208) with (236), it is obvious that the conditions (208) have the consequence, Fig. 2,

$$T = 0, \quad \text{for} \quad r \leqslant R_{B.i} \quad ; \tag{237}$$

or,

$$T = 0, \quad \text{within} \quad I_{i.1} \quad . \tag{238}$$

Because of the relation (237) and because T is harmonic in the space enclosed by D, (214), the harmonic potential T is equal to zero not only in $I_{i.1}$ but also in whole the interior of the space enclosed by the surface D, beyond the subspace $I_{i.1}$. This fact is evidenced by the procedure of the harmonic upwards continuation of the harmonic potential T from the spatial area $I_{i.1}$ upwards into the area $I_{i.2}$,

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(see Fig. 2). $I_i$ is the space enclosed by the surface of the Earth D. $I_{i.1}$ is the spatial domain enclosed by the interior Brillouin sphere with the radius $R_{B.i}$. Hence,

$$I_i = I_{i.1} + I_{i.2} \quad , \tag{239}$$

$$I_{i.2} = I_i - I_{i.1} \quad . \tag{240}$$

The following well-known theorem about the harmonic continuation is proved , for continuous gravitational potentials :

If T is harmonic in a domain $I_i$, and if T vanishes at all the points of a domain $I_{i.1}$ in $I_i$, then T vanishes at all the points of $I_i$, [12].

The spatial representation of T, (214), fulfills the Laplace differential equation in whole the infinite three-dimensional space,

$$\Delta T = 0 \quad . \tag{241}$$

Therefore, (242) is right,

$$T = 0, \text{ within } I_i \; . \tag{242}$$

On the strength of the equations (242) and (238), (239), the above cited theorem about the harmonic continuation leads to the relation, (238),

$$T = 0 \; , \text{ within } I_{i.2} \; , \tag{243}$$

and, further,

$$T = 0 \; , \text{ within } I_i \; . \tag{244}$$

This relation (244) has a consequence which is important for the aims here followed up.

In the theory of the potential of a surface distribution, it is shown that this potential is a continuous function within whole the

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three-dimensional space, even, in oase, the surface with the gravitating distribution m, (214), is crossed.

The testpoint of the potential function T, (214), may approach the point $P_D$ on the surface D from the interior domain and, in a second case, also from the exterior domain (that is to say from the side of $I_i$ resp. $I_a$). By these appoaches of the testpoint to the surface D, the values $(T_i)_D$ and $(T_a)_D$ are reached for the potential T. Because of the continuity of the potential T, (even if the surface D is crossed), the following equation is right, [12],

$$(T_i)_D = (T_a)_D \quad . \tag{245}$$

(244) leads to

$$(T_i)_D = 0 \quad . \tag{246}$$

(245) and (246) give

$$(T_a)_D = 0 \quad . \tag{247}$$

(247) is an expression for the boundary values of the exterior Dirichlet boundary value problem for the potential T. The solution of this problem is unique, [12]. Thus, by (247) and (241),

$$T \equiv 0 \text{ , within } I_a \quad . \tag{248}$$

$I_a$ is the domain exterior of the surface D.

For the derivatives of T in the direction of the exterior normal vector $\underline{n}$, Fig. 2, the values

$$\left(\frac{\partial T}{\partial n}\right)_{i.D} \quad \text{resp.} \quad \left(\frac{\partial T}{\partial n}\right)_{a..} \tag{249}$$

follow, approaching the surface point D from the side of the interior domain $I_i$, resp. from the side of the exterior domain $I_a$. The jump relation, [12], for

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$$\frac{\quad}{\partial} \qquad (250)$$

at the surface D gives, (214),

$$\left(\frac{\partial T}{\partial n}\right)_{i.D} - \left(\frac{\partial T}{\partial n}\right)_{a.D} = 4 \pi m . \qquad (251)$$

Since the T values are equal to zero everywhere in $I_i$ and $I_a$, (244) and (248), the normal derivatives of T are equal to zero, too.
Thus, the two terms on the left hand side of (251) are both equal to zero also.

Hence, with (251),

$$m = 0 , \text{ on } D . \qquad (252)$$

The relations (231) (232) (234) (252) reveal

$$f = 0 , \text{ on } D . \qquad (253)$$

Summarizing the above deliberations from (214) to (253), the conditions (208) (209) lead not only to the relations (237) (238), but also to the consequences shown by (253) (210).

Therewith, the completeness of the systems of the functions $v_i$ and $w_i$ is proved, (10) (11) (27) (28).

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## 6.10. The convergence property derived by the completeness of the system.

A regular function f has the following representation in the system of the functions $w_n$ ( $\varphi$ , $\lambda$ ), (78) (170).

$$f(\varphi, \lambda) = \sum_{n=1}^{\infty} g_n w_n(\varphi, \lambda), \tag{254}$$

and in the system of the functions $v_n$ ( $\varphi$ , $\lambda$ ), (191) (195), (2o7a),

$$f(\varphi, \lambda) = \sum_{n=1}^{\infty} W_n v_n(\varphi, \lambda) . \tag{255}$$

As it is shown in the textbooks, the convergence of (254) and (255) is demonstrated on the foundation of the completeness of the systems of the $w_n$ ( $\varphi$ , $\lambda$ ) resp. $v_n$ ( $\varphi$ , $\lambda$ ), a characteristic proved in the preceeding paragraph 6. 9.. The completeness is equivalent with the fulfillment of the Parseval relation, (79), [6] [10] [14],

$$\| f \|^2 = \sum_{n=1}^{\infty} g_n^2 . \tag{256}$$

This relation (256) leads to the uniform convergence of the series (254), it was proved by the detailed developments connected with the relations (79) to (136b), - after an exchange of $e_n$ ( $\varphi$, $\lambda$ ) and $w_n$ ( $\varphi$, $\lambda$ ). (119) and (120), (170), give

$$f(\varphi, \lambda) = \sum_{n=1}^{Q} g_n w_n(\varphi, \lambda) + \varepsilon_{11} , \tag{257}$$

with

$$\varepsilon_{11} \longrightarrow 0 , \text{ if } Q \longrightarrow \infty . \tag{258}$$

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According to the Cauchy convergence criterion, the relations from (125) to (136b) allow the following modification of the statement about the uniform convergence of (257), using the completeness of the $w_i$, (253)(256).

Theorem 13:

The series (254) is convergent, because, after the choice of a positive number,

$$\varepsilon_{15.0} > 0 , \qquad (259)$$

an integer $Q_0 = Q_0(\varepsilon_{15.0})$ can be found such that for the integer Q,

$$Q > Q_0(\varepsilon_{15.0}) , \qquad (260)$$

and for all the integers $Q^*$,

$$Q^* \geq 1 , \qquad (261)$$

the subsequent inequation follows,

$$\Big| g_{Q+1} w_{Q+1}(\varphi, \lambda) + g_{Q+2} w_{Q+2}(\varphi, \lambda) + \dots + g_{Q+Q^*} w_{Q+Q^*}(\varphi, \lambda) \Big| < \varepsilon_{15.0} . \qquad (262)$$

The relation (262) meets the fact that the property of the linear independence of the functions $v_n(\varphi, \lambda)$ is explained only for a limited number of functions of the $v_n$ system. The functions $w_n(\varphi, \lambda)$ are derived from the functions $v_n(\varphi, \lambda)$, (262), (23) (24) (25) (27).

As to the uniform convergence of (255), the orthonormalized functions of (257),

$$w_n(\varphi, \lambda) \quad , \quad (n = 1, 2, \dots, Q) , \qquad (263)$$

can be expressed by the linear independent functions

$$v_n(\varphi, \lambda) \quad , \quad (n = 1, 2, \dots, Q) . \qquad (264)$$

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This exchange of the base functions happens by means of the system (27). Along these lines, (27) (257), the equations (257) and (258) turn into the following shape,

$$f(\varphi, \lambda) = \sum_{n=1}^{Q} W_n v_n(\varphi, \lambda) + \varepsilon_{11} \; ; \qquad (265)$$

here, the inequations

$$\left|\varepsilon_{11}\right| < \left|\varepsilon_{11.0}\right| , \quad \left|\varepsilon_{11.0}\right| > 0 , \qquad (266)$$

determine the sufficient great integer $Q$ , ( see (182) (183) ).

The two relations (265) (266) given above prove the uniform convergence of the series development (196) which is here to be investigated. This proof given in the above paragraphs 6.9. and 6.10. is free of any consideration about the amount of a determinant of infinite dimension.

## 6.11. The theorem of Picone.

In the external space of a body, a system of harmonic functions

$$U_n(x, y, z) = U_n(\underline{x}) , \qquad (267)$$

$$n = 1, 2, 3, \ldots , \qquad (268)$$

may be defined. These functions have the character of base functions. x, y, z are rectangular Cartesian co-ordinates. The individual functions $U_n(x, y, z)$ fulfill the Laplace differential equation,

$$\Delta U_n(x, y, z) = \Delta U_n(\underline{x}) = 0 . \qquad (269)$$

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Let the star-shaped surface D of the body be described by a regular function,

$$\underline{\underline{x}}_D = \underline{\underline{x}}_D (p, q) \quad , \tag{270}$$

it depends uniquely on the radius vector. p and q denote Gaussian parameters on the surface D. For testpoints situated especially on the surface D the harmonic functions $U_n (\underline{\underline{x}})$, (267), change to the two-parameter functions $\xi_n$,

$$U_n (\underline{\underline{x}}_D) = \xi_n = \xi_n (p, q) \quad . \tag{271}$$

Now, an important property is introduced about the functions $\xi_n$: They have to construct a complete system of base functions

$$\xi_n (p, q) \quad , \tag{272}$$

$$n = 1, 2, 3, \ldots, \tag{273}$$

in the space of the regular functions.

Furthermore, let the following harmonic gravitational potential U be given in the external space of the body,

$$U = U (\underline{\underline{x}}) = U (x, y, z) \quad ; \tag{274}$$

with

$$\Delta U = 0 \tag{275}$$

in the space exterior of the surface D. The boundary values of the potential U on the surface D are described by the regular function

$$\eta = \eta (p, q) \tag{276}$$

giving

$$U (\underline{\underline{x}}_D) = \eta (p, q) \quad . \tag{277}$$

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Then, according to Picone's theorem, [2], the uniform convergence of the following series expansion is secured in the external space,

$$U = U(\underline{\underline{x}}) = \sum_{n=1}^{\infty} u_n U_n(\underline{\underline{x}}) . \qquad (278)$$

$u_n$ are the constant coefficients. Moreover, if the testpoint approaches the boundary surface D from the side of the exterior space, this series expansion tends to the function

$$U(\underline{\underline{x}}_D) = \eta(p, q) = \sum_{n=1}^{\infty} u_n \xi_n(p, q) . \qquad (279)$$

Changing over to our applications, the following substitutions have to take place, (5) (6) (10) (11) (81) (196) (205) (207),

$$U_n(\underline{\underline{x}}) = \left(\frac{1}{r}\right)^n u_n(\varphi, \lambda) , \qquad (280)$$

$$p = \varphi , \qquad (281)$$

$$q = \lambda , \qquad (282)$$

$$U_n(\underline{\underline{x}}_D) = \xi_n(p, q) = v_n(\varphi, \lambda) , \qquad (283)$$

$$U(\underline{\underline{x}}) = W(r, \varphi, \lambda) , \qquad (284)$$

$$U(\underline{\underline{x}}_D) = \eta(\varphi, \lambda) = W_D(\varphi, \lambda) , \qquad (285)$$

$$u_n = W_n . \qquad (286)$$

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The proof of the completeness of the system

$$v_n(\varphi, \lambda) = \xi_n(\varphi, \lambda) \quad (287)$$

was demonstrated by the relations (208) to (253) of the paragraph 6.9..

Therefore, the proof of the convergence of the series development (278) - brought about by means of the theorem of Picone and the completeness of the $\xi_n$ function system - is also a proof of the convergence of the spatial spherical harmonics development (6). The convergence of the surface series development (279) corroborates the convergence of the series developments (191) (265) valid for the testpoints at the surface of the Earth D and for testpoints in the exterior of the body of the Earth, (2o5) (2o7a) .

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## 7. The uniqueness of the Molodenskij boundary value problem.

Finally, a by-product of the above derivations should be mentioned. By means of (205), the proof of the uniqueness of the solution of the Molodenskij boundary value problem is uncomplicated.

This problem has the following definition: Along the real surface of the Earth shaped by the topographical heights the free-air anomalies are given as boundary values. They depend on the perturbation potential T by the fundamental differential equation of the physical geodesy,

$$\Delta g_T = - \frac{\partial T}{\partial r} - \frac{2}{r} T \quad . \qquad (288)$$

The solution of the Molodenskij boundary value problem, - in its original shape -, consists in the inversion of (288): The perturbation potential T along the real surface of the Earth, - accounting for the topography -, is to be determined in terms of the free-air anomalies $\Delta g_T$ of the gravity. The uniqueness of this solution is the question here to be investigated.

Since (205) is valid, the perturbation potential T has the following formula in the three-dimensional space described by the co-ordinates $r, \varphi, \lambda$

$$T = \sum_{l=2}^{\infty} t_l \left(\frac{1}{r}\right)^{l+1} u_l^*(\varphi,\lambda) \quad . \qquad (289)$$

(289) is free of the degrees $l = 0$, $l = 1$. The above expression $u_l^*(\varphi,\lambda)$ specifies the spherical harmonics of degree l. The constant coefficients $t_l$ are the Stokes constants. The meaning of the suffix l of (289) is not the same as the meaning of the suffix n of (205), it is obvious. The suffix n of (205) begins with the integer 1, since it is the custom to begin the numeration of the rows and columns of a matrix with the number 1, (47).

The convergent series development (289) is valid in the whole exterior space of the Earth's surface D .

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The formulas (288) and (289) give (for testpoints along the surface of the Earth, D )

$$\Delta g_T = \sum_{l=2}^{\infty} (l - 1)\, t_l \left(\frac{1}{r}\right)^{l+2} u_l^*(\varphi, \lambda) \quad , r = t = r_D . \tag{290}$$

The uniqueness of the solution demands that the constraint

$$\Delta g_T = 0 \quad , \quad r = t = r_D \quad , \tag{291}$$

for the free-air gravity anomalies along the surface of the Earth has necessarily the consequence

$$T = 0 \quad , \tag{292}$$

for testpoints on the surface of the Earth and in the exterior space.

In the investigation into whether this Molodenskij problem has a unique solution, the relation (290) and the constraint (291) lead to

$$0 = \sum_{l=2}^{\infty} (l - 1)\, t_l \left(\frac{1}{r}\right)^{l+2} u_l^*(\varphi,\lambda) \quad , r = t = r_D . \tag{293}$$

The multiplication with the non-vanishing value t leads to

$$0 = \sum_{l=2}^{\infty} (l - 1)\, t_l \left(\frac{1}{t}\right)^{l+1} u_l^*(\varphi,\lambda) \quad . \tag{294}$$

With

$$v_l^*(\varphi, \lambda) = \left[ \left(\frac{1}{r}\right)^{l+1} u_l^*(\varphi,\lambda) \right]_D \quad , \tag{295}$$

(294) turns to

$$0 = \sum_{l=2}^{\infty} (l - 1)\, t_l\, v_l^*(\varphi, \lambda) \quad . \tag{296}$$

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The relations from (293) to (296) are valid along the surface of the Earth, D.

The equation (294) can be considered as the representation of the Dirichlet type boundary values; in this case, the boundary values have the peculiarity to be equal to zero along the surface of the Earth. If a potential is equal to zero along a closed boundary surface, this potential follows to be equal to zero also in whole the exterior space of this boundary surface. This fact is proved in the potential theory, and it is generally accepted, [12].

Therefore, the boundary values (294) lead to the fact that the spatial harmonic potential function I which fulfils the Laplace equation

$$I = \sum_{l=2}^{\infty} (l - 1)\, t_l \left(\frac{1}{r}\right)^{l+1} u_l^{*}(\varphi, \lambda) \qquad (297)$$

is equal to zero on the surface of the Earth, D, and in the exterior space of it. $\Delta I = 0$. Along the exterior Brillouin sphere with the radius $\bar{R}$, the equation (298) is valid,

$$I = I(r = \bar{R}, \varphi, \lambda) = 0 = \sum_{l=2}^{\infty} (l-1)\, t_l \left(\frac{1}{\bar{R}}\right)^{l+1} u_l^{*}(\varphi, \lambda). \qquad (298)$$

The orthogonality relations (59) are also valid for the spherical harmonics $u_l^{*}(\varphi, \lambda)$, they result by (298)

$$t_l = 0, \ (l = 2, 3, \dots) . \qquad (299)$$

All the Stokes constants $t_l$ are equal to zero according to (299), if the condition for the uniqueness of the solution of the Molodenskij boundary value problem is fulfilled, (291) (293). The potential values of T follow, necessarily, to be also equal to zero, (289). Thus, (291) leads to (292).

Consequently, the solution of the considered type of the boundary value problem of Molodenskij is unique.

Obviously, this above considered boundary value problem is identical with even that version of the boundary value problem which maps the telluroid points into their image points on the Earth's surface by a shift along the geocentric radius vector.

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8. A short proof of the convergence of the spherical harmonics series development of the gravitational potential in the exterior domain of the Earth's body.

The above chapter E contains rather long and extensive investigations about the convergence of the spherical harmonics series development for the gravitational potential of the Earth. Finally, a rather short proof of this matter is to be added, [4] [5].

In [1], page 84-85, and in [4], page 177, a yet more short and instructive proof of the convergence can be found.

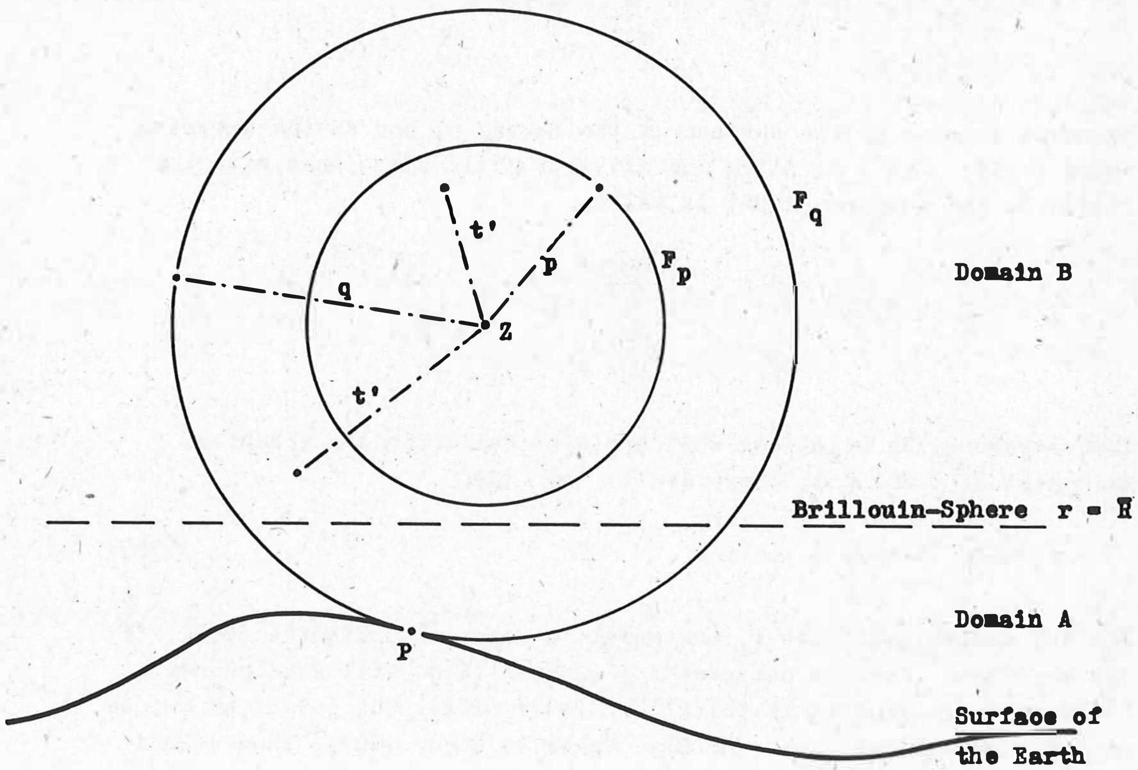


Fig. 3: The concentric spheres $F_p$ and $F_q$ with the common center Z. $F_p$ is in the exterior domain of the Brillouin sphere. $F_q$ touches the surface of the Earth in the point P.

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The real gravitational potential of the Earth is W, (1) (2). In the exterior domain, B, of the Brillouin sphere the convergent development (300) is valid,

$$W = \sum_{n=0}^{\infty} w_n \left(\frac{R}{r}\right)^{n+1} \overset{*}{u}_n(\varphi, \lambda), \text{ (domain B).} \tag{300}$$

$w_n$ are the Stokes constants and $\overset{*}{u}_n(\varphi, \lambda)$ are the spherical harmonics. In the exterior of the surface of the Earth, (domain A + B), the expression (301) is valid, Fig. 3,

$$W = U_C + V_C \text{ , (domain A + B),} \tag{301}$$

$$U_C = \sum_{n=0}^{C} w_n \left(\frac{R}{r}\right)^{n+1} \overset{*}{u}_n(\varphi, \lambda), \text{ (domain A + B) ,} \tag{302}$$

$U_C$ is a sum.

$V_C$ in the domain A, that is the function which is to be determined. (300) and (302) show that

$$V_C \longrightarrow 0, \text{ if } C \longrightarrow \infty \text{ , (domain B) .} \tag{303}$$

The Laplace differential equation is valid for W and $U_C$ in whole the domain A + B, it is well-known,

$$\Delta W = 0, \text{ (domain A + B),} \tag{304}$$

$$\Delta U_C = 0, \text{ (domain A + B) .} \tag{305}$$

(301) (304) and (305) lead to

$$\Delta V_C = 0, \text{ (domain A + B) .} \tag{306}$$

In the exterior domain B, a point Z is chosen. It is the center of two concentric circles, $F_p$ and $F_q$, (see Fig. 3). The sphere $F_p$ with the radius t' = p is situated completely in the domain B. The sphere $F_q$

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with the radius t' = q is situated in the domains A and B, it touches the surface of the Earth in the point P.

Because of (306), the potential $V_C$ has the following convergent series development along the sphere $F_p$, [12],

$$V_{C.p} = \sum_{i=0}^{\infty} v_{C.i} \left(\frac{p}{q}\right)^i \overset{*}{u_i} (\bar{\varphi}, \bar{\lambda}) \quad ; \tag{307}$$

and for the surface of the sphere $F_q$, the uniform convergent series

$$V_{C.q} = \sum_{i=0}^{\infty} v_{C.i} \overset{*}{u_i} (\bar{\varphi}, \bar{\lambda}) \, . \tag{308}$$

$\bar{\varphi}$ and $\bar{\lambda}$ refer to the center Z. The convergence property of (307) reveals, [12],

$$V_{C.p} = \sum_{i=0}^{N} v_{C.i} \left(\frac{p}{q}\right)^i \overset{*}{u_i} (\bar{\varphi}, \bar{\lambda}) + \varepsilon_{16} (N) \, , \tag{309}$$

with

$$\varepsilon_{16} (N) \longrightarrow 0, \text{ if } N \longrightarrow \infty \, . \tag{310}$$

Further,

$$V_{C.q} = \sum_{i=0}^{N} v_{C.i} \overset{*}{u_i} (\bar{\varphi}, \bar{\lambda}) + \varepsilon_{17} (N) \, , \tag{311}$$

with

$$\varepsilon_{17} (N) \longrightarrow 0, \text{ if } N \longrightarrow \infty \, . \tag{312}$$

The orthogonality relation of the $\overset{*}{u_i}$ functions(having the shape of (12)) and the relation (307) give

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$$v_{C.i} = \left(\frac{q}{p}\right)^i \int_{\bar{\varphi} = -\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{\bar{\lambda} = 0}^{2\pi} V_{C.p}\, u_i^{*}(\bar{\varphi}, \bar{\lambda}) \cos\bar{\varphi}\, d\bar{\varphi}\, d\bar{\lambda}, \qquad (313)$$

for

$$i = 0, 1, 2, \ldots, N. \qquad (314)$$

The equations (303),(313) and (314) have the consequence

$$v_{C.i} \longrightarrow 0, \text{ if } C \longrightarrow \infty, \qquad (315)$$

for

$$i = 0, 1, 2, \ldots, N. \qquad (316)$$

Hence, considering $V_{C.q}$, the first term on the right hand side of (311) is a sum of N + 1 terms. Since the normalized spherical harmonics $u_i^{*}(\bar{\varphi}, \bar{\lambda})$ have limited amounts, the combination of (315) and (316), and (311) results (317)

$$\sum_{i=0}^{N} v_{C.i}\, u_i^{*}(\bar{\varphi}, \bar{\lambda}) \longrightarrow 0, \text{ if } C \longrightarrow \infty \quad . \qquad (317)$$

Thus, for a sufficient great integer C, by means of (311) (312) (317),

$$\left| V_{C.q} \right| < \left| \varepsilon_{18}(C) \right| , \qquad (318)$$

with

$$\left| \varepsilon_{18}(C) \right| > 0 . \qquad (319)$$

The relations (318) and (319) are equivalent to the following statement:

$$V_{C.q} \longrightarrow 0, \text{ if } C \longrightarrow \infty. \qquad (320)$$

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Thus, for the surface point P, the function $V_C$ tends to zero if C tends to infinity. This fact proves the convergence of (300) in the domain A. Hence,

$$W = \sum_{n=0}^{\infty} w_n \left(\frac{R}{r}\right)^{n+1} u_n^{*}(\varphi, \lambda), \text{ (domain A + B)}. \qquad (321)$$

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