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Geodetic boundary value problems I

von

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Contents

A. Smoothed downwards continuation and the Bjerhammar sphere

Contents Page

Summary

The perturbation potential of the gravity field of the Earth is superposed with the potential of the gravitation force of the standard density mountain masses. This superposition leads to a smoothed potential. The associated model free-air anomalies are also smoothed; they are subsided from the surface of the Earth down to the Bjerhammar sphere of the test point, along the lines of the harmonic downward continuation. The terms that arise by the downward continuation are negligible in most cases since the considered model free-air anomalies are smoothed. Further, this superposition leads to the fact that the plane topographical reduction of the gravity must be added to the usual free-air anomalies of the real perturbation potential. The other supplements to the Stokes theory are not of importance since they are within the noise of the dates of the gravity field in most cases. There is an insight of clear transparency into these supplementary terms.

Zusammenfassung

Das Störpotential des Schwerefeldes der Erde wird superponiert mit dem Gravitationspotential der Gebirgsmassen, denen hier die Standarddichte zugeordnet wird. Das so erhaltene Potential ist geglättet, die zugehörigen Freiluftanomalien sind ebenfalls geglättet. Diese Freiluftanomalien werden mittels der harmonischen Fortsetzung von der Erdoberfläche zur Bjerhammar-Kugel des Aufpunktes nach unten herabgesenkt. Die Glättung läßt die durch die harmonische Fortsetzung entstehenden Glieder sehr klein werden. Durch die Superposition tritt die ebene Geländereduktion der Schwere als additives Glied zu den üblichen Freiluftanomalien des realen Störpotentials hinzu. Weitere Ergänzungen zur Stokes-schen Theorie brauchen nicht berücksichtigt zu werden, weil sie

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meistens innerhalb der Genauigkeit der Schwerenetze liegen. Diese ergänzenden Glieder haben einen mathematischen Ausdruck, der einen klaren Einblick in ihre Größe gestattet.

Резюме

Потенциал возмущения накладывается на потенциал видимых горных массивов. Этот потенциал сглажен; образующиеся соответствующие аномалии силы тяжести опускаются путем гармонического продолжения вниз с поверхности Земли на сферу Бьерхаммара в заданной точке. Сглаживание допускает очень сильное уменьшение получающихся в результате гармонического продолжения членов и вводит топографическую поправку силы тяжести в качестве аддитивного члена при аномалиях потенциала возмущения.

1. Introduction

At first, some considerations about the train of ideas connected with the Bjerhammar sphere seem to be advisable. The sphere of the Earth $\varkappa$ which is situated in the level of the oceans can be introduced as a Bjerhammar sphere with the radius R, e.g.. Further, any geocentric sphere $\varkappa'$ which does contain any test point P at the surface σ of the Earth can be introduced as a Bjerhammar sphere also. This sphere has the radius $R+\eta_p$, if η_p is the height of the surface point P above the sphere $\varkappa$ (cf. [6], [7]).

The sentence of Keldysch-Lavrentiev or Runge-Krarup contains the existence theorem for the Bjerhammar sphere : Any function A , harmonic outside the Earth's surface and continuous outside and on it , may be uniformly approximated by harmonic functions B regular outside an arbitrarily given sphere inside the Earth , in the sense that for any given $\varepsilon > 0$, the relation $|A - B| < \varepsilon$ holds everywhere outside and on the Earth's surface. But, this is an existence theorem only. It does not contain any detailed information about the structure of the functions B and the constant parameters of them. Beyond it, over and above this existence theorem, a complete and satisfactory expression for the function B is represented by the series development in spherical harmonics for the concerned potential; see also chapter D.

On the principles connected with the Bjerhammar spheres $\varkappa$ and $\varkappa'$, a certain harmonic potential field V is introduced in the exterior space of the sphere $\varkappa$. In this context, this space exterior of $\varkappa$ is presupposed to be free of masses. V has the following convergent series development in spherical harmonics, valid in the exterior space of $\varkappa$.

$$V = \sum_{n=2}^{\infty} V_n \left(\frac{R}{r}\right)^{n+1} \Theta_n(\varphi,\lambda), \qquad r \geq R. \tag{1}$$

V fulfills the Laplace differential equation,

$$\Delta V = \frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V}{\delta z^2} = 0 , \qquad r \geq R. \tag{2}$$

V_n are the Stokes constants. r, φ, λ are the spatial polarcoordinates, the origin of this system is the gravity center of the Earth. $\Theta_n(\varphi,\lambda)$ symbolizes all the spherical harmonics of the degree n and of the order m, $0 \leq m \leq n$. The representation (1) of the potential V is valid in the space situated between the surface of the Earth σ and the

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sphere $\varkappa$, i.e. Φ_i. In the space Φ_i, the potential V is not related to the real geopotential W. V has only a theoretical significance in the space Φ_i. However, the potential V and the representation (1) is valid also in the exterior space Φ_a of the surface σ of the Earth. In the space Φ_a, the potential V is a sufficient approximation for the geopotential W. This fact is fundamental for the definition of V.

The geopotential field W has in Φ_a a convergent series development in spherical harmonics, similar as V,

$$W = \sum_{n=2}^{\infty} W_n \left(\frac{R}{r}\right)^{n+1} \Theta_n(\varphi,\lambda), \qquad \text{in } \Phi_a . \tag{3}$$

$$\Delta W = 0 , \qquad \text{in } \Phi_a . \tag{4}$$

Several independent mathematical proofs about the convergence of (3) in Φ_a are published in the literature (cf. [1], [2], [9]) , see also chapter D. The objections against the convergence of (3) turned out to be not convincing and not valid (cf. [2], [15]). The convergence is secured also in the whole exterior space of an oblate rotation ellipsoid. For, as is well-known, the gravitation potential has a convergent series development in terms of Lamé functions which is valid in the exterior space of a rotation ellipsoid (cf. [11], [12]). The associated 3 independent variable parameters can be expressed into the geocentric radius r, the geocentric latitude φ and the longitude λ , i.e. the geocentric polar-coordinates, by means of absolute convergent series developments (cf. [11], [12], [13]). In this context, the sentence has to be taken into account that the product of two absolute convergent series developments is again an absolute convergent series (cf. [13]). Thus, a convergent series development in terms of r, φ, λ is necessarily obtained for the exterior potential of the ellipsoid. This series must necessarily be a development in spherical harmonics as the expression (3), because otherwise the Laplace differential equation (4) cannot be fulfilled (cf. [11]). Indeed, if a function of r, φ, λ does satisfy the Laplace differential equation and if this function does go to zero for $r \to \infty$, in this case, this function has necessarily the structure of the right hand side of (3). Thus, the considerations of [15] are not corroborated.

The convergent series development (3) can be replaced by a sum,

$$W = \sum_{n=2}^{\alpha} W_n \left(\frac{R}{r}\right)^{n+1} \Theta_n(\varphi,\lambda) + \varepsilon , \qquad \text{in } \Phi_a . \tag{5}$$

α is a fixed positive integer.
It can be taken for granted that $|\varepsilon|$ is sufficient small and smaller than a certain upper bound $|\Gamma|$,

$$|\varepsilon| < |\Gamma| , \tag{6}$$

if $\alpha = \alpha(\varepsilon)$ becomes sufficient great. Therefore, the term ε can be neglected in our applications, (5), without loss of precision. Thus,

$$W = \sum_{n=2}^{\alpha} W_n \left(\frac{R}{r}\right)^{n+1} \Theta_n(\varphi,\lambda), \qquad \text{in } \Phi_a . \tag{7}$$

DOI: https://doi.org/10.2312/zipe.1986.084

2. The analytical representation of the perturbation potential

After these considerations, the perturbation potential T can be represented in the exterior space Φ_a of the surface σ of the Earth by an expression of the following shape,

$$\mathfrak{R} = \sum_{n=2}^{\infty^*} T_n \left(\frac{R}{r}\right)^{n+1} \Theta_n(\varphi,\lambda), \qquad \text{in } \Phi_a ; \tag{8}$$

$$T = \mathfrak{R}, \qquad \text{in } \Phi_a . \tag{9}$$

The developments (8) and (9) describe the perturbation potential T in the form of a sum which is valid in the exterior space Φ_a. The error of the expression (8) is arbitrary small. $\mathfrak{R}$ is a harmonic function in Φ_a,

$$\Delta \mathfrak{R} = 0, \qquad \text{in } \Phi_a . \tag{10}$$

The expression (8) of $\mathfrak{R}$ can be computed also for test points which are situated in the space Φ_i between the surface σ and the sphere $\varkappa$,

$$\mathfrak{R} = \sum_{n=2}^{\infty^*} T_n \left(\frac{R}{r}\right)^{n+1} \Theta_n(\varphi,\lambda), \qquad \text{in } \Phi_i ; \tag{11}$$

$$\Delta \mathfrak{R} = 0, \qquad \text{in } \Phi_i . \tag{12}$$

The expression (11) can not be identified with the perturbation potential T in Φ_i. It has not a direct relation to the real geopotential in Φ_i. $\mathfrak{R}$ is in Φ_i a harmonic, computable and continuous function which is of theoretical importance only.

The deductions from (8) to (12) can be summarized by the following relations,

$$\mathfrak{R} = \sum_{n=2}^{\infty^*} T_n \left(\frac{R}{r}\right)^{n+1} \Theta_n(\varphi,\lambda), \qquad r \geq R, \tag{13}$$

$$\Delta \mathfrak{R} = 0, \qquad r > R, \tag{14}$$

$$T = \mathfrak{R}, \qquad \text{in } \Phi_a . \tag{15}$$

3. The boundary value problem for the Bjerhammar sphere

Thus, in view of this $\mathfrak{R}$ field in the exterior of the $\varkappa$ sphere, the Bjerhammar sphere $\varkappa'$ with the radius $R + \eta_p$ can be introduced as the boundary surface for the Stokes boundary value problem with the surface point P as the test point,

$$(\mathfrak{R})_p = (T)_p = \frac{R+\eta_p}{4\pi} \iint_{\omega} (\Delta g_{\mathfrak{R}})_{\varkappa'} \, S(\psi) \, d\omega . \tag{16}$$

ω is the unit sphere, $S(\psi)$ the Stokes function, ψ the spherical distance and $\Delta g_{\mathfrak{R}}$ is the free-air anomaly,

$$(\Delta g_{\mathfrak{R}})_{\varkappa'} = - \left(\frac{\partial \mathfrak{R}}{\partial r} + \frac{2}{R+\eta_p} \mathfrak{R}\right)_{\varkappa'} . \tag{17}$$

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4. The superposition with the potential of the mountain masses

The functions Ω and Δg_{Ω} are not smoothed. Therefore, it is of fundamental importance now to replace these functions by smoothed expressions. For this purpose, the gravitational potential B of the mountain masses above the sea level is introduced (cf. [3], [4], Fig. 1). The standard density of $\delta = 2.65$ g cm^{-3} is attributed to these masses. The potential B is a harmonic function in the exterior space Φ_a,

$$\Delta B = 0 , \qquad \text{in } \Phi_a . \tag{18}$$

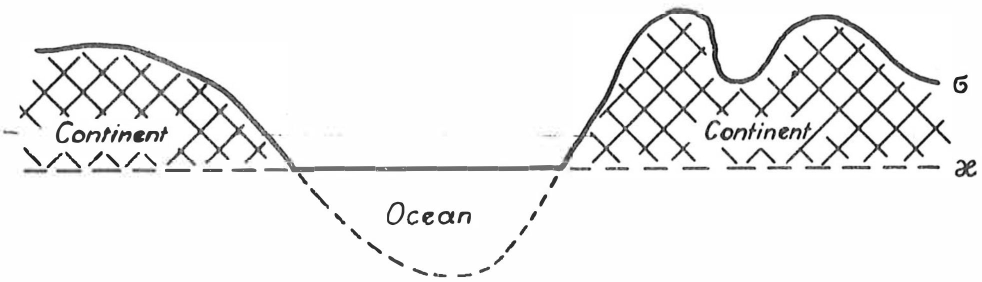

Figure 1: The visible mountain masses. The crosshatched area above the sea level is the area of the visible mountain masses.

The difference of the two potentials T and B is the potential M,

$$M = T - B. \tag{19}$$

The relations (13) - (15) lead to the following equations,

$$\Psi = \sum_{n=2}^{\infty^{**}} M_n \left(\frac{R}{r}\right)^{n+1} \Theta_n(\varphi,\lambda) , \qquad r \geq R , \tag{20}$$

$$\Delta \Psi = 0 , \qquad r > R , \tag{21}$$

$$M = \Psi , \qquad \text{in } \Phi_a . \tag{22}$$

The free-air anomalies in the harmonic potential field Ψ have the following relation for the points on the Bjerhammar sphere $\varkappa'$ which has the radius $R + \eta_p$, (cf. [6],[7]),

$$(\Delta g_{\Psi})_{\varkappa'} = - \left(\frac{\partial \Psi}{\partial r} + \frac{2}{R+\eta_p} \Psi\right)_{\varkappa'} , \qquad \text{on } \varkappa' . \tag{23}$$

The potential field Ψ is introduced in the exterior space of the sphere $\varkappa$, (cf. (20)). Further, whole the surface of the Earth σ is considered to be covered with the free-air anomalies of this potential field Ψ, i.e. $(\Delta g_{\Psi})_{\sigma}$.
Q is a point moving over this surface σ , it has the variating height η_Q above the sphere $\varkappa$. Thus, the free-air anomalies on σ depend from Ψ by

DOI: https://doi.org/10.2312/zipe.1986.084

$$(\Delta g_{\Psi})_{\sigma} = -(\frac{\partial \Psi}{\partial r} + \frac{2}{R+\eta_Q} \Psi)_{\sigma} \quad , \quad \text{on } \sigma . \tag{24}$$

The boundary value problem of Stokes is applied to $\varkappa'$ as the boundary sphere. The solution has the following rigorous equation, (16),

$$(\Psi)_P = \frac{R+\eta_P}{4\pi} \iint_{\omega} (\Delta g_{\Psi})_{\varkappa'} \, S(\psi) \, d\omega . \tag{25}$$

Now, the correpondence between the $(\Delta g_{\Psi})_{\sigma}$ -values, (cf. (24)), and the $(\Delta g_{\Psi})_{\varkappa'}$ -values, (cf. (23)), is to be considered. In this context, the free-air anomalies on σ are mapped on $\varkappa'$ by a shift along the geocentric radius r. The free-air anomalies change their value by the amount of s_M when this shift through the potential field Ψ is carried out. The corresponding anomalies have the relation

$$(\Delta g_{\Psi})_{\varkappa'} = (\Delta g_{\Psi})_{\sigma} + s_M . \tag{26}$$

(22) and (26) give

$$(\Delta g_{\Psi})_{\varkappa'} = (\Delta g_M)_{\sigma} + s_M . \tag{27}$$

(22), (25) and (27) lead to

$$M = \frac{R+\eta_P}{4\pi} \iint_{\omega} \left\{ (\Delta g_M)_{\sigma} + s_M \right\} S(\psi) \, d\omega . \tag{28}$$

(28) is valid for test points at the surface of the Earth σ . The relation (19) is introduced into the Stokes equation (28) in order to replace the potential M by the two potentials T and B,

$$T - B = \frac{R+\eta_P}{4\pi} \iint_{\omega} \left\{ (\Delta g_T)_{\sigma} - (\Delta g_B)_{\sigma} + s_M \right\} S(\psi) \, d\omega . \tag{29}$$

5. The potential of the mountain masses and the condensation method

Now, it is necessary to express the potential B and its radial derivative $\frac{\partial B}{\partial r}$ in (29) as functions of the heights η . The terms linear in the height are separated (cf. [3], [4]), and the following relations are abtained,

$$(B)_{\sigma} = \Lambda_1 + \Lambda_2 + [B]'' , \tag{30}$$

$$(\frac{\partial B}{\partial r})_{\sigma} = \Lambda_3 + \Lambda_4 + [\frac{\partial B}{\partial r}]'' , \tag{31}$$

$$(\Delta g_B)_{\sigma} = -\Lambda_3 - \Lambda_4 - \frac{2}{r}(\Lambda_1 + \Lambda_2) - [\frac{\partial B}{\partial r}]'' - \frac{2}{r}[B]'' . \tag{32}$$

(32) is transformed to

$$(\Delta g_B)_{\sigma} = -\Lambda_3 - \Lambda_4 - \frac{2}{R}(\Lambda_1 + \Lambda_2) - [\frac{\partial B}{\partial r} + \frac{2}{R} B]'' + 2B \frac{\eta_Q}{R^2} . \tag{33}$$

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The expressions for the linear Λ terms have the following shape,

$$\Lambda_1 = 4\pi f \delta R \eta_P , \tag{34}$$

$$\Lambda_2 = f \delta R^2 \iint_\omega (\eta_Q - \eta_P) \frac{1}{e_o} d\omega_Q , \tag{35}$$

$$\Lambda_3 = - 4\pi f \delta \eta_Q , \tag{36}$$

$$\Lambda_4 = - f \delta R^2 \iint_\omega (\eta_Y - \eta_Q) \frac{1}{e_o^2} \sin \frac{\psi}{2} d\omega_Y ; \tag{37}$$

with

$$e_o = 2 R \sin \frac{\psi}{2} . \tag{38}$$

(34) and (35) refer to the test point P. (36) and (37) are in relation to the point Q.

The Helmert condensation method proves the validity of the equation (39)(cf.[3],[4]),

$$\Lambda_1 + \Lambda_2 = - \frac{R}{4\pi} \iint_\omega [\Lambda_3 + \Lambda_4 + \frac{2}{R}(\Lambda_1 + \Lambda_2)] S(\psi) d\omega . \tag{39}$$

(30) - (39) change (29) into

$$T - \Lambda_1 - \Lambda_2 - [B]'' = \frac{R + \eta_P}{4\pi} \iint_\omega \{(\Delta g_T)_\sigma + \Lambda_3 + \Lambda_4 + \frac{2}{R}(\Lambda_1 + \Lambda_2) + + [\frac{\partial B}{\partial r} + \frac{2}{R} B]'' - 2B \frac{\eta_Q}{R^2} + s_M\} S(\psi) d\omega . \tag{40}$$

Some rearrangements of (40) by (39) give the desired relation for T,

$$T = \frac{R + \eta_P}{4\pi} \iint_\omega \{\Delta g_T + C\} S(\psi) d\omega + \Xi , \tag{41}$$

$$\Xi = [B]'' + \frac{R}{4\pi} \iint_\omega (- 2B \frac{\eta_Q}{R^2} + s_M) S(\psi) d\omega - \frac{\eta_P}{R} B . \tag{42}$$

Δg_T stands in (41) for $(\Delta g_T)_\sigma$.

C is the plane topographical reduction of the gravity (cf. [3], [4]). It has the relation

$$C \cong [\frac{\partial B}{\partial r} + \frac{2}{R} B]'' \geq 0 . \tag{43}$$

The amount of the first term on the right hand side of (42), $[B]''$, is negligible, (cf. [4]). The term

$$\frac{R}{4\pi} \iint_\omega 2B \frac{\eta_Q}{R^2} S(\psi) d\omega \tag{44}$$

in the expression for Ξ can be omitted also, it is very small, (cf. [3] , [4]). In most cases, the value of

$$\frac{\eta_P}{R} B \tag{45}$$

can be neglected also (cf. [3], [4]).

DOI: https://doi.org/10.2312/zipe.1986.084

Hence,

$$\Xi \cong \frac{R}{4\pi} \iint_{\omega} s_M \, S(\psi) \, d\omega \,. \qquad (46)$$

Thus, the problem of the evaluation of the amount of s_M is left over as a remaining task now to be solved. s_M can be interpreted as a term that is to be added to the free-air anomalies Δg_T in (41), similar as C.

6. The vertical shift of the smoothed free-air anomalies

As to the determination of s_M, the potential M, (19), comes into being by the gravitation force of the rock density anomalies, i.e. the deviations of the rock density from the standard density $\delta = 2.65$ g cm^{-3}. Further, the gravitation force of the isostatic mountain roots is also one of the sources that bring the potential M into existence. As it is well-known, these gravitation forces that give rise to the potential M are also the main sources of the Bouguer anomalies of the gravity.

Consequently, the free-air anomalies Δg_M of the potential field M will have amounts and structures along the surface of the Earth which are in the vicinity of these values in the field of the Bouguer anomalies, see chapter B. Both these systems of anomalies will have about the same amplitudes and about the same wave lengthes. Above all, it is sure that the cross correlation of the Δg_M values and the topographical heights is much more small than the cross correlation abtaind from the Δg_T values and the heights. Thus, it can be taken for granted that the superposition of the potential T with the potential B leads to a smoothing of the free-air anomalies. A more detailed description and analytical representation of the Δg_M values is intended to be given in the future at another place, see chapter B. Here, a rough first estimation of the s_M value, (26), shall suffice to demontrate the principle ideas.

s_M has the relation

$$s_M = (\Delta g_\psi)_{\varkappa'} - (\Delta g_\psi)_\sigma \,. \qquad (47)$$

The Bouguer anomaly map of the area of the Swiss Alps offers an excellent opportunity for an estimation of the s_M values (cf. [10]).

A north-south profile across the Swiss Alps shows that the Bouguer anomalies can be approximated there in the mean by a wave of about 200 km length and an amplitude of about b = 60 mgal.
The curvature of the globe $\varkappa$ can be neglected within an area of about 200 km diameter before the background of the here introduced approximation. Thus, a potential U of the shape of (48) can be considered as a regional representation of the potential M, (cf. [8]),

$$U = U_o \cos(\beta x) \cdot e^{-\beta z} \,, \quad -100 \text{ km} \leq x \leq +100 \text{km}. \qquad (48)$$

DOI: https://doi.org/10.2312/zipe.1986.084

U_o is the constant amplitude of the potential U, β is the constant wave length. The x coordinate is horizontal in the north-south direction, z is the height above the sphere $\varkappa'$. U is not variable in the east-west direction.

Now, it is presupposed that the test point P is situated at sea level for the evaluation of s_M. Thus, the Bjerhammar sphere $\varkappa'$ changes over to the sphere $\varkappa$. Hence,

$$s_M = (\Delta g \psi)_{\varkappa} - (\Delta g \psi)_{\sigma} \quad . \tag{49}$$

(48) fulfills the Laplace differential equation

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \quad . \tag{50}$$

The free-air anomaly in the field of the potential U can be brought into the following form,

$$\Delta g_U = -\frac{\partial U}{\partial z} - \frac{2}{R+z} U \quad . \tag{51}$$

The here discussed example has the following constants,

$$\beta = \frac{2\pi}{X} = 0.016 \text{ km}^{-1} \quad , \tag{52}$$

$$X = 400 \text{ km} \ll R \quad . \tag{53}$$

The derivations (48) – (53) give the following expression for the free-air anomaly in the U potential field,

$$\Delta g_U \cong -\frac{\partial U}{\partial z} = b \cos(\beta x) \cdot e^{-\beta z} \quad , \tag{54}$$

with

$$b = U_o \beta \cong 60 \text{ mgal} \quad . \tag{55}$$

The vertical change of Δg_U is here of special interest,

$$(\Delta g_U)_{\varkappa} - (\Delta g_U)_{\sigma} = b \cos \beta x - b \cos(\beta x) \cdot e^{-\beta z_\sigma} \quad . \tag{56}$$

The heights of the mountains z_σ do not surpass a value of about 4 km. Thus,

$$0 \leq z_\sigma \leq H = 4 \text{ km}. \tag{57}$$

(52) and (57) lead to

$$0 \leq \beta z_\sigma \leq 0.064 \ll 1 \quad . \tag{58}$$

(56) and (58) show that the exponential series

$$e^{-u} = E(-u) = 1 - \frac{u}{1!} + \frac{u^2}{2!} - + \ldots \tag{59}$$

transforms the relation (56) into

$$(\Delta g_U)_{\varkappa} - (\Delta g_U)_{\sigma} \cong b \beta z_\sigma \cos \beta x \quad . \tag{60}$$

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(59) is convergent for every value of u (cf. [13]). The topographical heights z of the considered north-south profile across the Swiss Alps can be approximated by the following function (cf. [10]),

$$z_\sigma = H \cos \beta x , \quad -100 \text{ km} \leq x \leq +100 \text{ km}. \tag{61}$$

(49), (60) and (61) give

$$s_M = b \beta H \cos^2 \beta x = 4 \cos^2 \beta x \; [\text{mgal}] . \tag{62}$$

The average value of s_M in the range of the considered profile follows to be

$$\bar{s}_M = \frac{2}{X} b \beta H \int_{x=-\frac{1}{4}X}^{+\frac{1}{4}X} \cos^2 \beta x \, dx . \tag{63}$$

Hence,

$$\bar{s}_M = \frac{1}{2} b \beta H . \tag{64}$$

(52), (55), (57) and (64) give

$$\bar{s}_M = 2 \text{ mgal} . \tag{65}$$

7. The solution of the geodetic boundary value problem

Thus, the amount of $\bar{s}_M$ seems to be within the noise of the dates of the gravity net of the Alps. Probably, s_M will be negligible in most cases. It follows that also Ξ , (46), can generally be omitted.
Thus, the equation (41) changes to

$$T = \frac{R + \eta_P}{4\pi} \iint_\omega \left\{ \Delta g_T + C \right\} S(\psi) \, d\omega . \tag{66}$$

Consequently, the extension of the geodetic boundary value problem of the Stokes type to the Molodenskii type happens by the addition of the plane topographical reduction value C to the usual free-air anomalies. The residual term Ξ , (41), (42), has a closed expression. It has a mathematical representation that gives an insight of clear transparency into the amount of Ξ . (66) corroborates the results abtained in [3]. In the low mountain ranges and in the high mountains, the C values surpass the measurement errors considerably. The C values can reach more than 10 mgal and at the summit of the Fuji-san mountain on the Japanese islands the value of C is even equal to 135.8 mgal. Thus, the C term cannot be neglected generally.

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8. The vertical shift of the non-smoothed free-air anomalies

If the Bjerhammar sphere boundary value problem does not take advantage of the smoothing effect of the superposition with the mountain masses, in this case, the real perturbation potential T is to be treated, instead of the potential M, (19).
In this original shape of the problem, just the values T and Δg_T must undergo the harmonic downward continuation, as can be found in the literature (cf. [5], [6], [7]).
Δg_T is not smoothed, it is rugged, as against to Δg_M.
The following easily understandable relations are obtained (cf. (8), (9), (47)),

$$\Delta g_T = - \frac{\partial T}{\partial r} - \frac{2}{r} T \, , \tag{67}$$

$$s_T = (\Delta g_\Omega)_{\varkappa'} - (\Delta g_\Omega)_\sigma \, . \tag{68}$$

Now, the amounts and the structures of the terms C, s_T and s_M are to be compaired with each other.

The harmonic vertical continuation of Δg_T from the surface σ of the Earth through the mass-free space Φ_i down to the Bjerhammar sphere $\varkappa'$ is governed by the following inhomogeneous integral equation of the first kind (cf. [6], [7], [14]),

$$v(Q) = \iint_{\varkappa'} K(Q,Y) \, w(Y) \, d\omega'_Y \, , \tag{69}$$

with the kernel function

$$K(Q,Y) = \frac{(R+\eta_Q)^2 - (R+\eta_P)^2}{4\pi \, (R+\eta_Q)} \, \frac{1}{|(\underline{x}_Q)_\sigma - (\underline{x}_Y)_{\varkappa'}|^3} \, , \tag{70}$$

$$v(Q) = (\Delta g_\Omega(Q))_\sigma = (\Delta g_T(Q))_\sigma \, , \tag{71}$$

$$w(Y) = (\Delta g_\Omega(Y))_{\varkappa'} \, . \tag{72}$$

The function v is known, the function w is unknown.
$(\underline{x}_Y)_{\varkappa'}$ is the position vector of the point Y on the sphere $\varkappa'$. Y is the moving point of the integration. $(\underline{x}_Q)_\sigma$ is the placement vector of the points Q at the surface σ of the Earth. The determination of w(Y) in terms of v(Q) by means of (69) involves an iteration procedure (cf. [6], [14]). The expression for s_T obtained by (13) - (17), (67) - (72), is in close neighborhood to the corresponding expression for s_T derived by means of the height gradient of the Δg_T values. The latter way was followed in [5] and it did lead to the KG(Δg_T) values.

The claim that the s_T value of (68) has an amount and structure similar as KG (Δg_T) is evidenced by the subsequent considerations. The term KG(Δg_T) has the following relation (cf. [5]),

$$KG(\Delta g_T)_Q \cong - \left(\frac{\partial \Delta g_T}{\partial \eta}\right)_Q (\eta_Q - \eta_P) \, . \tag{73}$$

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The detailed formula is

$$(KG(\Delta g_T))_Q \cong -(\eta_Q - \eta_P)\frac{1}{2\pi}\iint_{\varkappa} \frac{(\Delta g_T(Y))_\sigma - (\Delta g_T(Q))_\sigma}{e_o^3}\, d\varkappa_Y \quad . \qquad (74)$$

e_o is the horizontal distance between the two points Y and Q. The kernel functions of (69) and (74) have closely related structures, a property that is reflected also in s_T and KG (Δg_T). Further, KG (Δg_T) approximates the value of s_T.

This fact is evidenced by a modification of (69) and (70) wich consists in a passing to the limit of $\eta_Q - \eta_P \dashrightarrow 0$.
If the heights η are small, in this case both the terms s_T, (68), and KG(Δg_T), (74), are in the main identical.

Hence, in a first approximation and if the mountains are not too high, the term s_T can be replaced by KG(Δg_T), (cf. [5], [14]).

The numerical values of KG(Δg_T) in the area of the Harz mountain are discussed in [5]. The figures 4 - 7 of the publication [5] show the amounts of

$$\tau = \frac{KG(\Delta g_T)}{\eta_Q - \eta_P} \quad . \qquad (75)$$

To visualize the values of C and s_T, here to be discussed, a profile was drawn in the east-west direction through the Harz mountain across the Brocken summit which has a height of 1140 m. The figures 2 - 5 show the course of some interesting functions within the range of this profile. In the figure 2, the ϑ_1 values represent the s_T values for a Bjerhammar sphere situated at the sea level, i.e. $\eta_P \doteq 0$ m,

$$\vartheta_1 = \eta_Q \tau = KG(\Delta g_T)_1 , \qquad (76)$$

$$KG(\Delta g_T)_1 = -\eta_Q \frac{1}{2\pi}\iint_{\varkappa} \frac{(\Delta g_T(Y))_\sigma - (\Delta g_T(Q))_\sigma}{e_o^3}\, d\varkappa_Y \quad . \qquad (77)$$

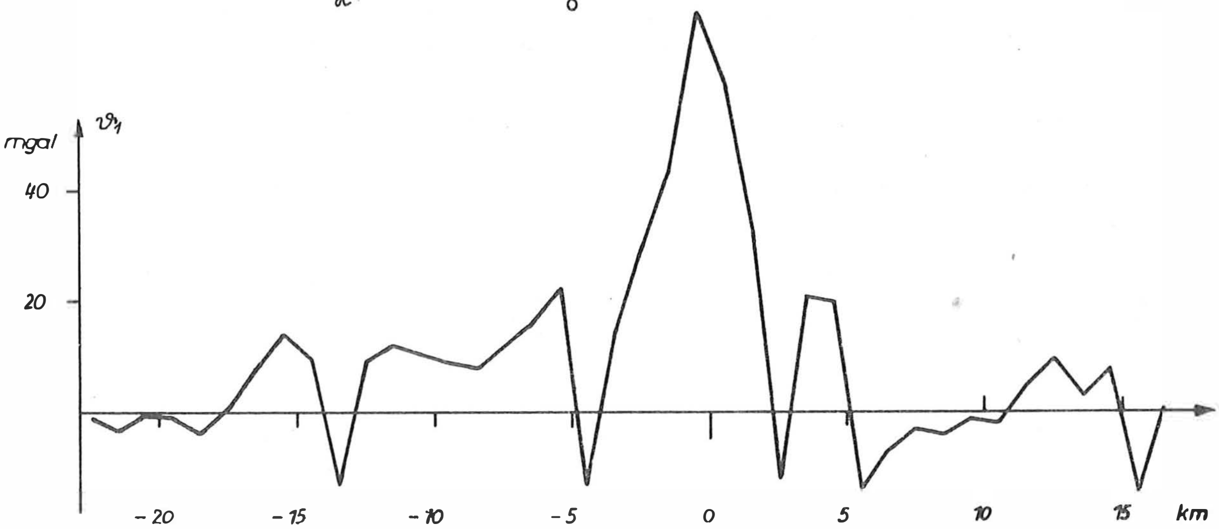

Figure 2: The ϑ_1 values of the Harz mountain profile (cf. (76),(77)). ϑ_1 approximates the term s_T, (68), in case of a continuation down to the sea level. The zero point on the axis of abscissas refers to the Brocken summit.

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The figure 3 shows the ϑ_2 values,

$$\vartheta_2 = (\eta_Q - \eta_{P(Brocken)}) \quad \tilde{\tau} = KG(\Delta g_T)_2 \quad . \tag{78}$$

These ϑ_2 values have to be applied in case the test point is the Brocken summit of the Harz mountain and if consequently the Bjerhammar sphere has the height of this summit.

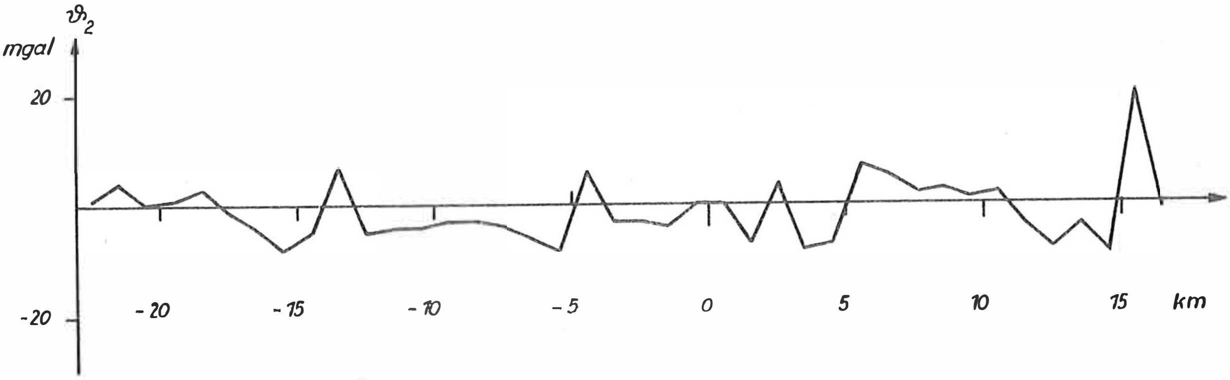

Figure 3: The ϑ_2 values of the Harz mountain profile (cf. (78)). They approximate the term s_T, (68), in case of a vertical harmonic continuation to the level of the Brocken summit.

9. The structure of the amounts of the plane topographical reduction of the gravity

The figure 4 shows the C values, i.e. the plane topographical reduction of the gravity. According to (66), the C values are to be applied instead of the downward continuation terms s_T.

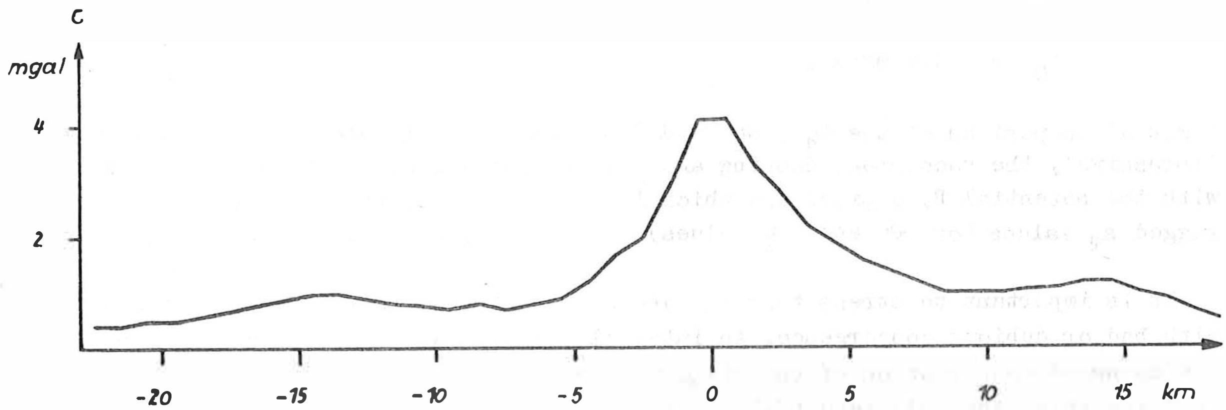

Figure 4: The C values of the Harz mountain profile, (plane topographical reduction).

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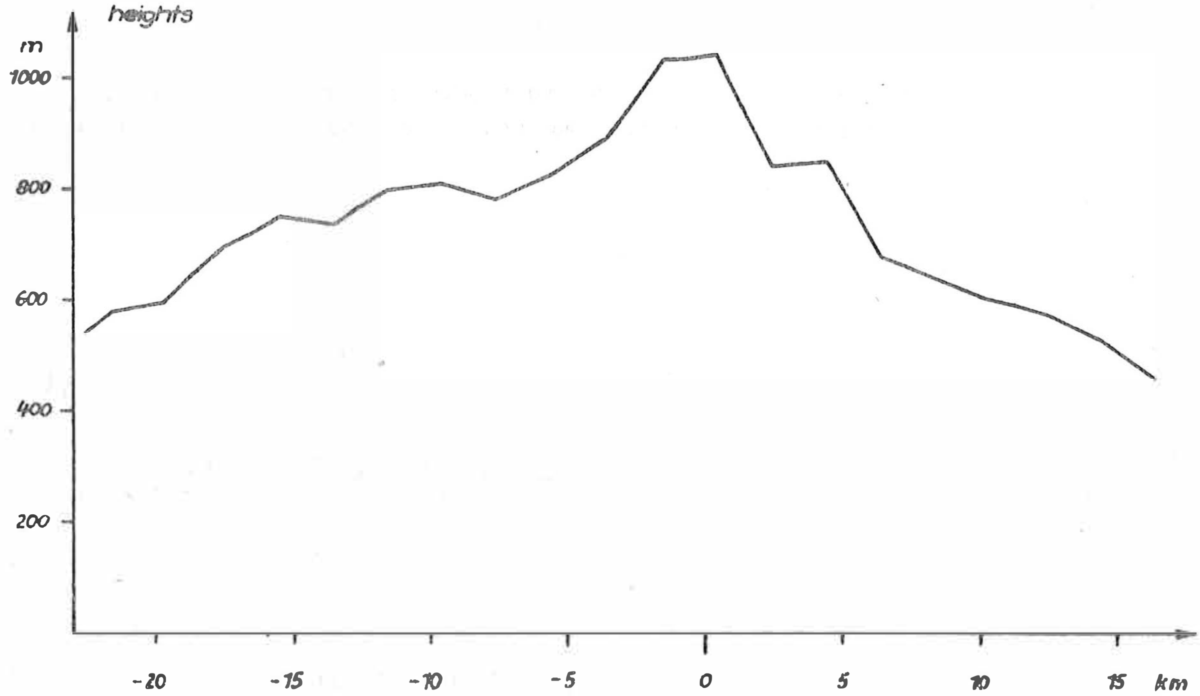

Figure 5: The topographical heights along the Harz mountain profile.

10. Instabilities

The mean square values for ϑ_1 , ϑ_2 and C are computed within the range of the Harz mountain profile. The following amounts are found,

$$\nu_{\vartheta_1} = 20 \text{ mgal} , \tag{79}$$

$$\nu_{\vartheta_2} = 6 \text{ mgal}, \tag{80}$$

$$\nu_C = 1.6 \text{ mgal} . \tag{81}$$

A mutual comparison of the ϑ_1 , ϑ_2 and C curves of the figures 2, 3 and 4 visualizes impressively the smoothing, damping and stabilizing effect caused by the superposition with the potential B, a procedure which is accompanied by the transition from the rugged s_T values (or ϑ_1 and ϑ_2 values) to the smoothed C values.

It is important to stress that the above derivations are free of series developments with bad or dubious convergence. An integral equation of the shape of (69) solves also the downward continuation of the Δg_{Ψ} values, (20) - (23).
In doing this, the relations (71) and (72) must be replaced by

$$p(Q) = (\Delta g_{\Psi}(Q))_{\sigma} = (\Delta g_M(Q))_{\sigma} , \tag{82}$$

and

$$q(Y) = (\Delta g_{\Psi}(Y))_{\varkappa} . \tag{83}$$

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Hence,

$$p(Q) = \iint_{\varkappa'} K(Q,Y)\ q(Y)\ d\varkappa'_Y \quad . \tag{84}$$

Therefore, the downward continuation procedures in the potential fields Ω and $\mathcal{V}$ are both governed by the same kernel function K(Q,Y), (69), (84). Thus, the instabilities inherent in these downward continuations have the same typical feature since they are caused by the same source. The downward continuatations of the Δg_M values have instabilities irrespective of the smoothed shape of these values. However, these instabilities are clearly without any importance for the M field if the component parts which have wave lengthes of 200 km and more are to be continued, as it is evident by (48) - (65). The instabilities are more in the fore in case that the short waves of only some kilometer length of the model anomalies Δg_M are to be continued downward. However, on the strength of the smoothing effect, the amplitudes of these short waves of the $(\Delta g_M)_\sigma$ values will be very small. In very many cases, these short wave amplitudes will be within the noise of the dates of the gravity field. Probably, this fact will allow to ignore these short waves in the downward continuation procedure.

Finally, it should be stated that the closed solution for the boundary value problem of Molodenskii derived by the author earlier at other places from the identity of Green is free of harmonic downward continuations as against to the here discussed Bjerhammar sphere boundary value problem. Thus, that earlier obtained closed solution is also free of instabilities (cf. [3], [4]), see also chapter B. In those earlier publications, the perturbation potential T and its radial derivative are considered as surface values which do not undergo any spatial shift represented by any series development. However, in the here discussed Bjerhammar sphere boundary value problem, the mathematical derivations are by far not so laborious as in [3] and [4].

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B. Numerical evaluations about the elimination of the iteration procedure term in the geodetic boundary value problem

Contents

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Summary

The iteration procedure term in the geodetic boundary value problem can be eliminated by a superposition of the pertubation potential with the potential of the mountain masses of standard density. The final solution is the Stokes relation in terms of the free-air anomalies supplemented by the plane topographical reduction of the gravity. Furthermore, 6 very small terms have to be added. They will never amount to more than about 5 cm in their impact on the height anomalies. Thus, for the first time and unprecedentedly, a closed solution is obtained finally. These 6 additional expressions are evaluated here precisely. The amount of them can be neglected, as the computations prove.

Zusammenfassung

Die Lösung des Randwertproblems der Geodäsie enthält einen iterativ zu bestimmenden Ausdruck. Dieser kann durch Überlagerung des Störpotentials mit dem Potential der Gebirgsmassen eliminiert werden. Als Lösung ergibt sich das Stokes-sche Integral, bei dem die Freiluftanomalien um die ebene Geländereduktion zu ergänzen sind. Es tritt ferner eine restliche Größe hinzu, die aus 6 sehr kleinen Gliedern besteht, deren Summe niemals mehr als etwa 5 cm erreicht. Die numerischen Beträge dieser 6 Glieder werden hier genau abgeschätzt. Sie sind vernachlässigbar klein.

Резюме

Решение краевой задачи геодезии содержит член, определяемыи итерационным методом. Последний исключается путем суперпозиции потенциала горных масс. В качестве решения получается интеграл Стокса, в котором аномалии свободного воздуха необходимо дополнить топографической поправкой. Кроме того, добавляется остаток, состоящий из 6 очень малых членов. Численные абсолютные значения 3-х их них точно оцениваются. Они пренебрежимо малы.

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1. Intoduction

The mathematical relation between the geopotential and the gravity is one of the most important problems of geodesy, as so as the inversion of this relation. The first relation gives the gravity in terms of the potential in a rather uncomplicated way, also for test points at the surface σ of the Earth. It is the theorem of Bruns, (cf. [13], [19]),

$$\Delta g_T = \left(- \frac{\partial T}{\partial r} - \frac{2}{r} T \right)_\sigma . \qquad (1)$$

Δg_T is the free-air anomaly at the surface σ of the Earth. T is the perturbation potential, r is the geocentric radius of the surface of the Earth σ.

In case of a spherical boundary surface $\varkappa$, the inversion of (1) leads to the Stokes integral, (cf. [13], [19]),

$$T_\varkappa = \frac{R}{4\pi} \iint_\omega \Delta g_T \, S(\psi) \, d\omega \quad . \qquad (2)$$

R is the radius of the globe of the Earth $\varkappa$, ω is the unit sphere. S (ψ) is the Stokes function. ψ is the spherical distance between the test point and the point variable in the integration.

However, in case the surface of the Earth σ is the boundary surface, the inversion of (1) is rather complicated since the irregular topographical heights are involved.

2. Critique of the present state of the solution of the geodetic boundary value problem

Earlier, 4 principally different methods were developed in order to find the solution of this problem, (Molodenskij, Moritz, Bjerhammar, Arnold), cf. [18], [19], [8], [7]. The main term of all of these 4 solutions is the Stokes integral, (2). However, these 4 methods are distinctly different if the supplementary expressions are considered in order to compute the surface height anomalies ζ,

$$\zeta = \frac{T}{\gamma} \quad . \qquad (3)$$

γ denotes the standard gravity at σ. These deviations from the Stokes integral reach presumably the amount of up to 1 or 2 m in the ζ value, (cf. [19], [21]). The components of the deflection of the vertical, ϑ_1 and ϑ_2, are presumed to have supplementary terms of about up to 1", (cf. [20]),

$$\vartheta_1 = - \frac{\partial T}{\partial x} \cdot \frac{1}{\gamma} \; , \quad \vartheta_2 = - \frac{\partial T}{\partial y} \cdot \frac{1}{\gamma} \quad . \qquad (4)$$

dx and dy are the horizontal differentials of arc in the south-north and west-east direction.

The individual authors have reached the following solutions for the inversion of (1).

Molodenskij has obtained his well-known series development for the perturbation potential T_σ at the surface of the Earth σ by the intermediary introduction of the potential of a surface distribution, [18],

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$$T_\sigma = T_\varkappa + \sum_{i=1}^{\infty} T_i \quad . \tag{5}$$

The T_i functions, (i=1,2, ...), are expressed in terms of Δg_F by the formulas of Molodenskij.

According to Moritz, the $T_\varkappa$ value at the left hand side of (2) has to be replaced by T_σ and, further, the Δg_T values on the right hand side of (2) by $(\Delta g_T)_P$, [19], with

$$(\Delta g_T)_P = U^{-1} \Delta g_T = D \Delta g_T = \Delta g_T + \sum_{i=0}^{\infty} D_i \Delta g_T \quad . \tag{6}$$

Δg_T is in (6) the surface value of the free-air anomalies, (1). D is the operator for the downwards continuation of the Δg_T values. This operator is represented by a series development, (6), which can be found explicitely in [19]. U is the operator for the upwards continuation. $(\Delta g_T)_P$ is the free-air anomaly at the Bjerhammar sphere $\varkappa_P$ with the radius R_P. P is the test point for which T is to be computed. The point P is situated on the line of intersection of the surface σ and of the Bjerhammar sphere, Fig. 1 and 3.

The solution according to Bjerhammar is situated within a certain vicinity to that of Moritz. However, Bjerhammar does compute the spherical $(\Delta g_T)_P$ values from the Δg_T values at the surface σ by means of an integral equation, instead of the Moritz series development, (cf. [8], [2], [3], [5], [12]),

$$\Delta g_T (Q) = \iint_{\varkappa_P} \Delta g_T (Q_{\varkappa_P}) \, K (Q, Q_{\varkappa_P}) \, d\varkappa_P \quad . \tag{7}$$

K is the kernel function of the integral equation (7), Q is a point at the surface of the Earth σ. $Q_{\varkappa_P}$ is the running point on the Bjerhammar sphere $\varkappa_P$ which has the radius $R_P = R + h_P$. h_P is the height of the point for which T_σ is to be computed. The point $Q_{\varkappa_P}$ is variable within the course of the integration according to (7).

Further, in the formerly published solution of the author, (cf. [1], [4], [7], [8]), the value of $T_\varkappa$ has also to be replaced by T_σ on the left hand side of (2) and Δg_T must be substituted by

$$\Delta g_T + KG \tag{8}$$

on the right hand side of (2), with

$$KG = C_1(T) = G\,(h_Q - h_P)\left(\frac{\partial \vartheta_1}{\partial \bar{x}} + \frac{\partial \vartheta_2}{\partial \bar{\bar{y}}} - \vartheta_1 \frac{\operatorname{tg} \varphi}{R}\right) \quad . \tag{9}$$

The mathematical deductions to reach the KG term are free of downwards and upwards continuations. G is here the global average values of the gravity, h is the height of the surface points above the globe of the Earth $\varkappa$. P is the certain already mentioned test point on σ for which T_σ is to be computed in terms of Δg_T. Q is the variable integration point. ϑ_1 and ϑ_2 are the components of the plumb-line deflection at the surface σ, (4). Thus, ϑ_1 and ϑ_2 are considered here as surface values which depend from two

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parameters only, e. g. the geographical latitude and longitude φ and λ , a fact that is to be stressed here.

$$\vartheta_1 = \vartheta_1 (\varphi , \lambda) ; \quad \vartheta_2 = \vartheta_2 (\varphi , \lambda). \tag{10}$$

$d\bar{x}$ and $d\bar{y}$ are the downwards projections of dx and dy, Fig. 1,

$$d\bar{x} = R \, d\varphi \quad , \tag{11}$$

$$d\bar{y} = R \cos\varphi \, d\lambda \quad . \tag{12}$$

The expression (9) depends on T. That is the function even to be determined. Thus, the solution of the author derived from the identity of Green along the lines of (8) and (9) does not lead to an equation which has the unknown T function explicitly only on the left hand side. The expression (9) has to be computed by repeated iterations. Consequently, the extension of the geodetic boundary value problem from the Stokes type to the Molodenskij type gives rise to the appearance of an iteration procedure term, (9). In a first approximation, in the lowlands, the ϑ_1 and ϑ_2 values of (9) can be derived from the spherical Stokes or Vening - Meinesz theory for the computation of KG according to (9).

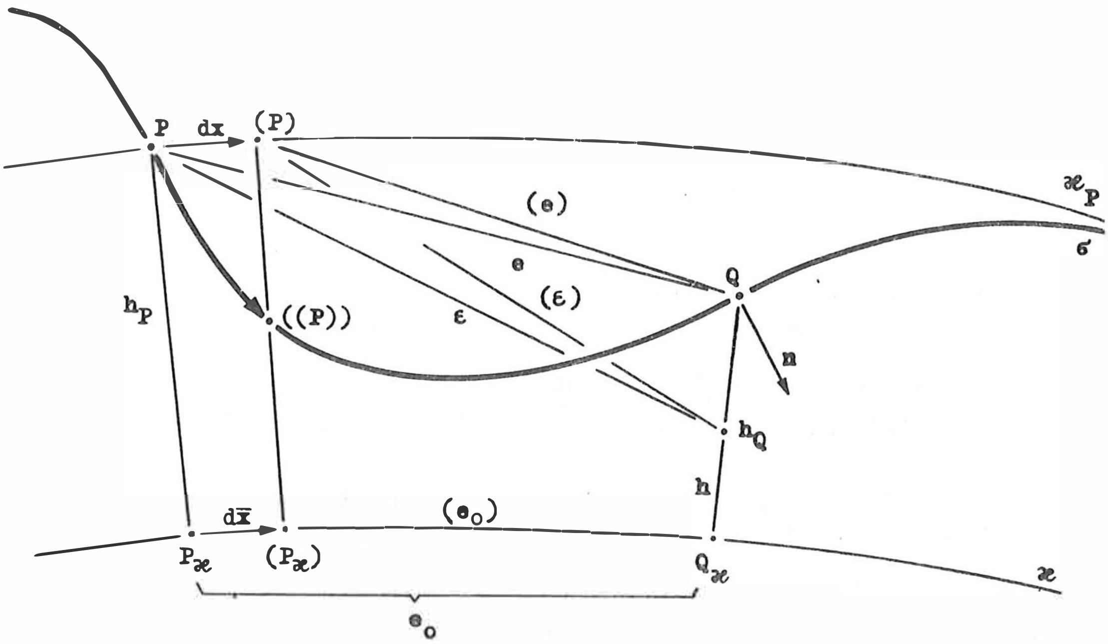

Fig. 1 : The straight distance e connects the test point P and the variable point Q on the surface of the Earth. The differentiation of e happens along the horizontal element of length dx, plotted at the point P. As opposed to this, if a surface function is shifted on σ from P to ((P)) along an infinitesimal length, in this case, the element of length $d\bar{x}$ is introduced as the denominator in the concerned differential quotient.

DOI: https://doi.org/10.2312/zipe.1986.084

Basing on a spherical theory, the term KG, (9), has the following approximative expression as function of Δg_T, (cf. [19]),

$$KG \cong (h_Q - h_P) \frac{R^2}{2\pi} \iint \frac{(\Delta g_T)_Z - (\Delta g_T)_Q}{e_0^3} \, d\omega_Z \, , \tag{13}$$

Z is the moving point in the integration of (13). e_0 has the following relation,

$$e_0 = 2 R \sin \frac{\psi}{2} \, . \tag{14}$$

ψ is here the spherical distance between the two points Z and Q.

The pros and cons of the 4 above cited methods developed to find a solution for the boundary value problem of Molodenskij are well studied and well-known; Molodenskij (cf. [5]), Moritz (cf. [6]), Bjerhammar (cf. [7]), Arnold (cf. [8])(cf. [9]). In this context, it is suitable to give a short critical comparison of these 4 methods:

The individual terms of the series development (5) contain instabilities, the convergence of (5) is an open question. Supposed, the convergence of (5) should really exist, in any case, this convergence will not be good and the series (5) will not have a rapid speed of convergence. During the last 20 or 30 years, there is not even one successful attempt to determine the residual term

$$\sum_{i=N}^{\infty} T_i \tag{15}$$

of the series development (5), N is a sufficient great integer. Thus, it is a matter of fact that the series development (5) cannot be governed. There is no mean to compute (5). In view of the determination of the decimeter geoid, it is indispensable to have a mathematical expression for the height anomalies ξ that has a guaranteed theoretical residual error of not more than some centimeters. Therefore, (5) is not a satisfactory solution of the boundary value problem.

The judgement about the solution according to the series (6) comes to the same result as that of the solution (5), since Pellinen has proved that the series developments (5) and (6) are equivalent, (cf. [19]). In the past, there was not even one successful step to evaluate the amount of the residual term of the series development (6),

$$\sum_{i=N}^{\infty} D_i \, \Delta g_T \, . \tag{16}$$

Nevertheless, the series (5) and (6) have led to some useful scientific findings. Indeed, theoretical investigations have shown that the first terms of these series developments, i. e. T_1 and $D_0 \, \Delta g_T$, are situated in a relative close vicinity of the plane topographical reduction of the gravity C, [13],

DOI: https://doi.org/10.2312/zipe.1986.084

$$C = f\varrho R_Q^2 \int_{\psi=0} \psi d\psi \int_{\alpha=0}^{2\pi} d\alpha \int_{z=0}^{Z} \frac{zdz}{[z^2 + (R_Q\psi)^2]^{3/2}} \geq 0 \quad . \tag{17}$$

C can reach considerable amounts, (cf. [17], [21]). f is the gravitational constant, ϱ is the standard density (ϱ = 2,65 [g cm^{-3}]). In the equation (17), ψ is the spherical distance between two points. The first point is moving in the course of the integration of (17) and the second point Q is the point for which C is to be determined. Z is the height difference between these two points, (cf. Fig. 3),

$$Z = h - h_Q \quad . \tag{18}$$

As to the solution according to (7), this form has the handicap that the inversion of the integral equation (7) is rather laborious. In the hilly countrysides and in the mountains, the differences $\Delta g_T - (\Delta g_T)_P$ variate in the form of relative short waves and relative great amplitudes. For the computation of the solution of (7), the numerical evaluations need a computation grid of relative small meshes.

The solution established by the relations (8) to (13) has about the same difficulties as that represented by (7). Here, in (8) - (13), an additional difficulty is involved since the relation (13) allows satisfactory results only for areas not too hilly. In the mountains, the relation (13) tolerates a lot of approximations which are not yet investigated.

3. The basic relations of the closed solution of the geodetic boundary value problem

The most recent investigations of the author show that the solution according to (8) and (9) has the advantage to be considerably developable by the superposition with the potential of the mountain masses B, (cf. [7], [9]). The same advantage can be brought to bear also in the solution along the lines of the Bjerhammar sphere, (cf. [8]). However, as to the series solutions according to (5) and (6), the superposition with the potential of the mountain masses does not bring considerable advantages there. The results of the most recent developments of the author about (8) and (9) can be summarized in the following way.

The amount of T at the surface σ is T_σ . It has the following equation, [7],

$$T_\sigma = \frac{R}{4\pi} \iint (\Delta g_T + C) \, S(\psi) \, d\omega + \sum_{i=1}^{6} \chi_i \quad . \tag{19}$$

The 6 terms χ_i can be neglected, they have an influence of less than some centimeters on the height anomalies ζ , (cf. (3)). They have certain mathematical expressions that give an insight of clear transparency into the amount of them, (cf. [7], [9]). In the here discussed geodetic applications, the relation (19) has the character of a closed solution; it has not the form of a series development. The relation (19) is of importance also for the solution of the mixed boundary value problem of geodesy, (cf. [6], [9]; see also chapter C). It is not possible to find the mixed boundary value

DOI: https://doi.org/10.2312/zipe.1986.084

problem without Stokes and Molodenskij.

The here preferred starting point for the derivation of the solution of the boundary value problem is the identity of Green for test points subsided down to the surface of the Earth (cf. [7], [15], [16]),

$$T_\sigma = \frac{1}{2\pi} \iint_\sigma \frac{1}{e} \frac{\partial T}{\partial n} d\sigma - \frac{1}{2\pi} \iint_\sigma T \frac{\partial 1/e}{\partial n} d\sigma \quad . \tag{20}$$

The direction of n is normal to the surface σ , positive into the interior of the Earth. Consequently, the perturbation potential T has a closed expression just from the beginning. The mathematical rearrangements which transform (20) into (19) are laborious. They can be found in [1],[7]. Here, some of the main ideas are to be sketched only.

The transition from $\frac{\partial T}{\partial n}$ to $\frac{\partial T}{\partial r}$ happens by the multiplication with a 3 x 3 rotation matrix, a series development is not needed. Two orthogonal tripods of unit vectors are introduced. The first tripod $\underline{\underline{n}}_1$, $\underline{\underline{n}}_2$, $\underline{\underline{n}}_3$ refers to the surface σ , $\underline{\underline{n}}_1$ and $\underline{\underline{n}}_2$ are tangential vectors of σ , and $\underline{\underline{n}}_3$ is both perpendicular to σ and positive into the exterior space. The second tripod x, y, z refers to the horizontal plane of the test point P, Fig. 1. The x, y plane is the horizontal plane and z is the vertical component, the positive direction of the z axis points upwards. Thus,

$$\begin{bmatrix} \frac{\partial T}{\partial n_1} \\ \frac{\partial T}{\partial n_2} \\ \frac{\partial T}{\partial n_3} \end{bmatrix} = \underline{\underline{A}} \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{bmatrix} \quad , \tag{21}$$

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{A}}\,\underline{\underline{A}}^{T} = \underline{\underline{E}} \quad , \tag{22}$$

$\underline{\underline{E}}$ is the unit matrix, the superscript T denotes the transposition. With

$$-\frac{\partial T}{\partial n} = \frac{\partial T}{\partial n_3} \tag{23}$$

and

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial z} \tag{24}$$

follows from (21)

$$\frac{\partial T}{\partial n} = a_1 \cdot \frac{\partial T}{\partial x} + a_2 \frac{\partial T}{\partial y} + a_3 \frac{\partial T}{\partial z} = \sum_{i=1}^{3} a_i \frac{\partial T}{\partial x_i} \quad , \tag{25}$$

$$\sum_{i=1}^{3} a_i^2 = 1 \quad , \tag{25 a}$$

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dx_i stands for dx, dy, dz, (i = 1,2,3). In the relation (20), e and $d\sigma$ depend from the shape of σ , Fig. 1. Thus, (20) transforms into

$$T_\sigma = \iint_{\varkappa} \Gamma_1(\sigma) \left(\frac{\partial T}{\partial n}\right)_\sigma d\varkappa - \iint_{\varkappa} \Gamma_2(\sigma)\, T_\sigma \, d\varkappa \quad . \tag{26}$$

(25) is introduced into (26),

$$T_\sigma = \iint_{\varkappa} \Gamma_1(\sigma) \sum_{i=1}^{3} a_i \frac{\partial T}{\partial x_i} d\varkappa - \iint_{\varkappa} \Gamma_2(\sigma)\, T_\sigma \, d\varkappa \quad . \tag{27}$$

Till the equation (40), the h values are the heights of the surface σ above the sphere $\varkappa$. If $h \rightarrow 0$, the transitions $\Gamma_1(\sigma) \rightarrow \Gamma_1(\varkappa)$ and $\Gamma_2(\sigma) \longrightarrow \Gamma_2(\varkappa)$ follow. The division of (27) into a spherical part and a supplementary height dependent term leads to two simultaneous equations,

$$T_\varkappa = \iint_{\varkappa} \Gamma_1(\varkappa) \left(\frac{\partial T}{\partial r}\right)_\sigma d\varkappa - \iint_{\varkappa} \Gamma_2(\varkappa)\, T_\varkappa \, d\varkappa \, , \tag{28}$$

$$\delta T = - \iint_{\varkappa} \Gamma_2(\varkappa)\, \delta T \, d\varkappa + \iint_{\varkappa} \left\{ \sum_{i=1}^{3} \Gamma_{3.i}(h) \left(\frac{\partial T}{\partial x_i}\right)_\sigma + \Gamma_4(h)\, T_\sigma \right\} d\varkappa , \tag{29}$$

$$\Gamma_2(\sigma) = \Gamma_2(\varkappa) - \Gamma_4(h) \quad , \tag{29 a}$$

$$T_\sigma = T_\varkappa + \delta T \quad . \tag{30}$$

(28) is equivalent to the Stokes solution, (2). The second term on the right hand side of (29) has a structure that gives rise to the iteration procedure term, since it does include the a priori unknown T_σ values and the derivatives of T in the x, y, z system. The first term on the right hand side of (29) can be neglected. This fact will be shown later on in the section 7.5 of this chapter of this publication. The first term on the right hand side (29) is denominated also by χ_2, at other places; see section 7.5 and the publication [7]. Some rearrangements of (28) and (29) lead to (19), [1] [7].

The principal ideas developed by the relations (21) - (30) can be represented also in another shape. The identity of Green, (20), can be written in the following form,

$$F_1 \left(\sigma , T, \frac{\partial T}{\partial n}\right) = 0 \quad . \tag{31}$$

(31) is linear in T and $\frac{\partial T}{\partial n}$. With

$$\sigma = \varkappa + \delta\sigma \tag{32}$$

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follows

$$F_1 \, (\varkappa + \delta\sigma \,,\, T, \, \frac{\partial T}{\partial n}) = 0 \, . \qquad (33)$$

(25) and (33) give

$$F_2 \, (\varkappa + \delta\sigma, \, T, \, \frac{\partial T}{\partial x} \,, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}) = 0 \,, \qquad (34)$$

or

$$F_3 \, (\varkappa + \delta\sigma, \, T, \left(\frac{\partial T}{\partial r}\right)_\sigma , \, \alpha_x \, (\delta\sigma) \, \frac{\partial T}{\partial x} , \, \alpha_y \, (\delta\sigma) \frac{\partial T}{\partial y} \,, \, \alpha_z \, (\delta\sigma) \frac{\partial T}{\partial r}) = 0 \, . \qquad (35)$$

(35) is divided into a spherical part and the residual expression which goes to zero if $h \longrightarrow 0$ or if $\delta\sigma \longrightarrow 0$. Hence, two silmultaneous equations are again obtained for the solution,

$$F_4 \, (\varkappa \,, \, T_\varkappa , \left(\frac{\partial T}{\partial r}\right)_\sigma) = 0 \quad , \qquad (36)$$

and

$$F_5 \, (\varkappa \,, \left(\frac{\partial T}{\partial r}\right)_\sigma ; \delta T) + F_6 \, (\varkappa \,, \, T_\sigma , \left(\frac{\partial T}{\partial r}\right)_\sigma ; \delta\sigma \,, \sum_{i=1}^{3} \alpha_i \, (\delta\sigma) \, \frac{\partial T}{\partial x_i} = 0 . \qquad (37)$$

(30), (36) and (37) lead to (19), (cf. [1], [7]). If $\delta\sigma = 0$, the equations $\alpha_i \, (\delta\sigma) = 0$ and $F_6 = 0$, $\delta T = 0$ follow. (36) represents the Stokes solution, (2).

The recent investigations of the author show that the solution of the boundary value problem can be written in the subsequent form, (cf. [7] equation (66)),

$$T = \frac{R}{4\pi} \iint_\omega [\Delta g_T + C_1 \, (T)] \, I \, (h) \, S \, (\psi) \, d\omega + \chi_6 \, (T) \quad , \qquad (38)$$

with

$$\chi_6 \, (T) = - \frac{1}{8\pi R} \iint_\omega \frac{\partial T}{\partial \psi} \, (h_Q - h_P) \left(\frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} + 2 \, \frac{dS}{d\psi} \right) d\omega \,, \qquad (39)$$

$$I \, (h) = 1 + \frac{1}{2} \, (\, 3 \, \frac{h_Q}{R} - \frac{h_P}{R}) \quad , \qquad (40)$$

$\frac{\partial}{\partial \psi}$ means the derivation in the radial and horizontal direction.
C_1 (T) is explained by (9), it is the iteration procedure term of the relation (38). It is advantageous to take measures to avoid this term since it does involve laborious and repeated iterations.

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As it was shown in [7],[9], the iteration procedure term can be eliminated in that the perturbation potential T is superposed with the potential B of the mountain masses. Or, to be more precise, this superposition transforms $C_1(T)$ into $C_1(T-B)$. The amount of $C_1(T-B)$ is smaller than the upper bound of the noise of the method, i. e. the random errors and the biases of the Δg_T field. Thus, along these lines, the solution of Bjerhammar and of the author according to (7) and (8) are developable, a preference that does not hold for the solution by (5) and (6).

4. The superposition with the potential of the mountain masses

Within the scope of this superposition, T is replaced by

$$M = T - B \tag{41}$$

in (38) - (40). Some rearrangements follow, they lead to (42), (cf. equation (77) of [7], [9]),

$$T = \frac{R}{4\pi} \iint_{\omega} [\Delta g_T + C]\, S(\psi)\, d\omega + \sum_{i=4,6,7,8} \chi_i \,, \tag{42}$$

C has the expression (17). Further,

$$\chi_4 = [B]'' = B - 4\pi f \varrho R h_P - f \varrho R^2 \iint_{\omega} (h_Q - h_P) \frac{1}{e_0} \, d\omega \,, \tag{43}$$

$$\chi_6 = - \frac{1}{4\pi R} \iint_{\omega} \frac{\partial M}{\partial \psi} (h_Q - h_P)\, S^*(\psi)\, d\omega \,, \tag{44}$$

$$\chi_7 = \frac{R}{4\pi} \iint_{\omega} \left(\left[\frac{\partial B}{\partial r} + \frac{2}{R} B \right]'' - C \right) S(\psi)\, d\omega \,, \tag{45}$$

$$\chi_8 = \frac{R}{4\pi} \iint_{\omega} G \left(\frac{\partial \mu_1}{\partial \bar{x}} + \frac{\partial \mu_2}{\partial \bar{y}} - \mu_1 \frac{\mathrm{tg}\, \varphi}{R} \right) (h_Q - h_P)\, S(\psi)\, d\omega \,. \tag{46}$$

$$\mu_1 = - \frac{\partial M}{\partial x} \frac{1}{\gamma} \,, \quad \mu_2 = - \frac{\partial M}{\partial y} \frac{1}{\gamma} \,; \tag{47}$$

$$S^*(\psi) = \frac{1}{2} \frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} + \frac{dS}{d\psi} \,. \tag{48}$$

Fig. 2 shows the course of S^* and of $\sin\psi \, S^*$. The expressions (42) to (46) consist of parts which are linear in the height h and of parts which are non - linear in h. The symbol [ ]'' denominates the regulation that the non - linear parts of the term within the brackets are considered only, (cf. [7]), their values are discussed later on.

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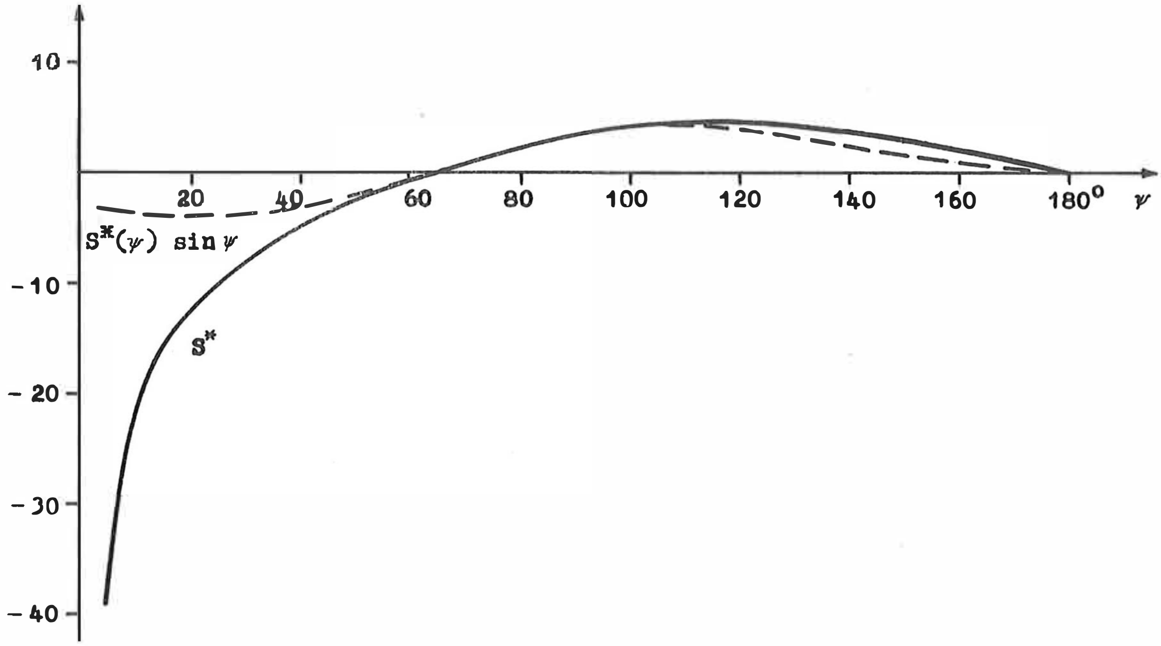

Fig. 2: The course of the functions $S^*(\psi)$ (solid line) and $S^*(\psi)\sin\psi$ (broken line), (cf. (44), (48)).

Within the course of the mathematical deductions that reveal the validity of (42) to (46), the standard density mountain masses above the sea level are condensed at the surface $\varkappa$ of the globe following the ideas of the Helmert condensation procedure (cf. [14]). The thus obtained spherical surface distribution ϱh gives rise to the potential B_c in the exterior space of $\varkappa$,

$$(B_c)_P = f\varrho \iint_{\varkappa} \frac{1}{e(P, Q_{\varkappa})}\, h\, d\varkappa \quad , \tag{49}$$

$$d\varkappa = R^2\, d\omega \quad . \tag{50}$$

ϱ is the standard density. e is the straight distance between the points P and $Q_\varkappa$. P is here a point in the exterior space of $\varkappa$ or a point situated on $\varkappa$. $Q_\varkappa$ is the point moving over $\varkappa$ in the integration, Fig. 1. $(B_c)_\varkappa$ denotes the potential of the condensation masses if the test point P is subsided down to the sphere $\varkappa$. It has the following rigorous Stokes equation, (2),

$$[B_c]_\varkappa = \frac{R}{4\pi} \iint_\omega \Delta g_{B_c}\, S(\psi)\, d\omega \quad , \tag{51}$$

with

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$$\Delta g_{B_c} = -\left(\frac{\partial B_c}{\partial r} + \frac{2}{R} B_c \right)_{\varkappa} \quad . \tag{52}$$

B_c and $\left[\frac{\partial B_c}{\partial r} \right]_{\varkappa}$ depend on the surface distribution ϱ h, (cf. [7], [16]),

$$B_c = \Lambda_1 + \Lambda_2 \quad , \tag{53}$$

$$\frac{\partial B_c}{\partial r} = \Lambda_3 + \Lambda_4 \quad , \tag{54}$$

with

$$\Lambda_1 = 4\pi f \varrho Rh \quad , \tag{55}$$

$$\Lambda_2 = f \varrho R^2 \iint_{\omega} (h_Q - h_P) \frac{1}{e_0} d\omega \quad , \tag{56}$$

$$\Lambda_3 = -4\pi f \varrho h \quad , \tag{57}$$

$$\Lambda_4 = -\frac{1}{2} f \varrho R \iint_{\omega} (h_Q - h_P) \frac{1}{e_0} d\omega \quad . \tag{58}$$

The relations (52) to (58) transform (51) into

$$\Lambda_1 + \Lambda_2 = -\frac{R}{4\pi} \iint_{\omega} \left[\Lambda_3 + \Lambda_4 + \frac{2}{R} (\Lambda_1 + \Lambda_2) \right] S(\psi) \, d\omega \; , \tag{59}$$

with

$$\Delta g_{B_c} = -\Lambda_3 - \Lambda_4 - \frac{2}{R} (\Lambda_1 + \Lambda_2) \; . \tag{60}$$

In order to avoid certain misunderstandings, some peculiarities connected with the potential B are to be discussed. The value of $\varrho = 2{,}65$ [g cm^{-3}] is the rigorous amount of the standard density. Therefore, the potential B can be computed precisely. It is a precisely defined model potential. The introduction of the potential B into (41) and (38) is not accompanied by a loss of precision in the result, by no means. The argumentation that the standard density ϱ should here better be replaced by the real geological density of the mountain masses, - in order to have a better approximation of the reality - , this argumentation is absolutely misleading. It does not meet the problem. Geological corrections of the standard density are not required. They cannot better the result. There is no need to introduce them. Needless to say that the introduction of geological corrections of the ϱ value would mean to carry

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out a lot of unnecessary work.

The potential B and the first derivatives of it at the point P on the surface σ can be expressed by the amounts of B_c, $\frac{\partial B_c}{\partial r}$, $\frac{\partial B_c}{\partial \bar{x}}$, $\frac{\partial B_c}{\partial \bar{y}}$ at the point $P_{\varkappa}$ on the sphere $\varkappa$ and by the addition of the supplementary terms $[B]''$, $[\frac{\partial B}{\partial r}]''$, $[\frac{\partial B}{\partial x}]''$, $[\frac{\partial B}{\partial y}]''$; (cf. Fig. 1). Accounting for (53) and (54), the following relations are obtained.

$$(B)_P = \Lambda_1 + \Lambda_2 + [B]'' , \tag{61}$$

$$(\frac{\partial B}{\partial r})_P = \Lambda_3 + \Lambda_4 + [\frac{\partial B}{\partial r}]'' . \tag{62}$$

The equations (1) and (60) reveal

$$\Delta g_B = \Delta g_{B_c} - [\frac{\partial B}{\partial r} + \frac{2}{r} B]'' + 2 \frac{h}{R^2} B . \tag{63}$$

The above relations (53) to (63) are needed to transform the equation (38) into (42). Indeed, T can be substituted by M in all the terms of (38), (cf. (41)), and then, thereafter, along the reverse procedure, on the other hand, M can be decomposed into T and B. Furthermore, B and $\frac{\partial B}{\partial r}$ can be substituted by (61) and (62). Finally, (42) is obtained and, by means of these deductions, the iteration term $C_1(T)$ is eliminated or, to be more precise, it is lowered down to the negligible amount $C_1(T - B) = C_1(M)$, (cf. (9), (38), (46)).

5. The relation between the iteration procedure term and the plane topographical reduction of the gravity

Before the detailed numerical evaluation of the small terms (43), (44), (45), (46) is executed, the relation between C and $C_1(T)$ is to be derived now explicitely, (cf. (17), (9)). To follow up this problem, T is in (38) replaced by B. Further on, the relations (53) - (63) are introduced and, after that, some rearrangements are undertaken. Finally, the relation wanted to have is found,

$$\frac{R}{4\pi} \iint_{\omega} [C - C_1(T)] S(\psi) d\omega = \chi_3^*(B) - \chi_4(B) + \chi_6(B) - \chi_8(M) . \tag{64}$$

The second, third and fourth term on the right hand side of (64) have the formulas (43), (44) and (46). The first term has the expression

$$\chi_3^*(B) = \frac{R}{4\pi} \iint_{\omega} [\frac{1}{2} (3 \frac{h_Q}{R} - \frac{h_P}{R}) \Delta g_B + 2 \frac{h_Q}{R^2} B] S(\psi) d\omega . \tag{65}$$

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The amount of the closed expression on the right hand side of (64) is very small. It can be neglected in most cases, as it will be seen by means of the extensive developments from (72) to (133) later on.

6. The closed solution for the geodetic boundary value problem

The relation (42) gives the surface value of the potential T and the height anomaly ζ, (3), as a solution of the boundary value problem of Molodenskij. In a similar way, the components of the plumb - line deflection ϑ_1 and ϑ_2 have the expressions, (4),

$$\left.\begin{matrix}\vartheta_1 \\ \vartheta_2\end{matrix}\right\} = \frac{1}{4\pi\gamma} \iint\limits_{\omega} [\Delta g_T + C] \frac{dS}{d\psi} \begin{Bmatrix}\cos\alpha \\ \sin\alpha\end{Bmatrix} d\omega - \frac{1}{\gamma} \sum_{i=4,6,7,8} \frac{\partial}{\partial x,y} \chi_i \quad . \tag{66}$$

α is here the clockwise counted azimuth.

The local south-north and west-east components of the slope of the topography ν_x and ν_y at the test point P do not appear in (66), see Fig. 1. These values have no effect on the deflections ϑ_1 and ϑ_2. These local values of ν_x and ν_y prove as the multiplication factors of some certain expressions of the following structure,

$$\int\limits_{\psi=0} f(\psi)\, d\psi \int\limits_{\alpha=0}^{2\pi} \cos\alpha \, d\alpha = 0 \; . \tag{67}$$

Therfore, the local amounts of ν_x and ν_y at the test point P have no effect on the computed values of T and ϑ_1 and ϑ_2 for the point P.

Now, the essential problem, the evaluation of the numerical amounts of χ_i and $\frac{\partial}{\partial x,y} \chi_i$, (i = 4,6,7,8), will be in the fore. It is to be proved that these values can be neglected in all the geodetic routine applications, (cf. (42) - (46), (66)),

$$\sum_{i=4,6,7,8} \chi_i \cong 0 \; , \tag{68}$$

$$\sum_{i=4,6,7,8} \frac{\partial}{\partial x,y} \chi_i \cong 0 \; . \tag{69}$$

Thus, the following formulas suffice for all the geodetic routine applications,

$$T = \frac{R}{4\pi} \iint\limits_{\omega} [\Delta g_T + C] \, S(\psi) \, d\omega , \tag{70}$$

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$$\left.\begin{array}{l}\vartheta_1\\ \\ \vartheta_2\end{array}\right\} = \frac{1}{4\pi\gamma}\iint_{\omega}[\Delta g_T + C]\,\frac{dS}{d\psi}\left\{\begin{array}{c}\cos\alpha\\ \\ \sin\alpha\end{array}\right\} d\omega \,. \tag{71}$$

7. The numerical evaluation of the residual term of the closed solution

The prove of the validity of the relations (68) and (69) is governed by the evaluation of the following amounts, (cf. (43),(45),(46),(9)),

$$\Xi_1 = [B]'' = (B)_\sigma - (B_c)_\varkappa \,, \tag{72}$$

$$\Xi_2 = \left[\frac{\partial B}{\partial r}\right]'' = \left(\frac{\partial B}{\partial r}\right)_\sigma - \left(\frac{\partial B_c}{\partial r}\right)_\varkappa \,, \tag{73}$$

$$\Xi_3 = \left[\frac{\partial B}{\partial x,y}\right]'' = \left(\frac{\partial B}{\partial x,y}\right)_\sigma - \left(\frac{\partial B_c}{\partial \bar{x},\bar{y}}\right)_\varkappa \,, \tag{74}$$

$$\Xi_4 = C_1(M) = G\,(h_Q - h_P)\left(\frac{\partial\mu_1}{\partial\bar{x}} + \frac{\partial\mu_2}{\partial\bar{y}} + \frac{\operatorname{tg}\varphi}{R}\,\mu_1\right) . \tag{75}$$

Later on, the amount of $\Xi_5 = \chi_6$ is evaluated also, (cf. (44)). The knowledge of the amounts of Ξ_i , (i = 1,2,3,4), allows the computation of the amounts of χ_4 , χ_7, χ_8 and that of

$$\frac{\partial}{\partial x,y}\;\chi_4,\,\chi_7\;,\chi_8 \;. \tag{75 a}$$

Consequently, the essential problem is now to find convenient mathematical expressions for these Ξ_i values in terms of the topographical heights h.

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7.1. The amount of the potential supplementary to the potential of the condensation method

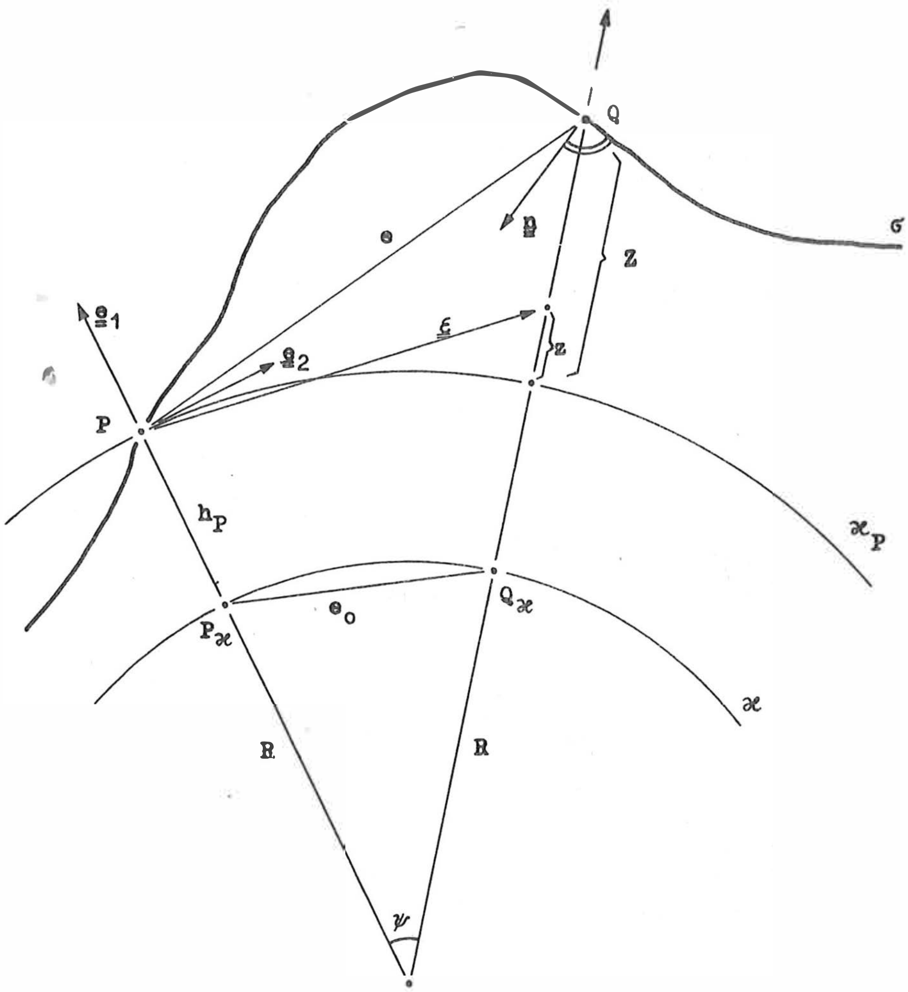

Fig. 3

Fig.3: The geocentric spheres $\varkappa$ and $\varkappa_P$. The surface of the Earth σ and the height h of σ above $\varkappa$. The height difference Z between the surface points Q and P, and the inclined straight distance from P to Q, e. The straight distance e_o between the points $P_\varkappa$ and $Q_\varkappa$ on $\varkappa$. The inclined straight distance ε which is variable with the height difference z above the point P. The two orthogonal unit vectors $\underline{\underline{e}}_1$ and $\underline{\underline{e}}_2$. The unit vector $\underline{\underline{n}}$ of the normal of the surface σ, positive into the interior.

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At first, the Ξ_1 value is to be computed, (cf. (72),(61),(53)).

$$\Xi_1 = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r=R}^{R+h_P+Z} \left(\frac{1}{\varepsilon}\right)_P r^2 \sin\psi \, dr \, d\psi \, d\alpha - 4\pi f \varrho R h_P -$$

$$- f\varrho R^2 \iint_{\omega} Z \left(\frac{1}{e_0}\right)_{P\varkappa} d\omega \quad . \tag{76}$$

Fig. 3 visualizes the following relation ,

$$\varepsilon^2 = (R + h_P)^2 + (R + h_P + z)^2 - 2 (R + h_P) (R + h_P + z) \cos\psi \quad . \tag{77}$$

Some rearrangements of (77) lead to

$$\varepsilon^2 = e_0^2 + z^2 + e_0^2 \left(\frac{2 h_P + z}{R} + \frac{h_P^2 + z h_P}{R^2} \right) \quad . \tag{78}$$

The abbreviations (79) and (80)

$$D_1 \cong \frac{e_0^2}{e_0^2 + z^2} \frac{z}{R} \ll 1 \quad , \tag{79}$$

and

$$D_2 \cong \frac{e_0^2}{e_0^2 + z^2} \cdot 2 \frac{h_P}{R} \ll 1 \quad . \tag{80}$$

transform (78) into

$$\frac{1}{\varepsilon} = \frac{1}{\sqrt{e_0^2 + z^2}} \left(1 - \frac{1}{2} D_1 - \frac{1}{2} D_2 \right) . \tag{81}$$

(76) and (81) and some additional mathematical considerations lead to

$$\Xi_1 = f\varrho \sum_{i=1}^{5} \iint_{\omega} \Theta_i \, d\omega \quad , \tag{82}$$

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with, (cf. [11]),

$$\Theta_1 = (R + h_P)^2 \operatorname{arsinh} \tau \quad , \tag{83}$$

$$\Theta_2 = \frac{1}{2} R e_0 \left(\frac{1}{\sqrt{1+\tau^2}} - 1 \right) \quad , \tag{84}$$

$$\Theta_3 = - R h_P \frac{\tau}{\sqrt{1+\tau^2}} \quad , \tag{85}$$

$$\Theta_4 = 2 R e_0 \left(\sqrt{1+\tau^2} - 1 \right) \quad , \tag{86}$$

$$\Theta_5 = - R^2 \tau \quad ; \tag{87}$$

$$\tau = \frac{Z}{e_0} = \frac{h_Q - h_P}{e_0} \quad . \tag{88}$$

The amount of Ξ_1 was computed for two simple models.

The first model has a test point P at the summit of a circular cone with the height h_P = 2 km and with a circular base of the radius c = 2 km. (88) gives the relation $\tau = -1$. The integration over the area of this cone, $0 \leq \alpha \leq 2\pi$ and $0 \leq e_0 \leq c$, reveals $\Xi_1 \frac{1}{G}$ = 3 cm, $\Xi_1 \frac{2}{R}$ = 10 μgal. These amounts have not to be taken into account.

The second model has the following parameters: $h_P = 0$, e_0 = 2 000 km, Z = 3 km, $R^2\Delta\omega = (600\ \text{km})^2$. For this example the above formulas give $\Xi_1 \frac{1}{G}$ = 0.4 cm, $\Xi_1 \frac{2}{R}$ = 1 μgal. These amounts are negligible.

7.2. The plane topographical reduction of the gravity and the free-air anomaly caused by the supplement potential of the condensation method

The expression

$$\left[\frac{\partial B}{\partial r} + \frac{2}{R} B \right]'' - C = \delta C \tag{89}$$

is the next function the amount of which is to be computed, (cf. (17),(45),(72),(73)). The first step to reach this aim is the evaluation of the amount of Ξ_2 , (cf. (73), (49), Fig. 1 and 3),

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$$\mathcal{E}_2 = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r'=R}^{R+h_P+Z} \left(\frac{\partial\, 1/\varepsilon}{\partial r}\right)_P r'^2 \sin\psi \, dr' \, d\psi \, d\alpha \; -$$

$$- f\varrho R^2 \iint_\omega h \left(\frac{\partial\, 1/e_0}{\partial r}\right)_{P_\varkappa} d\omega + 2\pi f\varrho h_P \quad . \qquad (90)$$

In the derivation of (90), the limit relations are observed that are valid for the potential of a surface distribution or for the potential of a single layer, (cf.[16]). For the expressions in the parentheses of the two integrands of (90), some special developments are needed in order to find them in terms of the heights differences z, (cf. Fig. 3). The following relations (91) and (92) can be deduced easily, as a view on Fig. 3 does show, accounting for $\varepsilon^2 = \underline{\underline{\varepsilon}}^2$,

$$\frac{\partial}{\partial r}\frac{1}{\varepsilon} = \frac{\underline{\underline{\varepsilon}}\;\underline{\underline{e}}_1}{\varepsilon^3} \quad , \qquad (91)$$

$$\underline{\underline{\varepsilon}} = [(R + h_P + z)\cos\psi - (R + h_P)]\,\underline{\underline{e}}_1 + (R + h_P + z)\sin\psi\;\underline{\underline{e}}_2 \quad . \qquad (92)$$

Some rearrangements of (91) and (92) change (91) into

$$\frac{\partial}{\partial r}\frac{1}{\varepsilon} = \frac{z - 2(R + h_P + z)\sin^2\frac{\psi}{2}}{(e_P^2 + z^2)^{3/2}}\left(1 - \frac{3}{2}D_3\right) \quad , \qquad (93)$$

with

$$e_P^2 = 2(R + h_P)\sin\frac{\psi}{2} \qquad (94)$$

and

$$D_3 \cong \frac{e_P^2}{e_P^2 + z^2}\cdot\frac{z}{R} \ll 1 \quad . \qquad (95)$$

(93) is introduced in the first integrand of (90). For the second integrand, the following substitution is recommended ,

$$\left(\frac{\partial}{\partial r}\frac{1}{e_0}\right)_{P_\varkappa} = -\frac{1}{2\,R\,e_0} \quad . \qquad (96)$$

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The relations (17), (82), (90) - (96) lead to the following representation of (89),

$$\delta C = \sum_{i=1}^{4} \delta_i C = \left[-\frac{\partial B}{\partial r} + \frac{2}{R} B \right]'' - C \quad . \tag{97}$$

$$\delta_1 C = f\varrho \int_{\psi=0}^{\pi} d\psi \int_{\alpha=0}^{2\pi} d\alpha \int_{z=0}^{Z} \Phi_1 \, dz \quad , \tag{98}$$

$$\delta_2 C = \frac{1}{2} f \varrho R \iint_{\omega} \frac{1}{e_0} \quad \Phi_2 \, d\omega \quad , \tag{99}$$

$$\delta_3 C = 2 f \varrho \frac{1}{R} \sum_{i=1}^{5} \iint_{\omega} \Theta_i \, d\omega \quad , \tag{100}$$

$$\delta_4 C = 4 \pi f \varrho h_P \frac{h_P}{R} \quad ; \tag{101}$$

with

$$\Phi_1 = \sin\psi \, (R_P + z)^2 \frac{z\,(1 - \frac{3}{2} D_3)}{(e_P^2 + z^2)^{3/2}} - \psi R_P^2 \frac{z}{(R_P^2 \psi^2 + z^2)^{3/2}} \quad , \tag{102}$$

$R_P = R + h_P$, and

$$\Phi_2 = \int_{z=0}^{Z} \left[1 - \frac{(R_P + z)^3}{R^3} \cdot \frac{e_o^3}{(e_P^2 + z^2)^{3/2}} (1 - \frac{3}{2} D_3) \right] dz \quad . \tag{103}$$

The numerical amounts of $\delta_1 C$ and $\delta_2 C$ are computed for 3 simple models.

The first model has the following parameters: $h_P = 0$, $R\psi = 5$ km, $Z = 2$ km, $R^2 \Delta\omega = (4 \text{ km})^2$. The integrals (98) and (99) are computed for this model according to the mean value theorem of the integral calculus. The results are $\delta_1 C = 0.5$ μgal, $\delta_2 C = 0.5$ μgal. These amounts are insignificant.

The second model is described as follows : $h_P = 0$, $R\psi = 4\ 000$ km, $Z = 3$ km, $R^2 \Delta\omega = (1\ 000 \text{ km})^2$.

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The following results are obtained, $\delta_1 C = -0.02\,\mu gal$, $\delta_2 C = -0.4\,\mu gal$. These amounts can be neglected.

Now, the computation of $\delta_3 C$ follows, according (100). A comparison with (82) shows

$$\delta_3 C = \frac{2}{R}\ \Xi_1 \quad . \tag{104}$$

This term has proved to be within the noise of the gravity measurements, (82).

Thus, it is guaranteed that the sum $\sum_{i=1}^{3} \delta_i C$ must not be taken into account, (cf. (97)).

As to $\delta_4 C$, the rather simple formula (101) leads to $\delta_4 C = 0.1$ mgal for $h_P = 2$ km . Therefore, also the term $\delta_4 C$ seems to be within the noise of the method in all the routine applications.

7.3. The deflections of the vertical caused by the supplement potential of the potential of the condensation method

The next step consists in the computation of the amount of Ξ_3 , (cf. (74)). Ξ_3 is of importance for the determination of the deflection of the vertical, (cf. (66)). (74) and (76) give, (cf. Fig. 1 and 3),

$$\Xi_3 = f\varrho \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \int_{r=R}^{R+h_P+Z} \left(\frac{\partial\, ^1/\varepsilon}{\partial\, x,\ y}\right)_P r^2 \sin\psi\, dr\, d\psi\, d\alpha -$$

$$- f\varrho R^2 \iint_{\omega} h \left(\frac{\partial\, ^1/e_0}{\partial\, \bar{x},\ \bar{\bar{y}}}\right)_{P_{\varkappa}} d\omega \, . \tag{105}$$

The question, as to whether a specific mathematical problem requires the introduction of dx, dy or $d\bar{x}$, $d\bar{y}$ instead, as the independent variables, this question was already discussed earlier in context with Fig.1.

The horizontal derivation of ε along the sphere $\varkappa_P$ and of e_0 along the sphere $\varkappa$ leads to the following developments, (cf. Fig. 1 and 3). The relation (78),

$$\varepsilon^2 = z^2 + e_0^2\,(1 + D_4) \quad , \tag{106}$$

with

$$D_4 = \frac{2\,h_P + z}{R} + \frac{h_P^2 + z\,h_P}{R^2} \cong \frac{2\,h_P + z}{R} \ll 1 \quad , \tag{107}$$

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gives

$$2\varepsilon \frac{\partial \varepsilon}{\partial \psi} \frac{\partial \psi}{\partial x} = 2\varepsilon \frac{\partial \varepsilon}{\partial x} = 2\, e_0 \frac{\partial e_0}{\partial \psi} \frac{\partial \psi}{\partial x} (1 + D_4) \quad . \tag{108}$$

Hence,

$$\frac{\partial \varepsilon}{\partial x} = \frac{\partial e_0}{\partial \psi} \frac{\partial \psi}{\partial x} \frac{e_0}{\varepsilon} (1 + D_4) \quad . \tag{109}$$

The equation (109) and

$$\frac{\partial \psi}{\partial x} = - \frac{1}{R + h_P} \cos \alpha \tag{110}$$

reveal

$$\frac{\partial \varepsilon}{\partial x} = - \cos \frac{\psi}{2} \cos \alpha \quad \frac{e_0}{\varepsilon} \ (1 + D_4)\ (1 - \frac{h_P}{R}) \quad . \tag{111}$$

On the sphere $\varkappa$, the following relations are valid,

$$\frac{\partial e_0}{\partial \psi} = R \cos \frac{\psi}{2} \quad , \tag{112}$$

$$\frac{\partial e_0}{\partial \bar{x}} = \frac{\partial e_0}{\partial \psi} \frac{\partial \psi}{\partial \bar{x}} \quad , \tag{113}$$

$$\frac{\partial \psi}{\partial \bar{x}} = - \frac{1}{R} \cos \alpha \quad , \tag{114}$$

$$\frac{\partial e_0}{\partial \bar{x}} = - \cos \frac{\psi}{2} \cos \alpha \ . \tag{114 a}$$

The above relations (111) and (114 a) lead to the following expressions for the two terms that are put in parentheses in the integrands of (105)

$$\left(\frac{\partial\, ^1/_\varepsilon}{\partial x,\ y} \right)_P = \cos \frac{\psi}{2} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} \frac{e_0}{\varepsilon^3} (1 + D_4 - \frac{h_P}{R}) \quad , \tag{115}$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$\left(\frac{\partial\, ^1/e_0}{\partial\, \bar{x}, \bar{y}} \right)_{P_{\varkappa}} = \cos \frac{\psi}{2} \begin{Bmatrix} \cos\alpha \\ \\ \sin\alpha \end{Bmatrix} \frac{1}{e_0^2} \quad . \tag{116}$$

The relations (115) and (116) are inserted into (105), the terms of the order $(^h/R)^2$ in D_4 are neglected and the following formulas for the computation of $\varXI_3$ are obtained,

$$\varXi_3 = f \varrho R \int_{\psi=0}^{\pi} \sin^2\psi \, d\psi \int_{\alpha=0}^{2\pi} \begin{Bmatrix} \cos\alpha \\ \\ \sin\alpha \end{Bmatrix} d\alpha \cdot \sum_{i=1}^{5} \varPsi_i \quad , \tag{117}$$

with

$$\varPsi_1 = \frac{(R + h_P)^3}{R} \cdot \frac{Z}{e_o^2 (e_o^2 + Z^2)^{1/2}} \quad , \tag{118}$$

$$\varPsi_2 = 3 R \left[\frac{1}{e_0} - \frac{1}{(e_o^2 + Z^2)^{1/2}} \right] \quad , \tag{119}$$

$$\varPsi_3 = - 3 R h_P \left(Z + \frac{2}{3} \frac{Z^3}{e_0^2} \right) \frac{1}{(e_o^2 + Z^2)^{3/2}} \quad , \tag{120}$$

$$\varPsi_4 = \frac{1}{2} R e_0^2 \left(\frac{1}{(e_o^2 + Z^2)^{3/2}} - \frac{1}{e_0^3} \right) \quad , \tag{121}$$

$$\varPsi_5 = - R^2 \frac{Z}{e_0^3} \quad . \tag{122}$$

The numerical amounts of $\varXi_3$ are computed for the two different Earth models.

The parameters of the first model treated above for the computation of $\delta_1 C$ and $\delta_2 C$, (cf. (98), (99)), resulted here the following amount:
$\varXi_3 \frac{\varrho''}{G} = 0.''3$; ($\alpha = 0^o$). Consequently, in extreme cases in the midst of the high mountains, it is possible that the amount of $\varXi_3 \frac{\varrho''}{G}$ effects the computed deflections of the vertical by more than 0.''1.

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The **parameters** of another model Earth, ($h_p = 0$, $R\psi = 100$ km, $Z = 3$ km, $R^2\Delta\omega = (40\ \text{km})^2$), lead to the amount $\Xi_3 \frac{\varrho''}{G} = 3'' \cdot 10^{-4}$. Thus, it can be concluded that the effect of the distant mountains on $\Xi_3 \frac{\varrho''}{G}$ is insignificant.

7.4. The theoretical error of the method for the elimination of the iteration procedure term

The next problem is the computation of the amount of Ξ_4 , (cf. (75)). To reach this aim, it is not intended here to compute the expressions for Ξ_4 in terms of the heights h of the mountain masses and in terms of the potential T. For the here following intentions, it is a great benefit that the amounts of the topographically reduced deflection components μ_1 and μ_2 can be taken directly from an austrian publication which contains the representation of a dense field of the μ_1 , μ_2 values for the area of the austrian Alps. This source brings in an excellent material for the here followed purposes, (cf. [10]). To be sure, the deflection values published in [10] enclose also the effects of the isostatic compensation masses, i. e. the mountain roots. However, there is no doubt, these mountain roots are situated in a depth of more than 30 km and, thus, they will have only a small and smoothed effect on the horizontal derivations of the deflections of the vertical at the surface σ .

For instance, a mountain mass of a base of 40 km square and of 2 km height has a well-defined isostatic mountain root. This root has a spatial extent of a bloc of $40 \times 40 \times 8.8\ \text{km}^3$, a density of $0.6\ \text{g cm}^{-3}$ and a center of gravity in a depth of 34.4 km below the sea level. The effect of this bloc shape mountain root alters the surface plumb-line deflections by not more than $3''$ if moving away from the epicenter of the mountain root as far as a distance of 40 km, as a short calculation does show. This source has a share in Ξ_4 that is not more than 0.4 mgal, for $h_Q - h_P = 1$ km, (cf. (75)). Consequently, for a rough estimation of the Ξ_4 values, it is admissable to introduce the topographically-isostatically reduced deflections of [10] instead of the topographically reduced μ_1 , μ_2 deflections.

In order to evaluate the amount of the differential quotient $\frac{\partial \mu_1}{\partial \bar{x}}$ which appears in the formula (75) of Ξ_4 , the μ_1 values situated within a narrow stripe along the meridian $\lambda = 14^\circ 30'$ are selected from the list in the austrian **publication**, (cf. [10]). These μ_1 values within the area $14^\circ 28' \leq \lambda \leq 14^\circ 32'$, $46^\circ 30' \leq \varphi \leq 48^\circ 30'$ are shown in Fig. 4.

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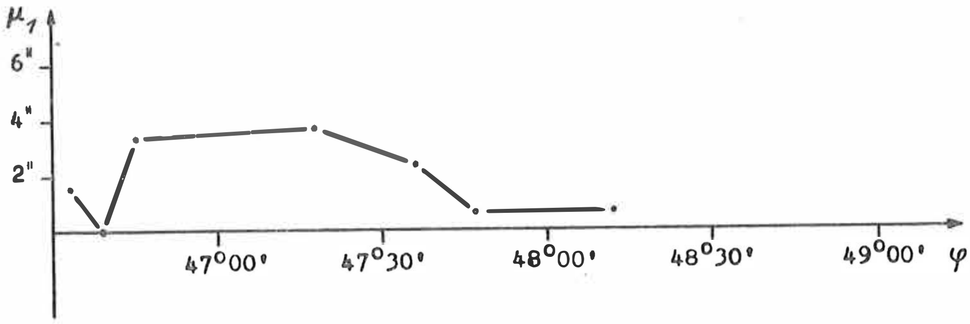

Fig. 4. The meridional components of the topographically - isostatically reduced plumb-line deflections along the meridian $\lambda = 14^\circ 30'$ in the area of the austrian Alps.

Fig. 4 visualizes that the μ_1 values variate over distances of about 200 km by an amount of about 4" for the extremely rugged topography of this sample area. According to (75), this fact influences the Ξ_4 term by not more than 0.1 mgal if $h_Q - h_P = 1$ km. Over distances of 20 km, the μ_1 values variate in the mean by about 2", (cf. Fig. 4). Hence, an effect of the amplitude of 0.5 mgal and of the wave length of 20 km follows for Ξ_4 , if $h_Q - h_P = 1$ km.

In a similar way, the μ_2 values are selected for a narrow stripe along the parallel of latitude of $\varphi = 47^\circ 00'$. For the area $46^\circ 58' \leq \varphi \leq 47^\circ 02'$, $12^\circ\ 0' \leq \lambda \leq 16^\circ\ 0'$, these topographically - isostatically reduced μ_2 values are shown by Fig. 5.

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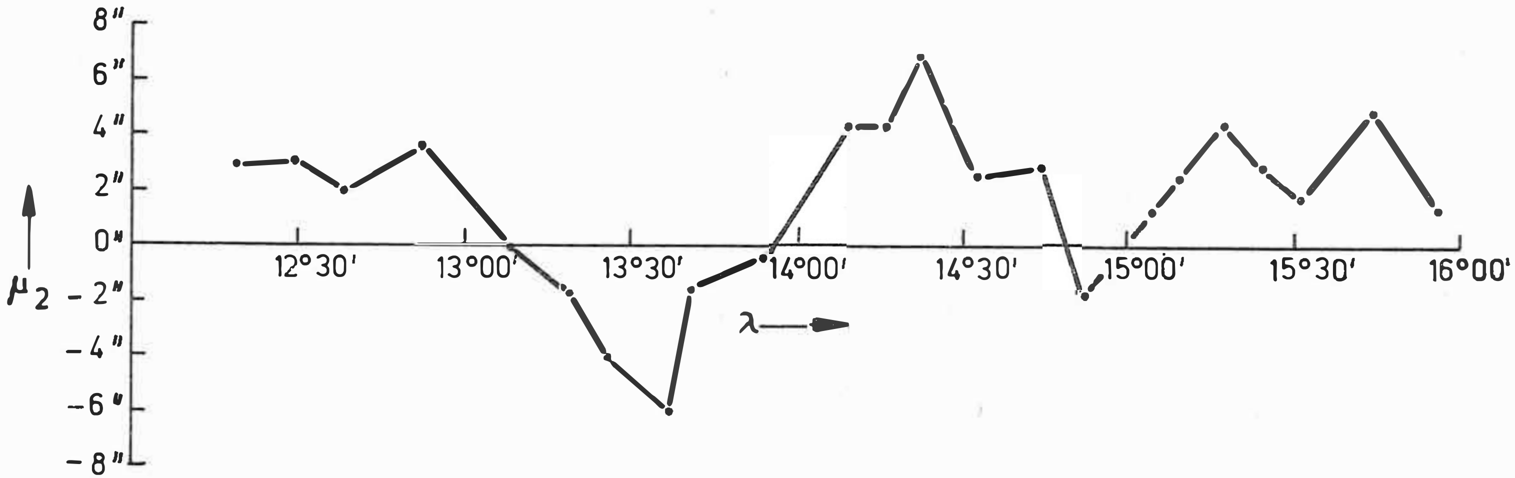

Fig. 5. The west-east components of the topographically - isostatically reduced plumb-line deflections along the parallel of latitude of $\varphi = 47^{o}oo'$ in the area of the austrian Alps

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As can be taken from Fig. 5, the μ_2 values variate over distances of 300 km by about 6". As a result of this fact, a shift of about 0.1 mgal follows for Ξ_4 if $h_Q - h_P = 1$ km. As to the shorter waves, the μ_2 values variate over a distance of 40 km likewise by about 6", (cf. Fig. 5). This fact influences the Ξ_4 amount by a wave of the amplitude of 0.8 mgal, in case $h_Q - h_P = 1$ km.

Finally, the last term in the expression for Ξ_4, (75), which has the shape

$$- G\,(h_Q - h_P)\,\frac{\mathrm{tg}\,\varphi}{R}\,\mu_1 \quad , \qquad (123)$$

is to be considered. For $\mathrm{tg}\varphi = 1$, $\mu_1 = 10''$, $h_Q - h_P = 1$ km, this term reaches about 10 μgal. Therefore, the expression (123) has an unimportant amount.

The numerical evaluation of the amount of $\Xi_5 = \chi_6$ or $\frac{1}{G}\chi_6$ and of the amount of the horizontal derivatives $\frac{1}{G}\frac{\partial}{\partial x,y}\chi_6$ turns out to be rather uncomplicated, (cf. (44), (68), (69)). For the circular area $0 \leq R\psi \leq 1\,000$ km around the test point P, the following numerical values can be introduced in the integrand of (44),
$\frac{1}{G}\frac{\partial M}{R\,\partial\psi} = \frac{0.05\text{ km}}{1\,000\text{ km}} = 5 \cdot 10^{-5}$, $h_Q - h_P = 1$ km, $|S^*(\psi)\sin\psi| = 3$, (cf. Fig. 2).
According to (44), these values reveal

$$\frac{1}{G}\chi_6 \longrightarrow 1\text{ cm}$$

for the integration over this inner circle. Now, the exterior area $R\psi \geq 1\,000$ km is considered. This part of the surface of the Earth is divided into about $5 \cdot 10^4$ compartments of the size $1^\circ \times 1^\circ$. The long periodic part of the horizontal derivative of $\frac{1}{G}M$ is governed by the effect of the isostatic mountain roots. According to Pratt, in the Himalaya and Tien Shan area, the deflections caused by the B potential amount to about 15". For this area, the T potential leads to deflections of about 5", (cf.[22]). Thus, the mountain roots have there an impact on the deflections of about 10". Here, the following parameters are introduced into (44), for the integration over $R\psi \geq 1\,000$ km,

$$\frac{1}{G}\frac{\partial M}{R\,\partial\psi} = 20'' = 1 \cdot 10^{-4} , \qquad h_Q - h_P = 1\text{ km}.$$

These values are understood as parameters that change as random variates in case of a transition from anyone compartment to another compartment. In the integrand of (44), the global mean value of $|S^*(\psi)|$ will be smaller than about 15, (cf. Fig. 2). Thus, an uncomplicated evaluation of the integral (44) reveals that the integration over the exterior area, $R\psi \geq 1\,000$ km, will contribute to the value of $\frac{1}{G}\chi_6$ an amount that is smaller than 0.7 cm.

The influence of the term χ_6 on the deflections of the vertical is

$$\frac{1}{G}\frac{\partial}{\partial x,y}\chi_6 = \frac{1}{G}\frac{\Delta\chi_6}{\Delta x, \Delta y} \quad ,$$

substituting the differential quotient by the corresponding difference quotient. For the evaluation of the amount of

$$\frac{1}{G}\frac{\partial}{\partial x,y}\chi_6 \quad ,$$

a difference quotient of the amount of

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$$\frac{1\ \text{cm}}{100\ \text{km}} = \frac{1}{G}\ \frac{\Delta\ \chi_6}{\Delta x, \Delta y}$$

is in keeping with the above evaluation of the amount of χ_6. Therefore, the impact of χ_6 on the deflection of vertical will not surmount the value of about

$$\varrho''\ \frac{1}{G}\ \frac{\partial}{\partial x, y}\ \chi_6 = 2'' \cdot 10^{-2} .$$

Summarizing, the term χ_6 is negligible in any case as so as the horizontal derivatives of χ_6.

7.5. The low degree harmonics of a height - dependent part of the closed solution

Furthermore, the evaluation of $\Phi_2(M)$ and of $\delta\Phi_2(M) = \chi_2(M)$ should not be forgotten, (cf. equations (33), (54), (56), (89 a) of [7]). $\Phi_2(M)$ has the expression

$$\Phi_2(M) = -\frac{1}{8\pi} \iint \frac{\partial M}{R\,\partial\psi} (h_Q - h_P) \frac{\cos\frac{\psi}{2}}{\sin^2\frac{\psi}{2}}\, d\omega \quad . \qquad (123\ a)$$

Compairing the expression for $\Phi_2(M)$, (123 a), with the Stokes integral, (2) (3), for the near surroundings of the test point P, the impact of $\Phi_2(M)$ on the height anomaly ξ can be interpreted as a shift of the average value of the free-air anomaly. It is a shift by the amount of

$$\mu\ (h_Q - h_P)\ \frac{G}{e_0} \quad . \qquad (123\ b)$$

In order to evaluate the amount of the expression of $\chi_2(M)$, the spherical harmonics of the function $\Phi_2(M)$ are of special interest, (cf. (123 a)), (cf. equation (54) of [7]). The low degree harmonics of $\Phi_2(M)$ are in the fore, $n \leq 10$. Thus, a division of the surface of the Earth into compartments of $20^o \times 20^o$ size is to be considered. The average value of (123 b) for the area of the inner circle, $0 \leq R\psi \leq 1\ 000$ km, is computed by

$$\mu = \frac{0.05\ \text{km}}{1\ 000\ \text{km}} ,$$

$$h_Q - h_P = 1\ \text{km} ,$$

$$e_0 = 500\ \text{km}.$$

These parameter values reveal an average amount of 0.1 mgal for the supplements to the free-air anomalies, (123 b). In computing the low degree Stokes constants in the spherical harmonics development for $\chi_2(M)$ according to the equation (54) of [7], this supplementary gravity amount of 0.1 mgal contributes not by its full amount, on no account, to the integrations over the central circle of $0 \leq R\psi \leq 1\ 000$ km coming up

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in this context. On the contrary, the reduced amount of

$$0.1 \, \frac{3}{2} \, \frac{1}{n-1} \text{ mgal}$$

is effective in the determination of the Stokes constants only. n is again the degree of the spherical harmonics. With n = 5 or n = 10, the supplementary effect caused by $\chi_2(M)$ follows to be of the order of 0.04 mgal or 0.02 mgal respectively. It can be neglected. The evaluation of χ_2 by the integration over the exterior area, $R\psi \geq 1000$ km, happens along similar ways as in case of χ_6 . The integration of (123 a) over the area $R\psi \geq 1000$ km amounts to a value not more than about 1 cm, it is easily found.

The function $\Phi_2(M)$ takes an impact on $\chi_2(M)$ by the spherical harmonics of low degree, $(n \leq 10)$, the coefficients of which are for $\chi_2(M)$ by the factor

$$\frac{3}{2} \, \frac{1}{n-1}$$

smaller than the corresponding coefficients of the harmonics development for $\Phi_2(M)$. Thus, $\chi_2(M)$ is smaller than 1 cm.

8. The difference between the refined Bouguer anomalies and the free-air anomalies of the model Earth

At last, at the end of the numerical calculations elaborated above, the free-air anomalies Δg_M of the potential field M at the points on the surface of the Earth σ are in the fore now. These anomalies are discussed in some recent publications of the author, (cf. [7] [8]). At the cited places, it was claimed that the Δg_M values are in the vicinity of the usual Bouguer anomalies, $\Delta g_{Bou.}$, as far as the continental areas are considered. A short derivation was given there. It shows this fact to be plausible. Here, a detailed formula for the difference $\Delta g_M - \Delta g_{Bou.}$ is to be derived now. The relations (1) and (41) lead to

$$\Delta g_M = -\left(\frac{\partial M}{\partial r} + \frac{2}{r} M\right)_\sigma \quad . \tag{124}$$

The equation (41) gives

$$\Delta g_M = \Delta g_T - \Delta g_B \quad , \tag{125}$$

with

$$\Delta g_B = -\left(\frac{\partial B}{\partial r} + \frac{2}{r} B\right)_\sigma \quad . \tag{126}$$

The relation (126) is rearranged expressing Δg_B in terms of the height h. In order to persue this aim, the equations (61) and (62) for B_P and $\left(\frac{\partial B}{\partial r}\right)_P$ are introduced, as so as

$$\frac{2}{r} = \frac{2}{R} - \frac{2}{R} \cdot \frac{h_P}{R} \quad . \tag{127}$$

Further, the following relation is taken into account,

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$$\delta C = 0 \quad , \tag{128}$$

it derives from (97) and from the **attached** computations about (98) - (103). The re-arranged equation (125) has the following form,

$$\Delta g_M = \Delta g_T + C + 4\pi f \varrho h_P (1 - 2 \frac{h_P}{R}) +$$

$$+ \frac{3}{2} f \varrho R (1 - \frac{4}{3} \frac{h_P}{R}) \iint_\omega (h_Q - h_P) \frac{1}{e_0} \, d\omega \quad . \tag{129}$$

Some simple transformations of (129) lead to

$$\Delta g_M = \Delta g_T - 2\pi f \varrho h_P + C + \frac{3}{2} f \varrho R \iint_\omega h_Q \frac{1}{e_0} \, d\omega \quad . \tag{130}$$

In the course of the derivation of (130), and in view of the here discussed applications, it makes no difference whether h_P is understood as the height above the ellipsoid or above the geoid, since this effect exerts an impact on ($\Delta g_M - \Delta g_{Bou.}$) that is obviously negligible. The Bouguer anomalies have the following well-known relation, (cf. [13]),

$$\Delta g_{Bou.} = \Delta g_T + C - 2\pi f \varrho h_P \quad . \tag{131}$$

Since C is added in (131), $\Delta g_{Bou.}$ is the so-called refined Bouguer anomaly, (cf.[19]). Thus, the difference between the two considered systems of anomalies has the subsequent mathematical relation,

$$\Delta g_M - \Delta g_{Bou.} = \beta \quad , \tag{132}$$

with

$$\beta = \frac{3}{2} f \varrho R \iint_\omega h_Q \frac{1}{e_0} \, d\omega \quad . \tag{133}$$

The course of the function β is considered and computed for a simple Earth model which consists of a mountain mass of 2 km height and of a base of 50 km square. The value of β , (133), depends from the distance of the mountain, as it is shown in Fig. 6.

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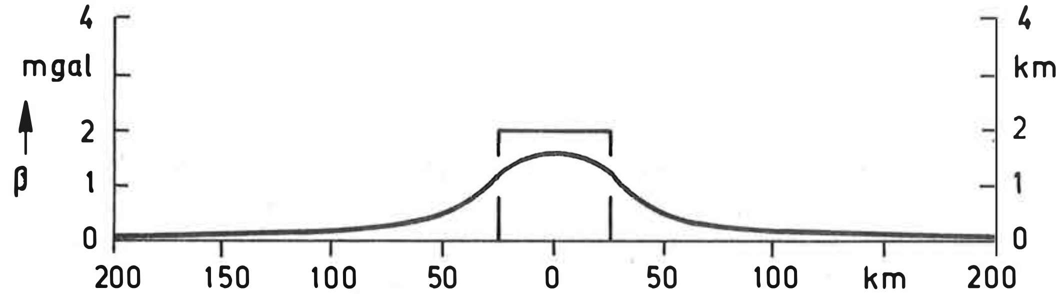

Fig. 6. The difference between the refined Bouguer anomalies and the model anomalies Δg_M . It is caused by a mountain bloc of a height of 2 km and of a base of 50 km square.

The β values are small and smoothed, the maximal amount is 1.6 mgal. It can be taken from Fig. 6. In the geodetic applications, the β values are not of direct interest, but just that effect is in the fore that the β values exert on the height gradient of the gravity anomalies Δg_M , (cf. [7] [8] [19]).

Furthermore, the β values have an impact on the computed horizontal variations of the plumb-line deflections, (cf. (75) and [7]). This impact is of direct interest also. Indeed , the effect of the β values will shift the Ξ_4 term by not more than about o.1 mgal, as a short computation shows by (13). Thus, this source has an impact that can be neglected, (cf. (75)).

9. Conclusion

Summarizing, the formulas (70) and (71) fulfill all the requirements of the geodetic routine applications.

Perhaps, in certain special applications, it is intended to have the formulas for T and ϑ_1 , ϑ_2 even more precise than the formulas (70) and (71) are. In this case, the addition of $C_1(M)$ to the free-air anomalies will be the next step to better the formulas (70) and (71). $C_1(M)$ consists of several constituents. In order to give a rough characterization of these constituents, the long wave part and the short wave part should be in the fore here. $C_1(M)$ has a long wave component caused by the isostatic mountain roots. This component can reach an amplitude of about 1 mgal. Further, $C_1(M)$ has a short wave part. It has a wave length that is not longer than about 40 km, and it has an amplitude of about 1 mgal. The existence of these short wave lengthes is connected with areas of high mountains. In the lowlands, $C_1(M)$ is connected with areas that have Bouguer anomalies of relative great amplitudes.

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Obviously, $C_1(M)$ is within the noise of the observed gravity field, in most cases.

Thus, as a result of all the above developments, it is possible to get control of the solution of the geodetic boundary value problem. There is an insight of clear transparency into the theoretical residual error. This residual is very small. The solution does meet all requirements of the present and of the future. The solution is free of any hypothesis about the density of the geological masses, and it is in keeping with the proved uniqueness of the problem, (cf. [4]).

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C. Considerations about the mixed boundary value problems of the geodesy

Contents

DOI: https://doi.org/10.2312/zipe.1986.084

Page

Summary

The second mixed boundary value problem has different types of boundary values on the oceans and on the continents. On the oceans, the gravity anomalies derived by terrestrial gravity measurements and the heights of the ocean above the ellipsoid derived by the satellite altimetry are introduced as given values. On the continents, only the free-air anomalies are the given boundary values. They are obtained by terrestrial gravity measurements. By the combination of these two types of boundary values, the second mixed boundary value problem is established in order to determine the geopotential in the exterior space of the Earth and along the surface of the Earth. The second mixed boundary value problem is governed by a linear inhomogeneous integral equation of the second kind. For a solution in the sub-space of the spherical harmonics of the degrees $n \geqq 2$, the uniqueness of the solution is proved and the compatibility conditions for the boundary values are derived. Further, a way to compute the solution is formulated. For a solution in the space of the spherical harmonics of the degrees $n \geqq 0$, a condition equation for a determinant is found, it must be fulfilled to secure the uniqueness of the solution.

Supplementary considerations show that the first mixed boundary value problem of the geodesy is governed by a linear inhomogeneous integral equation of the first kind. Here, on the oceans, the heights of the oceans above the ellipsoid are introduced as boundary values. They are derived by the methods of the satellite altimetry. Along the continents, the terrestrially derived free-air anomalies are the given boundary values. The uniqueness, the solution and the compatibility conditions are investigated, similar as for the second mixed boundary value problem. A criterion for a determinant is computed. The structure of this determinant proves the uniqueness of the solution in the space of the spherical harmonics of the degrees $n \geqq 0$ for the geographically given distribution of the oceans and the continents. In the sub-space of the spherical harmonics of the degrees $n \geqq 2$, the uniqueness is proved for an arbitrary distribution of the oceans and of the continents.

All the investigations refer to a spherical Earth. However, as to the first mixed boundary value problem, a solution of the problem is derived also for a non-spherical Earth shaped by the topography.

Zusammenfassung

Das zweite gemischte Randwertproblem hat auf den Ozeanen und auf den Kontinenten unterschiedliche Typen von Randwerten. Auf den Ozeanen hat man die Freiluftanomalien aus den ozeanischen Schweremessungen, ferner die Höhen des Ozeans über dem Erdellip-

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soid mit den Methoden der Satellitenaltimetrie. Auf den Kontinenten kennt man nur die Freiluftanomalien als Randwerte. Die Gesamtheit dieser beiden Typen von Randwerten bedeckt die gesamte Erdoberfläche. Sie führt zum zweiten gemischten Randwertproblem der Geodäsie. Aus ihm wird das Potentialfeld der Erde im Außenraum bestimmt. Die Lösung einer linearen inhomogenen Integralgleichung der zweiten Art ist beim zweiten gemischten Randwertproblem von zentraler Bedeutung. Die Eindeutigkeit der Lösung wird im Unterraum der Kugelfunktionen vom Grade $n \geq 2$ bewiesen. Ferner werden Kompatibilitätsbedingungen für die Randwerte aufgestellt. Es wird auch eine Methode zum Auffinden der Lösung angegeben. In ähnlicher Weise wird im Raum der Kugelfunktionen $n \geq 0$ ein Lösungsweg entwickelt. Dort wird auch eine Determinantenbedingung abgeleitet, aus der die Eindeutigkeit der Lösung folgt.

Eine kurze Betrachtung des gemischten Randwertproblems der ersten Art schließt sich an, indem die Eindeutigkeit, die Lösung und die Kompatibilitätsbedingungen betrachtet werden. Hier hat man die Höhen des Geoids über dem Ellipsoid als ozeanische Randwerte. Sie werden mittels der Satellitenaltimetrie erhalten. Auf den Kontinenten hat man wieder die Freiluftanomalien. Das erste gemischte Randwertproblem führt zu einer linearen inhomogenen Integralgleichung der ersten Art. Für die Eindeutigkeit des ersten gemischten Randwertproblems gilt im Raum der regulären Funktionen auch hier eine Determinatengleichung. Sie wird für die gegebene geographische Verteilung der Kontinente und Ozeane ausführlich numerisch behandelt. Das Kriterium zeigt, daß die betreffende Lösung eindeutig ist. Im Unterraum der Kugelfunktionen vom Grade $n \geq 2$ ist die Eindeutigkeit sogar für jede Verteilung der Ozeane und Kontinente gesichert.

Alle Untersuchungen setzen eine sphärische Erdfigur voraus. Die Lösung für das erste Randwertprobelem wird sogar für eine durch die topographischen Höhen ausgeformte Erde abgeleitet.

Резюме

Смешанная краевая проблема второго рода имеет на океанах и континентах различные типы краевых значений. На океанах воздушные аномалии исчисляются при помощи океанических гравитационных измерений, а высоты океана над эллипсоидом Земли - методами измерения высот посредством спутника. На континентах в качестве краевых значений фигурируют лишь воздушные аномалии. Эти два типа краевых значений в их совокупности покрывают всю земную поверхность. Они подводят ко второй смешанной краевой проблеме геодезии. На ее основе определяется потенциальное поле Земли во внешнем пространстве. Центральное значение в смешанной краевой проблеме имеет решение линейного неоднородного интегрального уравнения второго рода. Однозначность решения доказывается в подпространстве сферических функций степенью $n \geq 2$. Далее разрабатываются условия совместимости для краевых значений. Указывается также метод нахождения решения. Подобным же образом способ решения разрабатывается в пространстве сферических функций $n \geq 0$. Там же выводится детерминантное условие, из которого следует однозначность решения. К сказанному примыкает краткое описание смешанной краевой проблемы первого рода, в котором рас-

DOI: https://doi.org/10.2312/zipe.1986.084

сматриваются однозначность, решение и условия совместимости. В качестве океанических краевых значений здесь фигурируют морские высоты над эллипсоидом Земли. Эти значения получают методами измерения высот при помощи спутника. На континентах мы имеем дело снова с воздушными аномалиями. Первая смешанная краевая проблема подводит к линейному неоднородному интегральному уравнению первого рода. Для однозначности первой смешанной краевой проблемы и в этом случае применяется детерминантное уравнение /уравнение определителя/. Для данного географического распределения континентов и океанов оно получает подробную числовую обработку. Критерий показывает, что данное решение однозначно. В подпространстве сферических функций степени $n \geqq 2$ однозначность любого распределения океанов и континентов гарантирована. Все исследования предполагают сферическую форму Земли.

1. Introduction

The uniqueness and the compatibility conditions of the first mixed boundarey value problem of geodesy was investigated in an earlier publication, [5]. The boundary values of that problem consist in the values of the gravity potential of the Earth on the oceans, whereas on the continents the free-air anomalies of the gravity serve as boundary values, [3][4][25][26]. In that former publication [5], a condition for the uniqueness of the first mixed boundary value problem was formulated which can be applied to an arbitrary course of the coastline. Therewith, all the degrees of the spherical harmonics were included, (n= 0,1,2, ...). Meanwhile, certain numerical computations yielded that the uniqueness of the solution of this first mixed boundary value problem of n = 0,1,2, ... is secured in the case of the real geographical distribution of the continents and of the oceans, [8]. A thorough description of these computations about the concerned matrix criterion is given at the end of this report, chapter 5.3. Further, it was proved formerly that the uniqueness of the first mixed boundary value problem for solutions in the subspace of the harmonics of the second and higher degree is valid for every distribution of the continents and oceans.

However, the boundary values cannot be chosen arbitrarily in this special case of the first mixed boundary value problem which includes only the harmonics of the 2nd and higher degree. But, they must fulfill certain compatibility conditions which are in relation to the harmonics of the 0th and 1st degree, [5] [6]. Otherwise, the a priori given boundary values cannot be fulfilled by any solution of this problem a posteriori.

In the Stokes boundary value problem, such a compatibility condition for the boundary values does not appear because a shift of the globally distributed free-air anomalies by anyone of the spherical harmonics of the 0th or 1st degree is effective all over the Earth's surface. Thus, it is eliminated automatically since only the harmonics of the 2nd and higher degrees are envolved in the Stokes function, which is the kernel function which has to transform the boundary value function into the solution function.

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However, this situation turns out to be very different in case the first mixed boundary value problem is considered. Here, in this problem, a constant shift of the free-air anomalies is effective only on the continents. Therefore, such a shift is not eleminated automatically by the integral transformation established by the kernel function. Such a shift influences all the harmonics of the 2nd and higher degree which appear in the solution of the first mixed boundary value problem. In order to avoid the biases in the solution which come from this source, the boundary values must be reduced a priori to free them from the shares caused by the spherical harmonics of the 0th and 1st degree. The investigation into whether the compatibility conditions for these reduced boundary values are fulfilled or not must be executed, it gives a measure to find whether these reductions of the boundary values of the mixed type were choosen successfully or not, [6] [8], i. e. whether the heterogeneous boundary values are in keeping with the problem or not.

2. The boundary values of the mixed boundary value problems of geodesy

2.1. The representation of the boundary values in terms of the empirically given data

2.1.1. The empirical determination of the boundary values of the second mixed boundary value problem

Now, the second mixed boundary value problem is to be considered. The boundary values of the oceanic area, ω_s , are the radial derivatives of the perturbation potential T, i. e. the gravity deviations δg ,

$$\delta g = \alpha_1(\varphi, \lambda) = -\frac{\partial T}{\partial r} = (g - \gamma)_Q \quad , \tag{1}$$

on ω_s. α_1 is a regular function on ω_s , φ and λ are the geographical latitude and longitude. Since the flattening of the Earth is neglected here, the φ and λ values are here also equal to the geocentric latitude and longitude. r is in (1) the distance from the gravity center of the Earth. $g = g_\sigma = (g)_Q$ is the observed vertical intensity of the gravity at the running point Q on the surface of the Earth σ which is here the surface of the ocean, since the ω_s area is considered, (1). $(\gamma)_Q$ is the standard gravity at this point.

In the computation of the δg values, the $(\gamma)_Q$ values must be computed from the standard gravity γ_ε at the level ellipsoid ε and from the heights of the ocean surface above this ellipsoid ε ; these heights are equal to the N^* values obtained by the methods of the satellite altimetry, [23] [24],

$$(\gamma)_Q = \gamma_\varepsilon - \frac{2G}{R} N^*, \quad \text{on } \omega_s. \tag{2}$$

DOI: https://doi.org/10.2312/zipe.1986.084

The second term on the right hand side of (2) is derived from the free-air gradient of the standard gravity. G is the global mean value of the gravity and R is the mean radius of the globe.

Thus, the equations (1) and (2) give the following relation for the gravity deviation,

$$\delta g = (g)_Q - \gamma_\varepsilon + \frac{2G}{R} N^*, \quad \text{on } \omega_s \,. \tag{3}$$

The transition from the satellite altimetry data N^* to the sea surface topography N^{**} is a small step only. The geoidundulation N and the perturbation potential T have the well-known relation

$$T = G\,N\,. \tag{4}$$

The sea surface topography N^{**} derives from N^* and N by

$$N^{**} = N^* - N\,. \tag{5}$$

A shift of the geoid undulations N by the amount of N^{**} has an impact on the perturbation potential T by

$$T^{**} = G\,N^{**}\,. \tag{6}$$

N^{**} has the amount of about $\pm$ 0.5 m in the mean, [20].

The boundary values of the continental area, ω_c, are the free-air anomalies Δg_F. They can be expressed in terms of the perturbation potential T by the fundamental differential equation of the physical geodesy,

$$\Delta g_F = \alpha_2(\varphi, \lambda) = -\frac{\partial T}{\partial r} - \frac{2}{r} T = (g)_Q - (\gamma)_{\bar{Q}} \quad , \text{ on } \omega_c. \tag{7}$$

α_2 is a regular function of the geocentric longitude λ and latitude φ for the area ω_c. $\bar{Q}$ is the point which is situated perpendicular below the point Q and on the surface of the telluroid. The difference of the heights of the points Q and $\bar{Q}$ is the height anomaly. The height of the telluroid above the level ellipsoid, i. e. the mean Earth ellipsoid, is equal to the normal height. The last term on the right hand side of (7) does show the way to find Δg_F from the empirically given data. $(g)_Q$ is the measured value of the vertical intensity of the gravity. $(\gamma)_{\bar{Q}}$ is obtaind from the formula for the standard gravity if the measured normal height of the point Q above

DOI: https://doi.org/10.2312/zipe.1986.084

the geoid is introduced for the height of $\bar{Q}$ above the ellipsoid ε .

2.1.2. The empirical determination of the boundary values of the first mixed boundary value problem

As to the first mixed boundary value problem, the boundary values of the oceanic area, ω_s , are here the amounts of the perturbation potential T. The T values along the oceans are not directly obtained by empirical methods. But the heights of the ocean surface above the level ellipsoid ε , (i. e. N^*), that are the values derived empirically, by the methods of satellite altimetry. In a rigorous consideration, the N^* values must be transformed into the N values, i. e. the geoid undulations. The relations (4) and (5) give

$$T = G\,(N^* - N^{**}) \; . \tag{7a}$$

Thus, the sea surface topography N^{**} must be known to find T from N^* . However, within the scope of the here discussed problems, the N^{**} values are considered to be known or to be of negligible amounts. The N^{**} values can be determined empirically by a combination of the N^* values with satellite - to - satellite tracking; (see: Arnold, K. and D. Schoeps; Cosmic Interpolation of Terrestrial Potential Values, Gerlands Beitr. Geophysik 93 (1984), 409 - 422). Under these presuppositions, the T values are introduced as empirically given oceanic boundary values.

The continental boundary values of the first mixed boundary value problem are again the free-air anomalies Δg_F , as in case of the second mixed boundary value problem, (7). In context with the equation (7), the empirical way to reach the Δg_F values was described already.

2.2. The boundary values in terms of the perturbation potential

2.2.1. The boundary values of the second mixed boundary value problem in terms of the perturbation potential

According to the above explanatory lines about the 2nd mixed boundary value problem, the functions for the two different types of boundary values can be expressed in terms of the perturbation potential T by means of the following forms, (1), [4] [5] [25] [26] [28] [30],

$$\delta g = \alpha_{1.s}(\varphi, \lambda) = -\frac{\partial T}{\partial r} \; , \text{ on } \omega_s \; , \tag{8}$$

and, (7),

$$\Delta g_F = \alpha_{2.c}(\varphi, \lambda) = -\frac{\partial T}{\partial r} - \frac{2}{r}\,T \; , \text{ on } \omega_c \; . \tag{9}$$

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ω_s is the area covered by the oceans, ω_c is that covered by the continents. The gravity deviation δg derives by (3) empirically from both the system of the g values and the system of the N^* values. The free-air anomaly of the gravity Δg_F comes empirically from the g values and from the normal heights according to (7).

2.2.2. The boundary values of the first mixed boundary value problem in terms of the perturbation potential

On the oceans, the first mixed boundary value problem has the T values as boundary values. On the continents, the Δg_F values serve for this purpose. These two different types of boundary values can be expressed as functions of the T values by means of the following relations,

$$\alpha_{3.s}(\varphi,\lambda) = T \quad , \qquad \text{on } \omega_s \ , \tag{9a}$$

and

$$\Delta g_F = \alpha_{2.c}(\varphi,\lambda) = -\frac{\partial T}{\partial r} - \frac{2}{r} T \quad , \quad \text{on } \omega_c . \tag{9b}$$

3. The oceanic chief minor of the Stokes matrix

In this context, it is suitable to remember of some investigations about the solution of the first mixed boundary value problem since they are in close neighbourhood to the here intended developments about the second mixed boundary value problem. For a spherical Earth, the first mixed boundary value problem was found by a consideration of the Stokes integral, [12][15],

$$T = \frac{R}{4\pi} \iint\limits_{\omega} \Delta g_F \, S(\psi) \; d\omega \quad , \tag{10}$$

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi) \ . \tag{11}$$

ω is the unit sphere, P_n are the Legendre functions, ψ is the spherical distance.

In a more rigorous formulation of the boundary value problem, a non-spherical Earth is introduced. This refinement leads to supplementary terms of the amount of about up to 1 meter or 2 meter to be added to the height anomalies ζ obtained from the Stokes equation (10), [6][8]. In case, that the surface of the Earth σ is introduced as the boundary surface, instead of a spherical Earth, in this case, the boundary value problem of Molodenskij comes to the fore. It has the following solution, [6][8], for T at the surface σ ,

DOI: https://doi.org/10.2312/zipe.1986.084

$$T = \frac{R}{4\pi} \iint_{\omega} (\Delta g_F + C)\, S(\psi)\, d\omega , \qquad (12)$$

C is the plane topographical reduction of the gravity. The theoretical error of the solution for the Molodenskij problem is not greater than about some centimeters, (12), [6][8]. To be perfect, additional to C, a very small completing term comes from the Bouguer anomalies, it derives from the vertical gradient of the Bouguer anomalies in the free air.

On the strength of the equation (10), the T values can be computed from the free-air anomalies for test points distributed over whole the surface of the Earth, ω . Further, the use of the equation (10) can be restricted to test points which are situated in the oceanic area ω_s only.

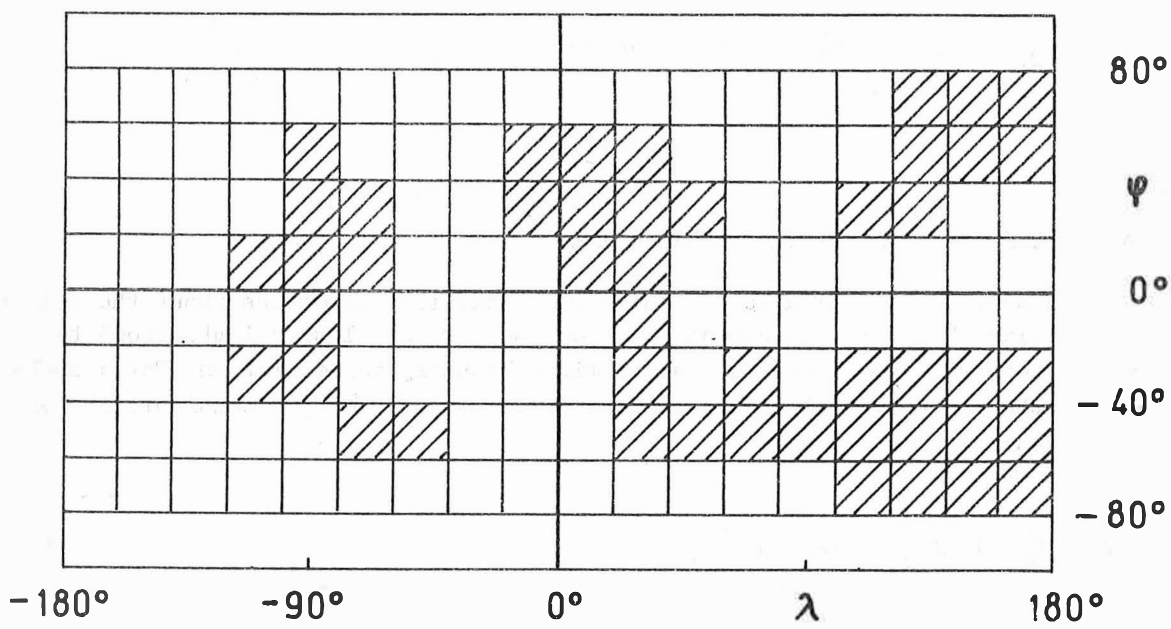

Fig.1. The model of the geographical distribution of the different types of the mixed boundary values on a map representation of the globe. The hatched areas are the oceans, ω_s , the residual white areas are the continents, ω_c .

For oceanic test points, the relation (10) changes into

DOI: https://doi.org/10.2312/zipe.1986.084

$$T_s = \frac{R}{4\pi} \iint_{\omega} \Delta g_F \, S(\psi) \, d\omega \, , \text{ on } \omega_s \, . \tag{13}$$

Or, dividing into the oceanic part and into the continental part of the integration,

$$T_s = \frac{R}{4\pi} \iint_{\omega_s} \Delta g_F \, S(\psi) \, d\omega + \frac{R}{4\pi} \iint_{\omega_c} \Delta g_F \, S(\psi) \, d\omega \quad , \text{ on } \omega_s \, . \tag{14}$$

T_s symbolizes the T values on ω_s. T on ω_s and Δg_F on ω_c are the empirically given values in the equation (14). Δg_F on ω_s is the function to be determined by the evaluation of (14). Thus, the solution of (14) leads to an inversion. (14) is a linear integral equation of the first kind for the determination of the free-air anomalies on the oceans which appear in the integrand of the first term on the right hand side of (14). S (ψ) is the kernel function valid for the area ω_s. In case, the test points and the running integration elements $d\omega$ are restricted to variate over the oceanic area only, in this case, the function S (ψ) is denominated by $S_{s.s}$. The function $S_{s.s}$ is positive definite, symmetric and closed, [5][8].

At first glance, the function $S_{s.s}$ seems to be discontinuous because the Stokes function, [15][12],

$$S(\psi) = \frac{1}{\sin\frac{\psi}{2}} - 6 \sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi \, \ln\left[\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right] \tag{15}$$

tends to infinity if ψ tends to zero. Indeed, S(ψ) can be written into the following form

$$S(\psi) = \frac{2}{\sin\psi} \, F(\psi) \quad , \tag{16}$$

F(ψ) derives by a comparison of (15) and (16). F(ψ) tends to the unit if ψ tends to zero, $F(\psi) \longrightarrow 1$ if $\psi \longrightarrow 0$, as can be found in the table of Lambert and Darling; [19] page 114. Therefore, S(ψ) tends to infinity as $\frac{2}{\psi}$ if ψ tends to zero; $S(\psi) \longrightarrow \frac{2}{\psi}$ if $\psi \longrightarrow 0$.

However, in the first term on the right hand side of (14), S(ψ) is multiplicated with the surface element of the unit sphere, i. e. $d\omega = \sin\psi \, d\psi \, d\bar{\alpha}$. Hence,

$$S(\psi) \, d\omega = 2 \, F(\psi) \quad d\psi \quad d\bar{\alpha} \quad , \tag{17}$$

DOI: https://doi.org/10.2312/zipe.1986.084

$\bar{\alpha}$ is the azimuth. Because of $F(\psi) \longrightarrow 1$ if $\psi \longrightarrow 0$, it is sure that the discontinuity of $S(\psi)$ for $\psi \longrightarrow 0$ is removable. Thus, the integrand of the first integral on the right hand side of (14) is equal to

$$\frac{R}{4\pi} \Delta g_F \, 2 \, F(\psi) \, d\psi \, d\bar{\alpha} \quad , \tag{18}$$

it is continuous over whole the globe.

The relation (10) can be transformed into the shape it has in the matrix calculus. A glance on the figure 1 shows that the surface of the Earth can be divided into 148 compartments, e. g. Such a division can be executed in such a way that all the 148 compartments have equal size $\Delta\omega$. The 47 hatched compartments of figure 1 belong to the oceanic area ω_s , and the 101 white compartments belong to the continental area ω_c . Such a division of the surface of the Earth ω into compartments of constant size $\Delta\omega$ turns the Stokes integral (10) into the matrix shape,

$$\underline{\underline{a}}_3 = \underline{\underline{S}} \; \underline{\underline{a}}_2 \, \Delta\omega \; . \tag{19}$$

The vector $\underline{\underline{a}}_3$ has the compartment mean values of $\frac{4\pi}{R}$ T as the components, and the vector $\underline{\underline{a}}_2$ has the mean values of the free-air anomalies Δg_F for the compartments as the components. $\underline{\underline{S}}$ is the matrix shape of the Stokes function, (11)(15). In case of a compartment division, as in figure 1, the two vectors $\underline{\underline{a}}_3$ and $\underline{\underline{a}}_2$ have 148 components and $\underline{\underline{S}}$ is here a square matrix of the size 148 · 148. In the equation (19), $\Delta\omega$ is a scalar.

The following equation (20) is the oceanic part of the matrix equation (19), as (14) is the oceanic part of (13),

$$\underline{\underline{a}}_{3.s} = \Delta\omega \, \underline{\underline{S}}_{s.s} \, \underline{\underline{a}}_{2.s} + \Delta\omega \, \underline{\underline{S}}_{s.c} \, \underline{\underline{a}}_{2.c} \; . \tag{20}$$

For the explanation of the meaning of the two matrices $\underline{\underline{S}}_{s.s}$ and $\underline{\underline{S}}_{s.c}$ of the equation (20), a compartment division of the surface of the Earth is introduced, as in figure 1.

The $\underline{\underline{S}}$ matrix is recommended to be written in a certain structure. At first, the components of the $\underline{\underline{a}}_3$ vector of (19) are considered which refer to the 47 oceanic compartments, Fig. 1. They are written ahead, they are followed by the 101 lines that refer to the continental test points. The relation (21) visualizes this structure of the elements of the symmetrical $\underline{\underline{S}}$ matrix, [31],

$$\underline{\underline{S}} = \begin{bmatrix} \bar{s}_{1.1} & \cdots & \bar{s}_{1.p} & \bar{s}_{1.p+1} & \cdots & \bar{s}_{1.q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{s}_{p.1} & \cdots & \bar{s}_{p.p} & \bar{s}_{p.p+1} & \cdots & \bar{s}_{p.q} \\ \bar{s}_{p+1.1} & \cdots & \bar{s}_{p+1.p} & \bar{s}_{p+1.p+1} & \cdots & \bar{s}_{p+1.q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{s}_{q.1} & \cdots & \bar{s}_{q.p} & \bar{s}_{q.p+1} & \cdots & \bar{s}_{q.q} \end{bmatrix} \; . \tag{21}$$

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The indices of the elements of $\underline{\underline{S}}$ which take the values i = 1, ... , p, (p= 47), even these indices refer to the oceanic compartments. Consequently, the remaining indices represent the continental compartments if they run through the values i = p + 1, ..., q, (q = 148).

The matrix (21) is divided into its oceanic and continental submatrices, i. e.

$$\underline{\underline{S}}_{s.s} = \begin{bmatrix} \bar{s}_{1.1} & \cdots & \bar{s}_{1.p} \\ \cdots & \cdots & \cdots \\ \bar{s}_{p.1} & \cdots & \bar{s}_{p.p} \end{bmatrix} , \quad (22)$$

$$\underline{\underline{S}}_{s.c} = \begin{bmatrix} \bar{s}_{1.p+1} & \cdots & \bar{s}_{1.q} \\ \cdots & \cdots & \cdots \\ \bar{s}_{p.p+1} & \cdots & \bar{s}_{p.q} \end{bmatrix} , \quad (23)$$

$$\underline{\underline{S}}_{c.s} = \underline{\underline{S}}^{T}_{s.c} , \quad (24)$$

$$\underline{\underline{S}}_{c.c} = \begin{bmatrix} \bar{s}_{p+1.p+1} & \cdots & \bar{s}_{p+1.q} \\ \cdots & \cdots & \cdots \\ \bar{s}_{q.p+1} & \cdots & \bar{s}_{q.q} \end{bmatrix} ; \quad (25)$$

the superscript T in the equation (24) denotes here the transposition of the matrix.

As it is well-known, the Stokes matrix $\underline{\underline{S}}$, (21), has the following submatrices, they are in the close **vicinity** of the chief minors of $\underline{\underline{S}}$, i. e. the chief minor determinants of $\underline{\underline{S}}$.

$$\underline{\underline{S}}_1 = \left[(\bar{s}_{1.1}) \right] , \quad (26)$$

$$\underline{\underline{S}}_2 = \begin{bmatrix} \bar{s}_{1.1} & \bar{s}_{1.2} \\ \bar{s}_{2.1} & \bar{s}_{2.2} \end{bmatrix} , \quad (27)$$

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$$\underline{\underline{S}}_3 = \begin{bmatrix} \bar{s}_{1.1} & \bar{s}_{1.2} & \bar{s}_{1.3} \\ \bar{s}_{2.1} & \bar{s}_{2.2} & \bar{s}_{2.3} \\ \bar{s}_{3.1} & \bar{s}_{3.2} & \bar{s}_{3.3} \end{bmatrix} , \qquad (28)$$

$$\underline{\underline{S}}_4 = \begin{bmatrix} \bar{s}_{1.1} & \bar{s}_{1.2} & \bar{s}_{1.3} & \bar{s}_{1.4} \\ \bar{s}_{2.1} & \bar{s}_{2.2} & \bar{s}_{2.3} & \bar{s}_{2.4} \\ \bar{s}_{3.1} & \bar{s}_{3.2} & \bar{s}_{3.3} & \bar{s}_{3.4} \\ \bar{s}_{4.1} & \bar{s}_{4.2} & \bar{s}_{4.3} & \bar{s}_{4.4} \end{bmatrix} , \qquad (29)$$

the other submatrices $\underline{\underline{S}}_5$, $\underline{\underline{S}}_6$, ... follow similarly.

The Stokes matrix, (21), has the following determinant

$$\det \underline{\underline{S}} = \begin{vmatrix} \bar{s}_{1.1} & \ldots & \bar{s}_{1.p} & \bar{s}_{1.p+1} & \ldots & \bar{s}_{1.q} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{s}_{p.1} & \ldots & \bar{s}_{p.p} & \bar{s}_{p.p+1} & \ldots & \bar{s}_{p.q} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{s}_{q.1} & \ldots & \bar{s}_{q.p} & \bar{s}_{q.p+1} & \ldots & \bar{s}_{q.q} \end{vmatrix} . \qquad (30)$$

det $\underline{\underline{S}}$ has the following 1st, 2nd, 3rd and 4th chief minors, i.e. the minors in principal position and of the indices

$1, \ldots , j, (j = 1, 2, 3, 4),$

$$\det \underline{\underline{S}}_1 = \begin{vmatrix} \bar{s}_{1.1} \end{vmatrix} , \qquad (31)$$

$$\det \underline{\underline{S}}_2 = \begin{vmatrix} \bar{s}_{1.1} & \bar{s}_{1.2} \\ \bar{s}_{2.1} & \bar{s}_{2.2} \end{vmatrix} , \qquad (32)$$

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$$\det \underline{\underline{S}}_3 = \begin{vmatrix} \bar{s}_{1.1} & \bar{s}_{1.2} & \bar{s}_{1.3} \\ \bar{s}_{2.1} & \bar{s}_{2.2} & \bar{s}_{2.3} \\ \bar{s}_{3.1} & \bar{s}_{3.2} & \bar{s}_{3.3} \end{vmatrix} \qquad (33)$$

$$\det \underline{\underline{S}}_4 = \begin{vmatrix} \bar{s}_{1.1} & \bar{s}_{1.2} & \bar{s}_{1.3} & \bar{s}_{1.4} \\ \bar{s}_{2.1} & \bar{s}_{2.2} & \bar{s}_{2.3} & \bar{s}_{2.4} \\ \bar{s}_{3.1} & \bar{s}_{3.2} & \bar{s}_{3.3} & \bar{s}_{3.4} \\ \bar{s}_{4.1} & \bar{s}_{4.2} & \bar{s}_{4.3} & \bar{s}_{4.4} \end{vmatrix} \qquad (34)$$

The other chief minors follow similarly.

The amounts of all the eigenvalues of the matrix $\underline{\underline{S}}$ are positive if the compartments $\Delta\omega$ are sufficient small, it was proved in [5]. Thus the matrix $\underline{\underline{S}}$ is positive definite, cf. (457) later on. The quadratic form related to the matrix $\underline{\underline{S}}$,

$$(\underline{\underline{w}}')^T \underline{\underline{S}} \, \underline{\underline{w}}' > 0 \quad , \qquad (35)$$

has always a positive value. The vector $\underline{\underline{w}}'$ is in the subspace of the spherical harmonics of 2nd and higher degree.

The chief minors of $\underline{\underline{S}}$ have also always positive values, [4][5][31],

$$\det \underline{\underline{S}} > 0 \quad , \qquad (36)$$

$$\det \underline{\underline{S}}_1 > 0 \quad , \qquad (37)$$

$$\det \underline{\underline{S}}_2 > 0 \quad , \qquad (38)$$

$$\det \underline{\underline{S}}_3 > 0 \quad , \qquad (39)$$

$$\det \underline{\underline{S}}_4 > 0 \quad , \qquad (40)$$

$$\dots \quad .$$

A view on the relations (19), (20), (21) shows that the oceanic matrix $\underline{\underline{S}}_{s.s}$ of (20) is related to a chief minor of $\underline{\underline{S}}$. Of course, det $\underline{\underline{S}}_{s.s}$ is the chief minor of the indices 1, ... , p, (p = 47), of $\underline{\underline{S}}$. It is the chief minor of the oceanic compartments or the chief minor of the oceanic elements of $\underline{\underline{S}}$, (21), which are ahead of this matrix. Therefore, the matrix $\underline{\underline{S}}_{s.s}$ of (20) is obviously positive definite and symmetric.

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It is only a small step from the integral relation (14) to the related matrix equation (20). In the equation (14), the test points are always situated in the oceanic area ω_s. In the first integral on the right hand side of (14), the integration covers the oceanic area ω_s. But, in the second integral, the integration happens for the continental area ω_c. This fact is expressed by the subscripts s and c. Along these lines, the integral relation (14) changes into the following form,

$$T_s = \frac{R}{4\pi} \iint_{\omega_s} \Delta g_F \, S_{s.s} \, d\omega + \frac{R}{4\pi} \iint_{\omega_c} \Delta g_F \, S_{s.c} \, d\omega \quad , \text{ on } \quad \omega_s \; . \tag{41}$$

According to the above considerations, the kernel function $S_{s.s}$ is symmetric, positive definite and closed, (457). $S_{s.s}$ goes to infinity if $\psi \longrightarrow 0$, but, this discontinuity of $S_{s.s}$ is removable, cf. (17)(18). In the theoretical investigations, $S_{s.s}$ can be replaced by $\overline{S}_{s.s}$ which is continuous, cf. (464)(468). $\overline{S}_{s.s}$ does meet the fact that in the numerical computations the function $S_{s.s}$ is replaced by its mean values for the compartments $\Delta\omega$. Therefore, it is possible to define and to compute the inverse of $\overline{S}_{s.s}$, (470)(472), and the corresponding matrix $\underline{\underline{\overline{S}}}_{s.s}$ and the inverse of it. From here, without loss of precision, it is possible to turn back to the inverse of the oceanic submatrix $\underline{\underline{S}}_{s.s}$, i. e. $\underline{\underline{S}}_{s.s}^{-1}$, attached to the introduced compartments.

Along these lines, the equation (20) leads to, [4][5],

$$\underline{\underline{a}}_{2.s} = \underline{\underline{S}}_{s.s}^{-1} \left(\frac{1}{\Delta\omega} \underline{\underline{a}}_{3.s} - \underline{\underline{S}}_{s.c} \, \underline{\underline{a}}_{2.c} \right) \; . \tag{42}$$

The equation (42) determines the unknown oceanic free-air anomalies $\underline{\underline{a}}_{2.s}$ in terms of the given boundary values of the first mixed boundary value problem, i. e. the boundary values $\underline{\underline{a}}_{3.s}$ and $\underline{\underline{a}}_{2.c}$, which represent the oceanic $\frac{4\pi}{R}$ T values and the continental Δg_F values.

The inversion of (41) has the following shape,

$$(\Delta g_F)_s = \frac{4\pi}{R} \iint_{\omega_s} S_{s.s}^{-1} \left[T_s - \frac{R}{4\pi} \iint_{\omega_c} S_{s.c} (\Delta g_F)_c \, d\omega \right] d\omega \; . \tag{43}$$

$(\Delta g_F)_s$ and $(\Delta g_F)_c$ are the free-air anomalies on the oceans and on the continents. The analogies between (42) and (43) are obvious.

As to the meaning of the function $S_{s.s}^{-1}$, a direct relation of the following form is introduced,

$$\vartheta^* = \iint_{\omega_s} S_{s.s} \, \vartheta \, d\omega \quad , \text{ on } \quad \omega_s \; . \tag{44}$$

In (44), the test points of ϑ^* and the surface elements $d\omega$ belong to the oceanic area ω_s only. Because the kernel function $S_{s.s}$, (41), is proved to be symmetric, positive definite and closed for regular functions, it is sure that (44) has an inversion of the following form, [1][5][22][29], (see also (462) to (477)),

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$$\vartheta = \iint_{\omega_S} S_{s.s}^{-1} \quad \vartheta^* \, d\omega \quad , \text{ on } \omega_S \ . \qquad (45)$$

$S_{s.s}$ and $S_{s.s}^{-1}$ can be replaced by $\bar{S}_{s.s}$ and $(\bar{S}_{s.s})^{-1}$ which are symmetric, continuous, positive definite and closed.

A view on the equations (41), (44), (45) shows that the solution of the first mixed boundary value problem consists in the solution of the inhomogeneous integral equation of the first kind (41). Later on, it will be shown that the second mixed boundary value problem leads to an integral equation of the second kind, [10][14][22][27]. Since a division of the globe into compartments is introduced, the inversion of (44) is always admissable in the here discussed applications and under the here valid conditions,[10]. T_S and $(\Delta g_F)_C$ are not introduced by rigorously given analytical expressions, they are empirically given mean values for certain finite elements.

The problem is : Find a harmonic solution that does meet the mixed boundary values within the tolerances. Perhaps, this is the original idea of the collocation, free of covariances.

4. The second mixed boundary value problem

4.1. The integral equation of the second kind of the second mixed boundary value problem

The perturbation potential T can be expressed in terms of the gravity deviation δg by the Hotine integral transformation which has the Hotine function as kernel function. The following relation is valid in the subspace of the harmonics of second and higher degree,

$$T' = \frac{R}{4\pi} \iint_{\omega} H'(\psi) \, \delta g \, d\omega \quad , \qquad (46)$$

and, in case, all the harmonics are included,

$$T = \frac{R}{4\pi} \iint_{\omega} H(\psi) \, \delta g \, d\omega \quad . \qquad (47)$$

The prime ( )' denotes in (46) that the spherical harmonics of the 0th and 1st degree are not taken into account. The Hotine function H depends on the spherical distance ψ by a closed expression,

$$H = \frac{1}{\sin\frac{\psi}{2}} - \ln\left(1 + \frac{1}{\sin\frac{\psi}{2}} \right) \ . \qquad (48)$$

H' derives from H by

$$H' = H - 1 - \frac{3}{2}\cos\psi \quad . \qquad (49)$$

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H has also a series development in terms of the Legendre functions P_n (cos ψ), similar as the function H', [11][12][13][15][17].

$$H = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n (\cos \psi) \quad , \tag{50}$$

$$H' = \sum_{n=2}^{\infty} \frac{2n+1}{n+1} P_n (\cos \psi) \quad . \tag{51}$$

The fundamental differential equation of the physical geodesy gives rise to the possibility to represent the perturbation potential T on the left hand side of (47) by the free-air anomalies Δg_F and by the gravity deviations δg, (7),

$$T = \frac{1}{2} R (\delta g - \Delta g_F) \quad , \tag{52}$$

or without the harmonics of the 0th and 1st degree,

$$T' = \frac{1}{2} R((\delta g)' - (\Delta g_F)') \quad . \tag{53}$$

Hence, by a combination of (46) and (53),

$$0 = (\delta g)' - \frac{1}{2\pi} \iint_{\omega} H' \, \delta g \, d\omega - (\Delta g_F)' \quad . \tag{54}$$

Now, in the application of the relation (54), the computations for the test points situated on the oceans are kept separate from the computations for the test points on the continents. Further, because of the different types of the oceanic and continental boundary values, a clear distinction is made between the integrations over the oceans and over the continents. Along these lines, the relations (8), (9), (54) lead to the following two equations,

$$0 = \alpha'_{1.s} - \frac{1}{2\pi} \iint_{\omega_s} H'_{s.s} \, \alpha'_{1.s} \, d\omega - \frac{1}{2\pi} \iint_{\omega_c} H'_{s.c} \, \alpha'_{1.c} \, d\omega - \alpha'_{2.s} \quad , \text{ on } \omega_s \, , \tag{55}$$

and

$$0 = \alpha'_{1.c} - \frac{1}{2\pi} \iint_{\omega_s} H'_{c.s} \, \alpha'_{1.s} \, d\omega - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \, \alpha'_{1.c} \, d\omega - \alpha'_{2.c} \quad , \text{ on } \omega_s \, . \tag{56}$$

$\alpha'_{1.s}$ and $\alpha'_{2.c}$ are the boundary values which are known from the very beginning, i. e.

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the gravity deviations on the oceans and the free-air anomalies on the continents. The prime denotes again the regulation that the spherical harmonics of the 0th and 1st degree are not involved. The subscripts at the kernel functions $H'_{s.s}$, $H'_{c.s}$, $H'_{c.c}$, $H'_{c.s}$ refer to the position of the endpoints of the spherical distance ψ which is the independent variable of these kernel functions, (48)(50)(51). If the first subscript is s resp. c, in this case, one of the endpoints of ψ is situated on the oceans resp. on the continents. An analogous regulation is valid for the second subscript of the kernel functions.

The unknown functions which are to be determined in the course of the investigations about the second mixed boundary value problem of the geodesy, that are the gravity deviations δg on the continents, i. e. the $\alpha'_{1.c}$ function. Afterwards, because the gravity deviations α'_1 or $(\delta g)'$ are known all over the globe, it is possible to compute the T' values all over the globe by (46). This is the solution of the second mixed boundary value problem. Therefore, the equation (56) is in the fore, in case, the unknown function $\alpha'_{1.c}$ is to be determined. As it will be found later on, the kernel function $H'_{c.c}$ of the relation (56) is symmetric, positive definite and closed for regular functions on ω_c . These properties are already found for the function $S_{s.s}$ which derives from the Stokes function and which is in the vicinity of the oceanic chief minor of the Stokes matrix.

After these considerations, the equation (56) can be brought into the following form,

$$0 = \beta'_{1.c} + \alpha'_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c}\, \alpha'_{1.c}\, d\omega \quad , \text{ on } \quad \omega_c \quad , \tag{57}$$

the inhomogeneity $\beta'_{1.c}$ of (57) has the following function in terms of the boundary values $\alpha'_{1.s}$ and $\alpha'_{2.c}$, as it can be found by a comparison of (56) and (57),

$$\beta'_{1.c} = -\frac{1}{2\pi} \iint_{\omega_s} H'_{c.s}\, \alpha'_{1.s}\, d\omega - \alpha'_{2.c} \; , \text{ on } \quad \omega_c \quad . \tag{58}$$

The relation (57) is an inhomogeneous linear integral equation of the second kind, [10] [14][18][22][27][29].

At this stage of the developments, it seems to be convenient to give some supplementary remarks about the fact that all the functions in the equations (57) and (58) refer to the subspace of the harmonics of 2nd and higher degree, e. g. the kernel functions, the unknown function $\alpha'_{1.c}$ and the other known functions. In (46), H' has the superscript ( )' by definition. It follows that T' and not T must appear on the left hand side of (46) necessarily, irrespective whether δg or $(\delta g)'$ is under the integral. Thus, it is well-understood that all the expressions of (55) and (56) have the superscript ( )', as so as the relations (57) and (58). However, the crucial problem is that the given functions of (58) are introduced as regular functions on the oceans only, as $\alpha'_{1.s}$, or as regular functions on the continents only, as $\alpha'_{2.c}$. But the property which is behind the superscript ( )' has a meaning for a globally given function only, since the spherical harmonics of 0th and 1st degree are defined by a global relation.

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The following equations are valid,

$$\alpha_1 = \frac{1}{4\pi} \sum_{i=1}^{4} Y_i(\varphi, \lambda) \iint_{\omega} \alpha_1 \; Y_i(\bar{\varphi}, \bar{\lambda}) \, d\omega + \alpha_1' \; , \tag{59}$$

$$\alpha_2 = \frac{1}{4\pi} \sum_{i=1}^{4} Y_i(\varphi, \lambda) \iint_{\omega} \alpha_2 \; Y_i(\bar{\varphi}, \bar{\lambda}) \, d\omega + \alpha_2' \; . \tag{60}$$

Y_i , (i = 1, 2, 3, 4), is a running denomination of all the four spherical harmonics of the 0th and 1st degree and order. The harmonics are normalized,

$$\iint_{\omega} Y_i^2 \, d\omega = 4\pi \; . \tag{61}$$

The separation of the share of the harmonics of 0th and 1st degree is possible for a globally given function only, this separation is not possible for the empirically given boundary values $\alpha_{1.s}$ since they are given on the oceans only. Further, the separation is not possible for the empirically given boundary values $\alpha_{2.c}$ which are distributed on the continents only. Therefore, in a more exact consideration, it has no meaning to fit the functions $\alpha_{1.c}$, $\alpha_{1.s}$, $\alpha_{2.c}$, $\beta_{1.c}$ in the relations (57) and (58) with the superscript ( )'. Such a procedure has no real scientific foundation.

Hence, the boundary values are now introduced as arbitrary regular functions for the concerned part of the surface of the Earth. All these complications which are brought to bear by putting the superscript ( )' or not, it is discussed later on thoroughly. These specialities will lead to the formulation of four compatibility conditions.

In any case, the inhomogeneous linear integral equation of the second kind, (57), is fundamental for the solution of the second mixed boundary value problem of the geodesy, and it is fundamental for the following discussions. The form free of the superscripts ( )' at the concerned functions is

$$0 = \beta_{1.c} + \alpha_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \; \alpha_{1.c} \, d\omega \; , \text{ on } \omega_c \; , \tag{62}$$

with

$$\beta_{1.c} = - \frac{1}{2\pi} \iint_{\omega_s} H'_{c.s} \; \alpha_{1.s} \, d\omega - \alpha_{2.c} \; , \text{ on } \omega_c \; . \tag{63}$$

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The $\alpha_{1.c}$ function has to be determined as a solution of the integral equation of the second kind (62). Afterwards, $\alpha_{1.c}$ is united with the $\alpha_{1.s}$ function which is known from the beginning since it is one type of the boundary values. Along these lines, the α_1 function is obtained for whole the globe of the Earth, i.e. the gravity deviations δg,

$$\delta g = \alpha_1 = \alpha_{1.s} \ , \text{ on } \ \omega_s \ , \tag{64}$$

$$\delta g = \alpha_1 = \alpha_{1.c} \ , \text{ on } \ \omega_c \ . \tag{65}$$

Besides of the peculiarities about the treatment of the spherical harmonics of the 0th and 1st degree and order, (59) (60), another special problem is to be considered, the question is the α_1 function obtained at the end of the considerations as the solution of the second mixed boundary value problem. The matter is as follows. The α_1 function is presupposed to be a regular function on ω. Thus, the amounts of α_1 must be continuous there. α_1 consists of the united values of $\alpha_{1.s}$ and $\alpha_{1.c}$. $\alpha_{1.s}$ is known by empirical methods from the beginning. $\alpha_{1.c}$ is known as the solution of the integral equation (62) and as the final result of the computations. Crossing the coasts, a jump of the α_1 values is not allowed to exist there. This fact leads to a constraint for the $\alpha_{1.s}$ and $\alpha_{1.c}$ values. If K is a point on the coastline, and if $(\alpha_{1.s})_K$ is the value which $\alpha_{1.s}$ does take approaching the point K from the seaside, and if $(\alpha_{1.c})_K$ is the value which $\alpha_{1.c}$ does reach approaching the same point K from the landside, in this case, the following continuity constraint must be fulfilled,

$$(\alpha_{1.s})_K = (\alpha_{1.c})_K \ . \tag{66}$$

The condition (66) must be observed for all the points K along the coastline.

One of the main questions which arise in context with the integral equation (62) that is the problem whether (62) has a unique solution for the $\alpha_{1.c}$ function or not. The investigation about the uniqueness of (62) is governed by the homogeneous form of it,

$$0 = \alpha_{1.c} - \frac{1}{2\pi} \iint\limits_{\omega_c} H'_{c.c} \ \alpha_{1.c} \ d\omega \ , \text{ on } \ \omega_c \ . \tag{67}$$

According to the derivations from (46) to (67), the inhomogeneous integral equation of the second kind that does govern the second mixed boundary value problem of the geodesy can be written in the following form,

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$$0 = \beta_{1.c} + \alpha_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \; \alpha_{1.c} \; d\omega \; , \text{ on } \omega_c \quad , \tag{67a}$$

with the inhomogeneity $\beta_{1.c}$,

$$\beta_{1.c} = - \frac{1}{2\pi} \iint_{\omega_s} H'_{c.s} \; \alpha_{1.s} \; d\omega \; - \; \alpha_{2.c} \; , \text{ on } \; \omega_c . \tag{67b}$$

The relation

$$\beta_{1.c} = 0 \tag{67c}$$

does lead to the homogeneous shape of (67a). The equations (67a) to (67c) are valid for considerations in the subspace of the spherical harmonics if the 2nd and higher degree.

In case, the validity of the developments is extended to a validity in the space of the spherical harmonics of all degrees, (n = 0,1,2, ...), the kernel functions $H'_{0.0}$ and $H'_{c.s}$ must be replaced by the corresponding functions derived from H, i.e. the functions $H_{c.c}$ and $H_{c.s}$. After the replacement of H' by H, the kernel function includes the harmonics of the 0th and first degree. Along these lines, the expressions (67a) to (67c) transform into the following form which is again an inhomogeneous integral equation of the second kind,

$$0 = \beta_{1.c} + \alpha_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H_{c.c} \; \alpha_{1.c} \; d\omega , \text{ on } \omega_c \quad , \tag{67d}$$

with the inhomogeneity

$$\beta_{1.c} = - \frac{1}{2\pi} \iint_{\omega_s} H_{c.s} \; \alpha_{1.s} \; d\omega \; - \; \alpha_{2.c} \; , \text{ on } \; \omega_c \quad . \tag{67e}$$

The homogeneous shape of (67d) has a $\beta_{1.c}$ value that is equal to zero.

4.2. The uniqueness of the solution of the second mixed boundary value problem of geodesy in the subspace of the spherical harmonics of 2nd and higher degree

The uniqueness of the solution of the second mixed boundary value problem of the geodesy can be investigated also independently by the homogeneous form of the fundamental integral equation, (67). In any case, the treatment of the uniqueness of this boundary value problem is governed by the constraint that

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the mixed boundary values have to be zero, (8)(9),

$$\alpha_{1.s}(\varphi, \lambda) = -\frac{\partial T}{\partial r} = 0, \text{ on } \omega_s \quad , \tag{68}$$

and

$$\alpha_{2.c}(\varphi, \lambda) = -\frac{\partial T}{\partial r} - \frac{2}{r} T = 0 \text{ , on } \omega_c \quad . \tag{69}$$

Thus, the integration of the product of the two functions δg and Δg_F over the unit sphere leads to the following equation,

$$\iint_{\omega} \delta g \quad \Delta g_F \, d\omega = \iint_{\omega} \alpha_1 \quad \alpha_2 \, d\omega = 0 \quad . \tag{70}$$

These above integrals are equal to zero because, with respect to (68) and (69), the subsequent expression is equal to zero,

$$\iint_{\omega_s} \alpha_{1.s} \quad \alpha_{2.s} \, d\omega + \iint_{\omega_c} \alpha_{1.c} \quad \alpha_{2.c} \quad d\omega = 0 \quad . \tag{71}$$

Now, the developments in spherical harmonics are introduced for δg and Δg_F in the equation (70).

The perturbation potential T has the following series development, [21],

$$T = \sum_{n=0}^{\infty} \bar{t}_n \left[\frac{R}{r} \right]^{n+1} X_n (\varphi, \lambda) \quad . \tag{72}$$

$\bar{t}_n$ are the Stokes constants. R is the radius of the globe and r is the distance from the center of gravity. $X_n(\varphi, \lambda)$ is a symbolism, it represents all the surface spherical harmonics of the degree n and of the order m = 0, ... , n; i.e. the functions $\bar{P}_{n.m}(\varphi) \cos m\lambda$ and $\bar{P}_{n.m}(\varphi) \sin m\lambda$, (m = 0,..., n), whereat the terms $\bar{P}_{n.m}(\varphi)$ stand for the normalized associated spherical harmonics, [15]. The series (72) is valid and uniformly convergent at the surface of the globe and in the exterior space, $r \geq R$, if T is a regular functions for $r = R$. On the surface of the globe of the Earth, the series development (72) and the relations (1) and (7) for the global functions $\delta g = \alpha_1$ and $\Delta g_F = \alpha_2$ lead to the following developments,

$$\alpha_1(\varphi, \lambda) = \frac{1}{R} \sum_{n=0}^{\infty} (n+1) \, \bar{t}_n \, X_n (\varphi, \lambda) = \sum_{n=0}^{\infty} \bar{a}_{1.n} \quad X_n (\varphi, \lambda) \quad , \tag{73}$$

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$$\alpha_2(\varphi, \lambda) = \frac{1}{R}\sum_{n=0}^{\infty}(n-1)\,\bar{t}_n\;X_n(\varphi, \lambda) = \sum_{n=0}^{\infty}\bar{a}_{2.n}\;X_n(\varphi, \lambda)\;. \qquad (74)$$

The equations (73) and (74) yield

$$\bar{a}_{1.n} = \frac{1}{R}(n+1)\,\bar{t}_n \quad , \qquad (75)$$

$$\bar{t}_n = \frac{R}{n+1}\,\bar{a}_{1.n} \quad , \qquad (76)$$

$$\bar{a}_{2.n} = \frac{1}{R}(n-1)\,\bar{t}_n \quad , \qquad (77)$$

$$\bar{t}_n = \frac{R}{n-1}\,\bar{a}_{2.n} \quad , \qquad (78)$$

$$\bar{a}_{2.n} = \frac{n-1}{n+1}\,\bar{a}_{1.n} \quad . \qquad (79)$$

The $X_n(\varphi, \lambda)$ expressions are considered as normalized orthogonal base functions,

$$\iint_{\omega} X_n(\varphi, \lambda) X_{\bar{n}}(\varphi, \lambda)\, d\omega = \begin{Bmatrix} 0, \text{ if } n \neq \bar{n} \\ \\ 4\pi\,, \text{ if } n = \bar{n} \end{Bmatrix} \quad . \qquad (80)$$

The terms α_1 and α_2 in (70) are substituted by the developments (73) and (74). Accounting for (80), (70)(73) and (74) lead to

$$0 = \sum_{n=0}^{\infty}(n^2 - 1)\,\bar{t}_n^2 \quad . \qquad (81)$$

In case, the kernel function H' is introduced, (51), the summation in (72) has to begin with the degree n=2. Thus, in the subspace of the harmonics of 2nd and higher degree, the following condition is valid for the proof of the uniqueness,

$$0 = \sum_{n=2}^{\infty}(n^2 - 1)\,\bar{t}_n^2 \quad , \qquad (82)$$

or

$$0 = 3\,\bar{t}_2^2 + 8\,\bar{t}_3^2 + 15\,\bar{t}_4^2 + \ldots \quad . \qquad (83)$$

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From (82), the constraints reveal necessarily,

$$\bar{t}_n = 0 \quad , \quad (n = 2, 3, 4, \ldots) \quad . \tag{84}$$

The potential T' has the development, (46)(72),

$$T' = \sum_{n=2}^{\infty} \bar{t}_n \left(\frac{R}{r}\right)^{n+1} X_n(\varphi, \lambda) \quad . \tag{85}$$

Thus, the condition for the uniqueness of the mixed boundary value problem of the second kind, i.e. the problem with the boundary values of the shape (68) and (69), leads necessarily to the equation, (84)(85),

$$T' = 0 \quad . \tag{86}$$

The above derivations show that (68) and (69) are followed by (84). Accounting for (84) and (85), it is proved that (86) is a necessary consequence of (68) and (69).

Thus, it can be concluded that the solution of the second mixed boundary value problem is unique in the subspace of the spherical harmonics of 2nd and higher degree.

The equation (81) does not allow a statement about the uniqueness of the solution of the second mixed boundary value problem in the space of the regular functions, i.e. the space of the harmonics of all the degrees n = 0,1,2,3, ... and of all the orders m = 0,1,2,3, ... , n. The reason is that in this case the right hand side of (81) has the following form

$$0 = -\bar{t}_0^{\,2} + 3\,\bar{t}_2^{\,2} + 8\,\bar{t}_3^{\,2} + 15\,\bar{t}_4^{\,2} + \ldots \quad . \tag{87}$$

This is again a quadratic form without mixed terms, as (83). But, (83) is positive definite and (87) is not positive definite. In order to fulfill (87), all the amounts of $\bar{t}_n$, (n = 0,1,2, ...), must not necessarily be equal to zero. (87) does not lead to an extension of the validity of the equation (86) over the space of all the spherical harmonics.

Now, the uniqueness of the second mixed boundary value problem of geodesy is to be proved by the consideration of the homogeneous form of the fundamental integral equation of the second kind (67). It has the following shape,

$$0 = \alpha_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \, \alpha_{1.c} \, d\omega \quad , \quad \text{on } \omega_c \quad . \tag{88}$$

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It is necessary to investigate into wether (88) has only the trivial solution, i.e. whether $\alpha_{1.c}$ has to be equal to zero over ω_c. In this case, a look on (68) and (46) does show that T' is zero on ω and in the exterior space of ω. Thus, if (88) allows only the trivial solution for $\alpha_{1.c}$,

$$\alpha_{1.c} = 0 \quad , \tag{89}$$

it can be taken for granted that the solution of the second mixed boundary value problem of the geodesy will be unique in the subspace of the spherical harmonics of 2nd and higher degree.

The test points of the relation (88) are situated on ω_c only and the integration in (88) covers also ω_c only. Now, to have the preferences of a global coverage and of global functions, a generalization of (88) is convenient. The area over which the test points move is extended from ω_c to ω, i.e. from the continents to whole the globe. And the integration area is extended in the same way from ω_c to ω.

Within the frame of this generalization, a function η is introduced on ω, it has the following properties,

$$\eta = \eta(\varphi, \lambda) = \eta_s = 0 \quad , \text{ on } \omega_s \ , \tag{90}$$

$$\eta = \eta(\varphi, \lambda) = \eta_c = \alpha_{1.c} \ , \text{ on } \omega_c \ . \tag{91}$$

Obviously, if η is equal to zero all over the globe ω then the criterion for the uniqueness, (89), is also valid on ω_c.

Further, a second global function η^* is defined on ω in the following way,

$$\eta^* = \eta^*(\varphi, \lambda) = \eta^*_s \ , \text{ on } \omega_s \ , \tag{92}$$

$$\eta^* = \eta^*(\varphi, \lambda) = \eta^*_c = 0, \text{ on } \omega_c \ . \tag{93}$$

The combination of the relations (51), (88) and (90) to (93) gives rise to the following equation which defines η^*,

$$\eta^*(P) = \eta(P) - \frac{1}{2\pi} \iint_{\omega} H'(P,\bar{P}) \ \eta(\bar{P}) \, d\omega \ , \text{ on } \omega \ . \tag{94}$$

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In (94), the points P and $\overline{P}$ move over whole the spherical surface ω . If the equation (94) is fulfilled, **in this case**, also the crucial equation (88) follows to be valid. Indeed, the test points on ω_s give

$$\eta_s^* = \eta_s - \frac{1}{2\pi} \iint_{\omega_s} H'_{s.s} \, \eta_s \, d\omega - \frac{1}{2\pi} \iint_{\omega_c} H'_{s.c} \, \eta_c \, d\omega, \quad \text{on} \quad \omega_s , \tag{95}$$

separating the integration over the oceans from that over the continents. In a similar way, the test points on ω_c lead to

$$\eta_c^* = \eta_c - \frac{1}{2\pi} \iint_{\omega_s} H'_{c.s} \, \eta_s \, d\omega - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \, \eta_c \, d\omega, \quad \text{on} \quad \omega_c . \tag{96}$$

(95) and (96) can be combined to (94).

The relations (90) to (93) are introduced into (96) and the following equation is obtained,

$$0 = \alpha_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \, \alpha_{1.c} \, d\omega \ , \quad \text{on} \quad \omega_c \ . \tag{97}$$

This is the crucial relation (88).

Along the same way, the oceanic relation (95) transforms into

$$\eta_s^* = - \frac{1}{2\pi} \iint_{\omega_c} H'_{s.c} \, \alpha_{1.c} \, d\omega \ , \quad \text{on} \quad \omega_s \ . \tag{98}$$

The amounts of η_s^* are not of direct interest, they derive by (98).

The expressions (90) to (93) show that the product of η and η^* over the sphere ω is equal to zero,

$$\iint_{\omega} \eta \ \eta^* \, d\omega = 0 \ . \tag{99}$$

Now, the function η^* in (99) is substituted by the expression on the right hand side of (94),

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$$0 = \iint\limits_{\omega} \eta^2 \, d\omega - \frac{1}{2\pi} \iint\limits_{\omega} \eta(P) \, d\omega \iint\limits_{\omega} H'(P,\bar{P}) \, \eta(\bar{P}) \, d\omega \quad , \text{ on } \omega \ . \tag{100}$$

According to (90) and (91), η is a bounded and regular function. Along the coastline, the η values have a discontinuity. Under these presuppositions, the function η on the globe ω can be represented by a development in the surface spherical harmonics $X_n(\varphi, \lambda)$, (72), of all degrees and orders, (n = 0,1, 2, ...). This representation has to fulfill the two constraints (90) and (91). Along the coastline, it takes the value $\frac{1}{2}\,\alpha_{1.c}$; i.e. the mean of the two values that are reached approaching the coast from the seaside and from the landside.

In this context, a theorem of E. W. Hobson is fundamental, [16] page 344; (see also the Dirichlet - Jordan criterion for Fourier series). It has the following expression word for word - in the terminology and symbolism of E. W. Hobson - : "The Laplace's series

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_0^{\pi} \int_{-\pi}^{+\pi} f(\theta', \phi') \, P_n(\cos\gamma) \sin\theta' \, d\theta' \, d\phi' \ ,$$

in which $f(\theta', \phi')$ has an absolutely convergent integral (Lebesgue) over the spherical surface, will converge at (θ, ϕ) to the value $f(\theta, \phi)$ if (θ, ϕ) is a point of continuity of the function with respect to (θ, ϕ), or to the value

$$\frac{1}{2} \left\{ f_1(\theta, \phi) + f_2(\theta, \phi) \right\} \ ,$$

if the point is such that there passes through it a line of discontinuity such that $f_1(\theta, \phi)$, $f_2(\theta, \phi)$ are the limits of the function at the point taken from the two sides of the line, provided that the function $\phi(\gamma)$, which is the mean value of the function $f(\theta, \phi)$, for each fixed value of γ over the small circle for which γ has that value, has bounded variation in the whole interval $(0, \pi)$ of γ." In the here discussed applications, the function $f(\theta, \phi)$ is always bounded, i.e. the perturbation potential T, the gravity deviations δg and the free-air anomalies Δg_F. Therefore, the use of the above theorem is admitted.

A bounded error or a bounded deviation in the functions η and/or η^* along the coastline cannot diminish the validity of (99), since the length of the coastline is bounded. The amount of the area of the surface of the coastline, i.e. the share of it on the surface of the unit sphere, is equal to zero. This is trivial, the coastline has the dimension of a length and not of a surface.

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Hence, it is allowed to introduce for η and η^* series developments in normalized surface spherical harmonics $X_n(\varphi, \lambda)$, (90) to (100). The concerned development for η is

$$\eta = \sum_{n=0}^{\infty} \bar{\eta}_n \; X_n(\varphi, \lambda) , \tag{101}$$

$$\iint_{\omega} X_n^2 \, d\omega = 4\pi . \tag{102}$$

The $\bar{\eta}_n$ values are the Stokes constants. The η function in the equation (100) is substituted by (101). H' is expressed in the equation (100) by the Legendre functions, (51). These Legendre functions $P_n(\cos\psi)$ can be developed by the decomposition formula, [15]. For normalized harmonics, it has the following shape,

$$P_n(\cos \psi_{P,\bar{P}}) = \frac{1}{2n+1} \; X_n(\varphi_P, \lambda_P) \; X_n(\varphi_{\bar{P}}, \lambda_{\bar{P}}) . \tag{103}$$

The relations (100) (101) (103) lead to the following equation,

$$0 = \sum_{n=0}^{\infty} \bar{\eta}_n^2 - \sum_{n=2}^{\infty} \frac{2}{n+1} \; \bar{\eta}_n^2 . \tag{104}$$

(104) can be brought into the following form

$$0 = \bar{\eta}_0^2 + \bar{\eta}_1^2 + \sum_{n=2}^{\infty} \frac{n-1}{n+1} \; \bar{\eta}_n^2 , \tag{105}$$

with

$$\frac{n-1}{n+1} > 0 , \; (n = 2,3,4, \ldots) . \tag{106}$$

The right hand side of (105) is a quadratic form without mixed terms. All the coefficients of (105) are positive. Thus, (105) is a positive definite quadratic form. The relations

$$\bar{\eta}_n = 0 , \; (n = 0,1,2,3,4, \ldots) , \tag{107}$$

result from (105) and (106).

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In case, the equation (105) would give no evidence about the value of $\bar{\eta}_1$, for instance, it is possible to carry out the following experiment of thoughts. In this context, the relation (90) turns out to be useful for the evaluation of the amount of $\bar{\eta}_1$. Indeed, (101) gives

$$\eta = \bar{\eta}_1 \cdot X_1(\varphi, \lambda) \quad , \tag{108}$$

all the other values $\bar{\eta}_n$, (n = 0,2,3,4, ...), are supposed to be equal to zero, for this experiment of thoughts. $\bar{\eta}_1$ represents the three Stokes constants of the degree n = 1 and of the order m = 0,1. $X_1(\varphi, \lambda)$ symbolizes the three corresponding surface spherical harmonics. If Y_i, (i = 2,3,4), is a running denomination of these spherical harmonics, as in (59) (60) (61), (108) can be put into the following form,

$$\eta = \sum_{i=2}^{4} \bar{\bar{\eta}}_i \cdot Y_i(\varphi, \lambda) \quad . \tag{109}$$

(90) and (109), (273) to (277), give

$$0 = \eta_s = \sum_{i=2}^{4} \bar{\bar{\eta}}_i \cdot Y_i(\varphi, \lambda) \text{ , on } \omega_s \text{ .} \tag{110}$$

It follows obviously that the three Stokes constants $\bar{\bar{\eta}}_i$, (i = 2,3,4), must necessarily be equal to zero, (see also [5], page 344),

$$\bar{\bar{\eta}}_i = 0 \text{ , } (i = 2,3,4). \tag{111}$$

The area ω_s has more than 3 points. (110) can be specialized for all the points of ω_s with the concerned parameter couple (φ, λ). This abundance of homogeneous determining equations for $\bar{\bar{\eta}}_i$, (i = 2,3,4), gives (111).
Thus,

$$\bar{\eta}_1 = 0 \text{ .} \tag{112}$$

A point in the midst of ω_s can be chosen as the pole of the (φ, λ) - system, since a rotation of the coordinate system does not influence the degree of the spherical - harmonic representation of a function, as η_s. The multiplication of (110) by the factors sin φ , cos λ , sin λ , (see (273) to (277)), - one after another - and a succeeding integration over λ along a cirle within ω_s around the pole does lead to (111).

Returning back to (107), the following relation is obtained,

$$\eta = 0 \quad . \tag{113}$$

(113) is valid for whole the globe ω . The equation (91) gives

$$\alpha_{1.c} = 0 \quad . \tag{114}$$

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(114) corroborates that the crucial homogeneous integral equation of the second kind (88) has the trivial solution only.

Thus, the inhomogeneous integral equation of the second kind, (57) (62), has a unique solution only. The second mixed boundary value problem for the sphere has a unique solution in the subspace of the spherical harmonics of second and higher degree.

The above investigations about the uniqueness are valid in the subspace of the spherical harmonics of 2nd and higher degree only. The concerned criterion equations (82), (87), and (104) can be generalized by the supplementary inclusion of the harmonics of 0th and 1st degree. However, in this case, a form is reached which does not allow the investigation of the uniqueness of the solution for this generalized problem. Indeed, the inclusion of the spherical harmonics of 0th and 1st degree necessitates in (100) the shift from H' to H, (50) (51). It follows that (104) must be replaced by

$$0 = \sum_{n=0}^{\infty} \bar{\eta}_n^{\,2} - \sum_{n=0}^{\infty} \frac{2}{n+1} \bar{\eta}_n^{\,2} \quad , \tag{115}$$

or

$$0 = - \bar{\eta}_0^{\,2} + \frac{1}{3} \bar{\eta}_2^{\,2} + \frac{1}{2} \bar{\eta}_3^{\,2} + \frac{3}{5} \bar{\eta}_4^{\,2} + \ldots \quad . \tag{116}$$

The right hand side of (116) is not positive definite. Therefore, the relation (116) does not allow the consequence that (113) must necessarily be fulfilled also in the space of the spherical harmonics of all degrees, (n = 0,1,2, ...). Along the lines of (116), the prove of the uniqueness of the solution of the second mixed boundary value problem for n = 0,1,2, ... is not possible.

Considering (70), the concerned generalized criterion equation is (81), instead of (82). (81) leads to

$$0 = - \bar{t}_0^{\,2} + 3\, \bar{t}_2^{\,2} + 8\, \bar{t}_3^{\,2} + 15\, \bar{t}_4^{\,2} + \ldots \quad . \tag{117}$$

This is not a positive definite quadratic form, similarly as (116). The consequences

$$\bar{t}_n = 0 \; , \quad (n = 0,1,2, \ldots) \quad , \tag{117a}$$

and

$$T = 0 \tag{117b}$$

are not allowed. However, by no means, (116) and (117) allow not the disproval of the uniqueness of this problem in the space of the harmonics of all degrees, (n= 0,1,2, ...). It is an open question till now. Later on, by the equations from (297) to (354), a criterion about the uniqueness of the second mixed boundary value problem in the space of all the harmonics of the degrees n = 0,1,2, ... will be derived. It allows to obtain a clear standpoint about this question for an arbitrary course of the coast-line.

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4.3. The kernel function of the integral equation of the second kind and the Hotine function and their property of being positive definite and closed

For the further deductions and for the numerical computation operations, it is of interest to investigate into whether the kernel functions $H'_{c.c}(P,\overline{P})$ and $H_{c.c}(P,\overline{P})$ are positive definite or not. $H'_{c.c}(P,\overline{P})$ and $H_{c.c}(P,\overline{P})$ are identical with the functions $H'(P,\overline{P})$and $H(P,\overline{P})$ in case that both the points P and $\overline{P}$ are situated only in the continental area ω_c, (50) (51) (57) (62). The kernel functions $H'_{c.c}$ and $H_{c.c}$ of the integral equation (62) are acknowledged as positive definite kernels if the following inequation formulas are fulfilled, [1] [2] [29],

$$\theta'(\varkappa,\varkappa) = \iint_{\omega_c} \varkappa(P) \quad d\omega_P \iint_{\omega_c} H'_{c.c}(P,\overline{P}) \quad \varkappa(\overline{P}) \quad d\omega_{\overline{P}} > 0 , \qquad (118)$$

and, as to $H_{c.c}$,

$$\theta(\varkappa,\varkappa) = \iint_{\omega_c} \varkappa(P) \quad d\omega_P \iint_{\omega_c} H_{c.c}(P,\overline{P}) \quad \varkappa(\overline{P}) \quad d\omega_{\overline{P}} > 0 . \qquad (119)$$

$\varkappa$ is an arbitrarily chosen regular function on the continents ω_c ,

$$\varkappa \not\equiv 0 . \qquad (120)$$

The area of validity of the inequation formulas (118) and (119) can be enlarged, it can be expanded from the continents ω_c to whole the globe ω . In the course of the realization of this conception, the function $\varkappa$ is now understood as a global function. It takes the following values on the globe ω ,

$$\varkappa = \varkappa_s = 0 , \quad \text{on } \omega_s , \qquad (121)$$

and

$$\varkappa = \varkappa_c , \quad \text{on } \omega_c . \qquad (122)$$

These considerations and the presuppositions connected with the formulas (121) (122) transform (118) into a global relation,

$$\theta'(\varkappa,\varkappa) = \iint_{\omega} \varkappa(P) \quad d\omega_P \iint_{\omega} H'(P,\overline{P}) \quad \varkappa(\overline{P}) \quad d\omega_{\overline{P}} > 0 . \qquad (123)$$

Similarly, (119) leads to (124),

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$$\Theta(\varkappa, \varkappa) = \iint_{\omega} \varkappa(P)\, d\,\omega_P \iint_{\omega} H(P, \overline{P})\; \varkappa(\overline{P})\, d\,\omega_{\overline{P}} > 0. \qquad (124)$$

In the transition from (118) to (123) and from (119) to (124), it is permissible to replace $H'_{c.c}$ by H' and $H_{c.c}$ by H. In the equations(123) and (124), the points P and $\overline{P}$ move over whole the globe ω . According to the formerly cited theorem of E. W. Hobson, [16] page 344, the function $\varkappa$ can be represented by a global surface spherical harmonic development in the inequation formulas (123) and (124). The introduction of the series development for η , (101), into the relation (100) was a similar procedure. Thus, the $\varkappa$ function in (123) and (124) is substituted by the series development, (121) (122),

$$\varkappa = \sum_{n=0}^{\infty} \overline{\varkappa}_n \cdot X_n(\varphi, \lambda) \quad , \text{on } \omega . \qquad (125)$$

This substitution will not modify the validity of (123) and (124), as the introduction of (101) into (100) did not modify the validity of (100). The functions H' and H in (123) and (124) are replaced by the developments (50) and (51),

$$H' = \sum_{n=2}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi) \quad , \qquad (126)$$

$$H = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi) \quad . \qquad (127)$$

The decomposition formula (103) substitutes the Legendre functions $P_n(\cos\psi)$ in (126) (127) by the normalized surface spherical harmonics $X_n(\varphi, \lambda)$,

$$P_n(\cos\psi_{P.\overline{P}}) = \frac{1}{2n+1} \; X_n(\varphi_P, \lambda_P)\; X_n(\varphi_{\overline{P}}, \lambda_{\overline{P}}) \quad . \qquad (128)$$

The relations (125) (126) (128) change (123) into the following shape,

$$\Theta'(\varkappa, \varkappa) = \iint_{\omega} \sum_{i=0}^{\infty} \overline{\varkappa}_i \cdot X_i(\varphi_P, \lambda_P)\, d\,\omega_P \cdot \left\{ \iint_{\omega} \sum_{j=2}^{\infty} \frac{1}{j+1} X_j(\varphi_P, \lambda_P) \cdot \right.$$

$$\left. \cdot X_j(\varphi_{\overline{P}}, \lambda_{\overline{P}}) \sum_{k=0}^{\infty} \overline{\varkappa}_k \; X_k(\varphi_{\overline{P}}, \lambda_{\overline{P}})\, d\,\omega_{\overline{P}} \right\} > 0 \quad . \qquad (129)$$

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The orthogonality relations (80) are introduced into (129) and the following relation is obtained,

$$\theta'(\varkappa,\varkappa) = \iint\limits_{\omega} \sum_{i=0}^{\infty} \bar{\varkappa}_i \; X_i(\varphi_P, \lambda_P)\, d\,\omega_P \; \sum_{j=2}^{\infty} \frac{1}{j+1} \; X_j(\varphi_P, \lambda_P)\; \bar{\varkappa}_j \;\cdot$$

$$\cdot \iint\limits_{\omega} X_j^{\,2}\,(\varphi_{\bar{P}}, \lambda_{\bar{P}})\; d\,\omega_{\bar{P}} \; > 0 \quad . \qquad (130)$$

Hence,

$$\theta'(\varkappa,\varkappa) = 4\pi \sum_{i=2}^{\infty} \bar{\varkappa}_i^{\,2} \frac{1}{i+1} \iint\limits_{\omega} X_i^{\,2}(\varphi_P, \lambda_P)\; d\,\omega_P > 0 \quad . \qquad (131)$$

This relation leads to

$$\theta'(\varkappa,\varkappa) = (4\pi)^2 \sum_{n=2}^{\infty} \frac{1}{n+1} \; \bar{\varkappa}_n^{\,2} \; > 0 \; , \qquad (132)$$

and, in case the harmonics of 0th and 1st degree are included,

$$\theta(\varkappa,\varkappa) = (4\pi)^2 \sum_{n=0}^{\infty} \frac{1}{n+1} \; \bar{\varkappa}_n^{\,2} \; > 0 \quad . \qquad (133)$$

The inequations (132) and (133) show that $H'_{c.c}$ and $H_{c.c}$ are positive definite kernel functions for the area ω_c, (118) (119).

At last, a special situation for (132) and (133) is to be touched. If the following relations about the function $\varkappa$ are valid,

$$\bar{\varkappa}_0 \neq 0 \; , \; \bar{\varkappa}_1 \neq 0 \; , \qquad (133a)$$

and

$$\bar{\varkappa}_i = 0 \; , \; (i = 2,3,4, \ldots) \; , \qquad (133b)$$

in this case, the operator $\theta'(\varkappa,\varkappa)$ is equal to zero, (120) (132). The relation (132) is not fulfilled, that seems to be the consequence of (133a) and (133b). But, the relations (133a) and (133b) are in contradiction to (121). Before the background of (133a) and (133b), the equation (121) cannot be fulfilled obviously unless the area of ω_s is zero, $\omega_s = 0$; see also (108) to (112). The relations (119) (120) (121) (122) lead therefore necessarily to the fact that

DOI: https://doi.org/10.2312/zipe.1986.084

$$\theta'(\varkappa, \varkappa) > 0 \ , \tag{133c}$$

$$\theta(\varkappa, \varkappa) > 0 \ . \tag{133d}$$

Thus, $H'_{c.c}$ and $H_{c.c}$ are positive definite kernel functions within ω_c, in any case and if $\omega_s \neq 0$; see also the developments about (139a) to (139d).

As regards the inequations (132) and (133), the function $\varkappa$ that appears in these formulas and that is defined by (121) (122), this function cannot be chosen absolutely arbitrary all over the globe. On the oceans, it is equal to zero. However, on the continents, the values of it are arbitrary.

Now, the argument domain is extended over whole the globe ω in order to prove that both the kernel functions H' and H are positive definite for the area ω. In this context, whole the globe ω is covered by a function $\xi = \xi(\varphi, \lambda)$. It is an arbitrarily chosen regular function on the globe ω,

$$\xi = \xi(\varphi, \lambda) \not\equiv 0 \ , \quad \text{on } \omega . \tag{134}$$

The operators θ'^{*} and θ^{*} are applied on the function ξ, and the following inequations are obtained,

$$\theta'^{*}(\xi, \xi) = \iint_{\omega} \xi(P)\, d\omega_P \iint_{\omega} H'(P,\overline{P})\, \xi(\overline{P})\, d\omega_{\overline{P}} > 0 \ , \tag{135}$$

and

$$\theta^{*}(\xi, \xi) = \iint_{\omega} \xi(P)\, d\omega_P \iint_{\omega} H(P,\overline{P})\, \xi(\overline{P})\, d\omega_{\overline{P}} > 0 \ . \tag{136}$$

ξ has a series development in spherical harmonics,

$$\xi = \sum_{n=0}^{\infty} \xi_n \, X_n(\varphi, \lambda) \ , \quad \text{on } \omega . \tag{137}$$

Along the way which did lead from (123) and (124) to (132) and (133), it is possible to transform (135) and (136) into the following shape,

$$\theta'^{*}(\xi, \xi) = (4\pi)^2 \sum_{n=2}^{\infty} \frac{1}{n+1} \, \xi_n^2 > 0 \ , \tag{138}$$

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$$\theta^*(\xi, \xi) = (4\pi)^2 \sum_{n=0}^{\infty} \frac{1}{n+1} \bar{\xi}_n^2 > 0 \quad . \tag{139}$$

The relations (135) and (136) are the conditions for the Hotine function H' and H that must be fulfilled in case they have the property of being positive definite for whole the area ω . A glance on (138) and (139) shows that (135) and (136) are right.

It is unnecessary to mention that - for a global coverage - H' is positive definite in the subspace of the harmonics of the 2nd and higher degree only, since it is defined only there, similarly as the Stokes function. Indeed, if

$$\bar{\xi}_0 \neq 0 , \quad \bar{\xi}_1 \neq 0 \quad , \tag{139a}$$

$$\bar{\xi}_i = 0 , \ (i = 2,3,4, \ldots) \quad , \tag{139b}$$

in this case, the operator $\theta'^*(\xi, \xi)$ is equal to zero, (138). Thus, (134) (135) do not lead to

$$\theta'^*(\xi, \xi) > 0 \quad . \tag{139c}$$

However, the presupposition about the subspace - per definitionem -

$$\bar{\xi}_0 = \bar{\xi}_1 = 0 \tag{139d}$$

leads to the right situation.

Hence, the Hotine function H' is positive definite for the subspace of the harmonics of 2nd and higher degree and for a global argument domain.

The Hotine function H has the property of being positive definite for the space of all the harmonics of the degree n = 0,1,2, ... , i.e. for all the regular functions and for a global argument domain.

This fact is also evidenced by the treatment of the operators $\theta'(\varkappa, \varkappa)$ and $\theta(\varkappa, \varkappa)$ which lead from (123) and (124) to (132) and (133). Indeed, in case that the amount of the oceanic area ω_s, (121), does undergo the passage to the limit $\omega_s \longrightarrow 0$, in this case, the consideration about the operators $\theta'(\varkappa, \varkappa)$ and $\theta(\varkappa, \varkappa)$ corroborate the considerations about the operators $\theta'^*(\xi, \xi)$ and $\theta^*(\xi, \xi)$. But, with (121), the passage to the limit $\omega_s \longrightarrow 0$ is not compatible with the procedure to conclude that the maintenance of

$$\bar{\varkappa}_i = 0 , \ (i = 2,3,4, \ldots) \quad , \tag{139e}$$

does necessarily lead to

$$\bar{\varkappa}_0 = \bar{\varkappa}_1 = 0 \quad , \tag{139f}$$

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see (110) to (113), and (133a) to (133d). Therefore, the operator $\theta'^{*}(\xi, \xi)$ has the property of being positive definite in the subspace of the spherical harmonics of 2nd and higher degree only.

The kernel functions $H'_{c.c}$ and $H_{c.c}$ of the integral equations of the second kind, (62) (67d), have not only the property of being positive definite. As to the properties of a kernel function, it is important to know whether it is a closed function or not. This is of interest for the problem of the uniqueness and of the inversion. Concerning the definition of this property of a function, the function $H'_{c.c}(P,\bar{P})$ is a closed one, if it fulfills the following condition. The two functions $\mu^{*}(P)$ and $\mu(P)$ are introduced as regular functions for the area of the continents. The kernel function $H'_{c.c}(P,\bar{P})$ transforms the function $\mu(\bar{P})$ into $\mu^{*}(P)$ by the following integral relation,

$$\mu^{*}(P) = \iint_{\omega_c} H'_{c.c}(P,\bar{P}) \; \mu(\bar{P}) \, d\,\omega_{\bar{P}} \quad . \tag{140}$$

The points P and $\bar{P}$ move over the area of the continents ω_c only. In case, the combination of the equation (141),

$$\mu^{*}(P) = 0 \; , \quad \text{on} \;\; \omega_c \; , \tag{141}$$

with the equation (140) leads necessarily to the fact that

$$\mu(P) = 0 \; , \quad \text{on} \;\; \omega_c \; , \tag{142}$$

in this case, the kernel function $H'_{c.c}(P,\bar{P})$ is said to be a closed function for the area ω_c and for regular functions, [29]. Thus, the following relation for $H'_{c.c}$,

$$0 = \iint_{\omega_c} H'_{c.c}(P,\bar{P}) \; \mu(\bar{P}) \, d\omega_{\bar{P}} \; , \tag{143}$$

must give the equation (142) for μ , in case the kernel function $H'_{c.c}$ is closed.

For the proof that the kernel function is closed, the relation (140) is transformed into a global shape, in close neighbourhood to the ideas connected with the transtion from (118) to (124). Along these lines, the generalized function μ has the following relations,

$$\mu = \mu_s = 0 \; , \quad \text{on} \;\; \omega_s \; , \tag{144}$$

$$\mu = \mu_c \; , \quad \text{on} \;\; \omega_c \; . \tag{145}$$

As regards μ^{*}, it has the following constraints,

DOI: https://doi.org/10.2312/zipe.1986.084

$$\mu^* = \mu^*_s \ , \qquad \text{on } \omega_s \ , \tag{146}$$

$$\mu^* = \mu^*_c = 0 \ , \text{ on } \omega_c \ . \tag{147}$$

The global product of μ and μ^* is equal to zero,

$$\iint_{\omega} \mu \, \mu^* \, d\omega = 0 \ . \tag{148}$$

The equation (140) changes over into the global shape,

$$\mu^*(P) = \iint_{\omega} H'(P,\overline{P}) \, \mu(\overline{P}) \, d\,\omega_{\overline{P}} \ , \tag{149}$$

as a look on the equations (144) to (147) does show. (148) and (149) give

$$\iint_{\omega} \mu(P) \, d\,\omega_P \iint_{\omega} H'(P,\overline{P}) \, \mu(\overline{P}) \, d\,\omega_{\overline{P}} = 0 \ . \tag{150}$$

In (150), H' is substituted by (126). The decomposition formula (128) and (123) are taken into account and the subsequent relation is obtained, (132),

$$\theta'(\mu,\mu) = (4\pi)^2 \sum_{n=2}^{\infty} \frac{1}{n+1} \, \overline{\mu}_n^{\,2} = 0 \ . \tag{151}$$

The global function μ has the following series development in spherical harmonics, according to the sentence of Hobson [16],

$$\mu = \sum_{n=0}^{\infty} \overline{\mu}_n \, X_n(\varphi,\lambda) \ . \tag{152}$$

If the function $\varkappa$ is replaced by μ, in (123), then the left hand side of (150) is obtained. It is equal to the operator $\theta'(\mu,\mu)$.

The computations from (129) to (132) demonstrate that, because of (150), the operator $\theta'(\mu,\mu)$ does fulfill the condition (151). Hence, the following Stokes constants $\overline{\mu}_n$ must obviously be equal to zero,

$$\overline{\mu}_n = 0, \ (n = 2,3,4, \ldots). \tag{153}$$

The supplementary relations

$$\overline{\mu}_0 = 0 \ , \quad \overline{\mu}_1 = 0 \tag{154}$$

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can be confirmed easily. (152) and (153) give

$$\mu = \bar{\mu}_0 \cdot X_0 + \bar{\mu}_1 \cdot X_1 \quad . \tag{155}$$

However, because of (144) and because ω_s has an area of finite amount,

$$\omega_s \neq 0 \quad , \tag{156}$$

it concludes obviously that (154) is right. See also (108) to (113). Thus,

$$\bar{\mu}_n = 0 \text{ , } (n = 0,1,2, \ldots) \text{ .} \tag{157}$$

(157) gives

$$\mu = 0 \quad , \quad \text{on } \omega \text{ ,} \tag{158}$$

and

$$\mu_c = 0 \quad , \quad \text{on } \omega_c \text{ .} \tag{159}$$

These developments above show that the condition (142) must be fulfilled necessarily if (143) is presupposed.

Thus, the function $H'_{c.c}$ is a closed function.

Now, the question is to be investigated whether the kernel function $H_{c.c}$ has also the property of being a closed function including the harmonics of 0th and first degree. This problem can be investigated by the methods already applied in the inversion of the function $H'_{c.c}$, (140) to (159). In this context, the function $H'_{c.c}$ has to be replaced by $H_{c.c}$ in the relations (140), (143), (149), (150). Thereby, the left hand side of (150) turns to the operator $\Theta(\mu,\mu)$ of the relation (124). Thus, the investigation into whether the kernel function $H_{c.c}$ is a closed function in the space of the regular functions along ω_c is governed by the criterion relation, (133),

$$\Theta(\mu,\mu) = (4\pi)^2 \sum_{n=0}^{\infty} \frac{1}{n+1} \bar{\mu}_n^{\,2} = 0 \quad . \tag{160}$$

A comparison with the definition of μ and $\bar{\mu}_n$, (144) (152), does demonstrate that the condition for the property of $H_{c.c}$ to be a closed kernel function, (see the corresponding relation (143)),

$$0 = \iint_{\omega_c} H_{c.c} (P,\bar{P}) \, \mu(\bar{P}) \, d\omega_{\bar{P}} \quad , \tag{161}$$

can never be fulfilled unless the function μ does vanish,

$$\mu(P) = 0 \quad , \quad \text{on } \omega_c . \tag{162}$$

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(160), (161), (162) prove the fact that the kernel function $H_{c.c}$ has the property of being a closed function.

Therefore, the kernel functions $H'_{c.c}$ and $H_{c.c}$ are symmetrical, positive definite and closed within their domain of definition. The singularity for $\psi = 0$ is removable, it is evidenced by developments similar to those deductions for the Stokes function, (16) (17) (18).

Extending the domain of definition from ω_c to ω, from a surface part to whole the surface, these properties, (symmetrical, positive definite and closed), are valid for the kernel function H also. And these properties are valid in the same way for the kernel function H' in the subspace of the harmonics of the 2nd and higher degrees.

As to the singularity for $\psi \longrightarrow 0$, in this context, see also the derivations from (461) to (477) about the Stokes functions. Similar considerations can be carried out for the Hotine function, in order to avoid the singularity of H, H', $H_{c.c}$, $H'_{c.c}$ for $\psi \longrightarrow 0$, without any loss of precision.

4.4. The continental chief minor of the Hotine matrix

The above developments about the second mixed boundary value problem of geodesy allow a representation in the matrix calculus also. The matrix calculus is more convenient for the further numerical calculations. In this context, the global unit sphere ω is divided into certain surface compartments of the uniform size $\Delta\omega$, Fig. 1. The 47 hatched compartments of figure 1 belong to the oceanic area ω_s and the 101 white compartments represent the continental area ω_c. For these compartments, the mean values of the functions

$$\delta g ,$$

$$\Delta g_F ,$$

$$\frac{4\pi}{R} T ,$$

are understood as the components of the vectors

$$\underline{\underline{a}}1 ,$$

$$\underline{\underline{a}}2 ,$$

$$\underline{\underline{a}}3 .$$

$$\underline{\underline{a}}1 = \begin{pmatrix} \cdots \\ (\delta g)_i \\ \cdots \end{pmatrix} , \qquad (163)$$

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$$\underline{\underline{a}}_2 = \begin{pmatrix} \cdots \\ (\Delta g_F)_i \\ \cdots \end{pmatrix} , \tag{164}$$

$$\underline{\underline{a}}_3 = \frac{4\pi}{R} \cdot \begin{pmatrix} \cdots \\ (T)_i \\ \cdots \end{pmatrix} ; \tag{165}$$

$$i = 1, 2, 3, \ldots , k , k + 1, \ldots , q . \tag{166}$$

The indices

$$i = 1, 2, \ldots , k , (k = 101) , \tag{167}$$

belong to the continental part, and

$$i = 102, 103, \ldots , q , (q = 148) , \tag{168}$$

refer to the oceanic compartments.

The expression (136) about the Hotine function H can be transformed into the matrix shape. The quadratic form $\Phi(\underline{\underline{x}} , \underline{\underline{x}})$ is reached,

$$\Phi(\underline{\underline{x}}, \underline{\underline{x}}) = \underline{\underline{x}}^T \underline{\underline{H}} \, \underline{\underline{x}} > 0 . \tag{169}$$

$\underline{\underline{H}}$ is the following matrix,

$$\underline{\underline{H}} = \begin{bmatrix} h_{1.1} & \cdots & h_{1.k} , & h_{1.k+1} & \cdots & h_{1.q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{k.1} & \cdots & h_{k.k} , & h_{k.k+1} & \cdots & h_{k.q} \\ h_{k+1.1} & \cdots & h_{k+1.k} , & h_{k+1.k+1} & \cdots & h_{k+1.q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{q.1} & \cdots & h_{q.k} , & h_{q.k+1} & \cdots & h_{q.q} \end{bmatrix} . \tag{170}$$

The vector $\underline{\underline{x}}$,

$$\underline{\underline{x}} = \begin{pmatrix} \cdots \\ x_i \\ \cdots \end{pmatrix} , \tag{171}$$

$$i = 1, 2, 3, \ldots , k , k + 1, \ldots , q , \tag{172}$$

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is an arbitrary vector which represents a regular function covering whole the globe ω . The components x_i refer to the individual compartments.The inequation formulas (136) (139) show that $\Phi(\underline{\underline{x}},\underline{\underline{x}})$ is a positive definite quadratic form. Thus, according to a theorem of the matrix calculus, and because the kernel function H is closed, (160),

$$\det \underline{\underline{H}} > 0 \quad . \tag{173}$$

As it is well-known, (173) leads to the fact that all the eigenvalues of the matrix $\underline{\underline{H}}$ are positive, [3]. Furthermore, because of (173), all the chief minors of the matrix $\underline{\underline{H}}$ have also positive amounts, [4][3].

$$\det \underline{\underline{H}}_1 > 0 \quad , \tag{174}$$

$$\det \underline{\underline{H}}_2 > 0 \quad , \tag{175}$$

$$\det \underline{\underline{H}}_3 > 0 \quad , \tag{176}$$

$$\ldots \quad .$$

$$\det \underline{\underline{H}}_1 = \begin{vmatrix} h_{1.1} \end{vmatrix} \quad , \tag{177}$$

$$\det \underline{\underline{H}}_2 = \begin{vmatrix} h_{1.1} & h_{1.2} \\ h_{2.1} & h_{2.2} \end{vmatrix} \quad , \tag{178}$$

$$\det \underline{\underline{H}}_3 = \begin{vmatrix} h_{1.1} & h_{1.2} & h_{1.3} \\ h_{2.1} & h_{2.2} & h_{2.3} \\ h_{3.1} & h_{3.2} & h_{3.3} \end{vmatrix} \quad . \tag{179}$$

Particularly, the continental chief minor of the matrix $\underline{\underline{H}}$ is positive, i.e. the determinant of the matrix

$$\underline{\underline{H}}_k = \underline{\underline{H}}_{c.c} = \begin{bmatrix} h_{1.1} & \ldots & h_{1.k} \\ \ldots & \ldots & \ldots \\ h_{k.1} & \ldots & h_{k.k} \end{bmatrix} \quad . \tag{180}$$

Thus, the continental chief minor is

$$\det \underline{\underline{H}}_{c.c} = \begin{vmatrix} h_{1.1} & \ldots & h_{1.k} \\ \ldots & \ldots & \ldots \\ h_{k.1} & \ldots & h_{k.k} \end{vmatrix} > 0 \quad . \tag{181}$$

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The matrix relations (173) and (181) are in keeping with (136) and (119).

In case, $\underline{\underline{m}}$ is an arbitrarily choosen vector of the dimension q = 148, representing a global function, the relation

$$\underline{\underline{m}}^* = \underline{\underline{H}} \ \underline{\underline{m}} \tag{182}$$

has the inversion

$$\underline{\underline{m}} = \underline{\underline{H}}^{-1} \ \underline{\underline{m}}^* \ . \tag{183}$$

$\underline{\underline{H}}^{-1}$ is the well-defined inverse matrix of $\underline{\underline{H}}$, (170).

In the same way, if $\underline{\underline{m}}_c$ is an arbitrarily choosen vector of the dimension k = 101, representing a continental function, in this case, the relation

$$\underline{\underline{m}}_c^* = \underline{\underline{H}}_{c.c} \ \underline{\underline{m}}_c \ , \tag{184}$$

has the inversion

$$\underline{\underline{m}}_c = \underline{\underline{H}}_{c.c}^{-1} \ \underline{\underline{m}}_c^* \ . \tag{185}$$

$\underline{\underline{H}}_{c.c}^{-1}$ is the well-defined inverse matrix of $\underline{\underline{H}}_{c.c}$, (180). The corresponding kernel function $H_{c.c}$ was proved to be closed, see (161) (162). Concerning the inversion of $\underline{\underline{H}}_{c.c}$, the study of the corresponding considerations about the inversion of the Stokes matrices $\underline{\underline{S}}$ and $\underline{\underline{S}}_{s.s}$ is recommended, cf. (462) to (477).

The developments about the continental chief minor of the matrix $\underline{\underline{H}}$, carried out by the relations (169) to (185), can be transferred to the matrix $\underline{\underline{H}}'$ which derives from the Hotine function H', (49) (51). It refers to the subspace of the spherical harmonics of the 2nd and higher degree. The matrix $\underline{\underline{H}}'$ has a shape similar to that of $\underline{\underline{H}}$, (170).

$$\underline{\underline{H}}' = \begin{bmatrix} h'_{1.1} & \cdots & h'_{1.k} & , h'_{1.k+1} & \cdots & h'_{1.q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h'_{k.1} & \cdots & h'_{k.k} & , h'_{k.k+1} & \cdots & h'_{k.q} \\ h'_{k+1.1} & \cdots & h'_{k+1.k} & , h'_{k+1.k+1} & \cdots & h'_{k+1.q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h'_{q.1} & \cdots & h'_{q.k} & , h'_{q.k+1} & \cdots & h'_{q.q} \end{bmatrix} , \tag{186}$$

$$i = 1, 2, 3, \ldots , k , k + 1 , \ldots , q ; \tag{187}$$

$$k = 101 \quad , \quad q = 148 \quad . \tag{188}$$

The elements $h'_{j.l}$ of the matrix $\underline{\underline{H}}'$ are computed from the function H' with the

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argument ψ , it is the spherical distance between the center points of the two compartments which have the indices i = j and i = l as attributes. In a similar way, the elements $h_{j.l}$ of the matrix $\underline{\underline{H}}$, (170), have to be computed from the function H, (48) (50).

Now, a vector $\underline{\underline{x}}'$ is introduced. $\underline{\underline{x}}'$ represents an arbitrary regular function described in the subspace of the spherical harmonics of 2nd and higher order. Whole the Earth is covered by the components of $\underline{\underline{x}}'$.

$$\underline{\underline{x}}' = \begin{pmatrix} \cdots \\ x_i' \\ \cdots \end{pmatrix} , \tag{189}$$

$$i = 1, 2, 3, \ldots , k , k + 1 , \ldots , q , \tag{190}$$

$$k = 101 , \quad q = 148 . \tag{191}$$

The expression (135) about the Hotine function H' is transformed into the matrix calculus. Thus, the quadratic form $\Phi'(\underline{\underline{x}}', \underline{\underline{x}}')$ is obtained, cf. (169),

$$\Phi'(\underline{\underline{x}}',\underline{\underline{x}}') = (\underline{\underline{x}}')^T \underline{\underline{H}}' \underline{\underline{x}}' > 0 . \tag{192}$$

The inequation formula (135) shows that $\Phi'(\underline{\underline{x}}',\underline{\underline{x}}')$ is a positive definite quadratic form. Since the corresponding kernel function H' was proved to be closed, positive definite and symmetrical, a well-known theorem of the matrix calculus shows that (192) does lead to the inequation

$$\det \underline{\underline{H}}' > 0 . \tag{193}$$

(193) has the consequence that all the eigenvalues of the matrix $\underline{\underline{H}}'$ are positive. This fact has the inference that all the chief minors of the matrix $\underline{\underline{H}}'$ have positive amounts also, [29] [31].

$$\det \underline{\underline{H}}_1' > 0 , \tag{194}$$

$$\det \underline{\underline{H}}_2' > 0 , \tag{195}$$

$$\det \underline{\underline{H}}_3' > 0 , \tag{196}$$

$$\ldots ,$$

$$\det \underline{\underline{H}}_1' = \left| h_{1.1}' \right| > 0 , \tag{197}$$

$$\det \underline{\underline{H}}_2' = \begin{vmatrix} h_{1.1}' & h_{1.2}' \\ h_{2.1}' & h_{2.2}' \end{vmatrix} > 0 , \tag{198}$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$\det \underline{\underline{W}}_3' = \begin{vmatrix} h_{1.1}' & h_{1.2}' & h_{1.3}' \\ h_{2.1}' & h_{2.2}' & h_{2.3}' \\ h_{3.1}' & h_{3.2}' & h_{3.3}' \end{vmatrix} > 0 \quad . \tag{199}$$

In the here discussed applications, the continental chief minor of the matrix $\underline{\underline{H}}'$ is in the fore. The concerned matrix is,

$$\underline{\underline{H}}_{c.c}' = \begin{pmatrix} h_{1.1}' & \dots & h_{1.k}' \\ \dots & \dots & \dots \\ h_{k.1}' & \dots & h_{k.k}' \end{pmatrix} \quad . \tag{200}$$

The rang defect of $\underline{\underline{H}}_{c.c}'$ is equal to zero. The matrix (200) leads to the continental chief minor,

$$\det \underline{\underline{H}}_{c.c}' = \begin{vmatrix} h_{1.1}' & \dots & h_{1.k}' \\ \dots & \dots & \dots \\ h_{k.1}' & \dots & h_{k.k}' \end{vmatrix} > 0 \quad . \tag{201}$$

The matrix relations from (192) to (201) harmonize with the developments (118), (123), (126), (132), (133c), (135), (138).

Because of (201), it is possible to compute the inverse matrix of $\underline{\underline{H}}_{c.c}'$, (200), i.e. $(\underline{\underline{H}}_{c.c}')^{-1}$. In case, $\underline{\underline{m}}'$ is an arbitrarily choosen vector of the dimension q = 148 which represents a regular function in the subspace of the harmonics of 2nd and higher degree, the relation

$$(\underline{\underline{m}}')^* = \underline{\underline{H}}' \; \underline{\underline{m}}' \quad , \tag{202}$$

is inverted by

$$\underline{\underline{m}}' = (\underline{\underline{H}}')^{-1} \, (\underline{\underline{m}}')^* \quad , \tag{203}$$

$(\underline{\underline{H}}')^{-1}$ is the inverse of (186). (202) and (203) conduct to analogous developments for $\underline{\underline{H}}_{c.c}'$. $\underline{\underline{m}}_c'$ is an arbitrarily choosen vector of the dimension k = 101, as μ_c is an arbitrarily choosen regular function in the area ω_c, (145). $\underline{\underline{m}}_c'$ represents a regular function in the area ω_c. Hense, the relation

$$(\underline{\underline{m}}_c')^* = \underline{\underline{H}}_{c.c}' \, \underline{\underline{m}}_c' \quad , \tag{204}$$

has the well-defined inverse equation

$$\underline{\underline{m}}_c' = (\underline{\underline{H}}_{c.c}')^{-1} \, (\underline{\underline{m}}_c')^* \quad . \tag{205}$$

(204) and (205) are in keeping with the developments from (140) to (159).

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As concerns the inversion of $\underline{\underline{H}}'_{c.c}$, the corresponding considerations about the inversion of the Stokes matrices $\underline{\underline{S}}$ and $\underline{\underline{S}}_{s.s}$ are recommended, cf. (462) to (477).

It is unnecessary to mention that the here discussed model of 101 continental and 47 oceanic compartments for the compartment division of the surface of the Earth ω is an example only. This example was treated here as a base for the fixation of the train of the ideas. The smaller the compartments $\Delta\omega$, the more precise the solution. The greater the numbers of k and q, the more detailed the result, this fact is self-explanatory.

4.5. The solution of the second mixed boundary value problem of the geodesy and its formulation in the matrix calculus

In the functional analysis, several different methods are well developed for the solution of an inhomogeneous integral equation of the second kind such as (67a) and (67d). The solution method that works with the eigenfunctions of (67a) does not seem to be convenient for the here treated applications. It will be difficult to find the eigenfunction that refer to the kernel function $H'_{c.c}$ or $H_{c.c}$. In case working with eigenfunctions, a continuous analytical function for the course of the coastline seems to be indispensable, this is a problem extremely difficult to solve, [10][14].

The iteration procedure or the method of the iterated kernel has probably a greater chance to be of use in the numerical solution of (67a), [10][14][18][22][27][29]. As it is well-known, the iteration procedure for (67a) works in the following way. The first step has the approximation

$$(\alpha_{1.c})_1 = -\beta_{1.c} \quad , \tag{206}$$

the second step is

$$(\alpha_{1.c})_2 = -\beta_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \, \beta_{1.c} \, d\omega, \text{ on } \omega_c \quad . \tag{207}$$

Thus, the general frame of the iteration procedure is

$$(\alpha_{1.c})_{i+1} = -\beta_{1.c} + \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \, (\alpha_{1.c})_i \, d\omega \quad , \quad \text{on } \omega_c \quad . \tag{208}$$

In the relation (208), the term $(\alpha_{1.c})_i$ can be expressed by $(\alpha_{1.c})_{i-1}$, the concerned expression is obtained substituting $(\alpha_{1.c})_{i+1}$ in (208) by $(\alpha_{1.c})_i$, and $(\alpha_{1.c})_i$ by $(\alpha_{1.c})_{i-1}$. A repeated application of this method leads to the fact that $(\alpha_{1.c})_{i+1}$ can be expressed in terms of $\beta_{1.c}$ only. Thus,

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$$(\alpha_{1.c})_{i+1} = -\beta_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} (H'_{c.c})^{(i)} \beta_{1.c} \, d\omega \text{ , on } \omega_c \, . \qquad (209)$$

$(H'_{c.c})^{(i)}$ is the i-th iteration kernel of $H'_{c.c}$. Along these lines, the following solution of the integral equations (67a) and (67d) can be obtained,

$$\alpha_{1.c} = -\beta_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} \Gamma \cdot \beta_{1.c} \, d\omega \text{ , on } \omega_c \, . \qquad (210)$$

Γ is the resolvent kernel. However, the very problem is open whether

$$(\alpha_{1.c})_i \longrightarrow \alpha_{1.c} \qquad (211)$$

$$\text{for } i \longrightarrow \infty \, .$$

In the here discussed applications, the convergence problem implicated in the iteration procedure, (208), has still to be investigated.

Concerning further details about the eigenfunction method and about the iterated kernel and the reciprocal kernel function, the study of one of the numerous textbooks about the theory of integral equations is recommended here, [10] [22][29].

Now, the integral equations (67a) and (67d) are to be transferred into the matrix calculus. Along this way, a method will be developed which seems to be conducive to a numerical solution of the problem. The vectors $\underline{\underline{a}}_1$ and $\underline{\underline{a}}_2$ and the Hotine matrix $\underline{\underline{H}}'$ transform the relation (54) or (62) into the matrix shape, (19)(163)(164)(165),

$$0 = \underline{\underline{a}}_1 - \underline{\underline{H}}' \, \Delta\omega^* \, \underline{\underline{a}}_1 - \underline{\underline{a}}_2 \text{ , on } \omega \text{ ;} \qquad (212)$$

$$\Delta\omega^* = \frac{1}{2\pi} \Delta\omega \, . \qquad (213)$$

The impact of the spherical harmonics of 0th and 1st degree on the vectors $\underline{\underline{a}}_1$ and $\underline{\underline{a}}_2$ and on the equation (212) will be discussed later, cf. (287)(288).

The components of the three vectors $\underline{\underline{a}}_1$, $\underline{\underline{a}}_2$, $\underline{\underline{a}}_3$ are divided into two parts accounting for the fact that they refer either to the area ω_c or ω_s respectively, (163)(164)(165).

$$\underline{\underline{a}}_1 = \begin{pmatrix} \underline{\underline{a}}_{1.c} \\ \underline{\underline{a}}_{1.s} \end{pmatrix} \, , \qquad (214)$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$\underline{\underline{a}}_2 = \begin{pmatrix} \underline{\underline{a}}_{2.c} \\ \underline{\underline{a}}_{2.s} \end{pmatrix} , \tag{215}$$

$$\underline{\underline{a}}_3 = \begin{pmatrix} \underline{\underline{a}}_{3.c} \\ \underline{\underline{a}}_{3.s} \end{pmatrix} . \tag{216}$$

The relations (214)(215) are introduced into the matrix shape of the integral equation, (212),

$$0 = \underline{\underline{a}}_{1.c} - \underline{\underline{H}}'_{c.c}\,\Delta\omega^*\,\underline{\underline{a}}_{1.c} - \underline{\underline{H}}'_{c.s}\,\Delta\omega^*\,\underline{\underline{a}}_{1.s} - \underline{\underline{a}}_{2.c} , \tag{217}$$

$$0 = \underline{\underline{a}}_{1.s} - \underline{\underline{H}}'_{s.c}\,\Delta\omega^*\,\underline{\underline{a}}_{1.c} - \underline{\underline{H}}'_{s.s}\,\Delta\omega^*\,\underline{\underline{a}}_{1.s} - \underline{\underline{a}}_{2.s} . \tag{218}$$

In the equations(217) and (218), the vectors $\underline{\underline{a}}_{1.s}$ and $\underline{\underline{a}}_{2.c}$ represent the given boundary values, i.e. the gravity deviations δg on the oceans and the free-air anomalies Δg_F on the continents. Whereas, the vectors $\underline{\underline{a}}_{1.c}$ and $\underline{\underline{a}}_{2.s}$ have components that are unknown in the beginning, i.e. the gravity deviations δg on the continents and the free-air anomalies on the oceans.

The matrix relation (217) can be brought into the following form, separating the known terms and the unknown terms,

$$\underline{\underline{a}}_{2.c} + \underline{\underline{H}}'_{c.s}\,\Delta\omega^*\,\underline{\underline{a}}_{1.s} = (\underline{\underline{E}} - \underline{\underline{H}}'_{c.c}\,\Delta\omega^*)\,\underline{\underline{a}}_{1.c} . \tag{219}$$

$\underline{\underline{E}}$ is the unit matrix of the same dimension as $\underline{\underline{H}}'_{c.c}$. The left hand side of (219) is known by the boundary values, it is denominated by $\underline{\underline{b}}_c$,

$$\underline{\underline{b}}_c = \underline{\underline{a}}_{2.c} + \underline{\underline{H}}'_{c.s}\,\Delta\omega^*\,\underline{\underline{a}}_{1.s} . \tag{220}$$

The matrix in the braces on the right hand side of (219) is symbolized by $\underline{\underline{K}}'_{c.c}$,

$$\underline{\underline{K}}'_{c.c} = \underline{\underline{E}} - \underline{\underline{H}}'_{c.c}\,\Delta\omega^* . \tag{221}$$

With (220) and (221), the matrix relation (219) turns into

$$\underline{\underline{b}}_c = \underline{\underline{K}}'_{c.c}\,\underline{\underline{a}}_{1.c} . \tag{222}$$

The relation (222) is the matrix version of the inhomogeneous integral equation of the second kind, (67a).

The homogeneous shape of (222) is

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$$0 = \underline{\underline{K}}'_{c.c} \; \underline{\underline{a}}_{1.c} \quad . \qquad (223)$$

The solution of (222) is unique if (223) leads to

$$\underline{\underline{a}}_{1.c} = 0 \quad . \qquad (224)$$

The determinant of $\underline{\underline{K}}'_{c.c}$ is positive definite if, (169)(182)(202)(204),

$$(\underline{\underline{m}}_c)^T \; \underline{\underline{K}}'_{c.c} \quad \underline{\underline{m}}_c > 0 \quad , \qquad (225)$$

as it is proved in the matrix calculus. The superscript T denominates the transposition. $\underline{\underline{m}}_c$ represents an arbitrary function on ω_c . Returning back to (100) and accounting for (90), it reveals that the right hand side of (100) is equal to

$$\Lambda' = \iint_{\omega_c} \eta^2 \, d\omega - \frac{1}{2\pi} \iint_{\omega_c} \eta(P) \, d\omega \iint_{\omega_c} H'(P,\bar{P}) \eta(\bar{P}) d\omega \quad . \qquad (226)$$

In case, the left hand side of (225) is multiplied by $\Delta\omega$, (213), the thus obtained quadratic form shows to be the matrix version of the Λ' expression, (226). Further, the derivations from (100) to (105) prove that Λ' is equal to

$$\Lambda' = 4\pi \left[\bar{\eta}_0^2 + \bar{\eta}_1^2 + \sum_{n=2}^{\infty} \frac{n-1}{n+1} \bar{\eta}_n^2 \right] > 0 \quad . \qquad (227)$$

Since Λ' is positive definite, (227), it follows that (225) is right. Thus, the determinant of the matrix $\underline{\underline{K}}'_{c.c}$ is positive definite,

$$\det \underline{\underline{K}}'_{c.c} > 0 \quad . \qquad (228)$$

Further, the rang defect of $\underline{\underline{K}}'_{c.c}$ is to be considered. Regarding the matrix equation (223), a look on (221) and (67) shows that (223) is the matrix version of (67). Earlier investigations did show that (67) is necessarily followed by the condition (114). If this situation is transferred into the matrix shape, the validity of (224) as a consequence of (223) is corroborated.

Thus, the rang defect of the matrix $\underline{\underline{K}}'_{c.c}$ is equal to zero. Therefore, because of (223)(224)(228), the matrix relation (222) can be inverted.

$$\underline{\underline{a}}_{1.c} = (\underline{\underline{K}}'_{c.c})^{-1} \; \underline{\underline{b}}_c \quad . \qquad (229)$$

$(\underline{\underline{K}}'_{c.c})^{-1}$ is the well-founded inverse of $\underline{\underline{K}}'_{c.c}$.

DOI: https://doi.org/10.2312/zipe.1986.084

(220) and (229) yield

$$\underline{\underline{a}}_{1.c} = (\underline{\underline{K}}'_{c.c})^{-1} \; (\underline{\underline{a}}_{2.c} + \underline{\underline{H}}'_{c.s} \; \Delta\omega^* \; \underline{\underline{a}}_{1.s}) \quad . \tag{230}$$

(230) is important. It is the matrix version of the solution of the second mixed boundary value problem of the geodesy in the subspace of the spherical harmonics of 2nd and higher degree. $\underline{\underline{a}}_{2.c}$ represents the known free-air anomalies on the continents and $\underline{\underline{a}}_{1.s}$ symbolizes the given boundary values on the oceans, i.e. the gravity deviations δg. (230) computes the δg values on the continents, i.e. the components of $\underline{\underline{a}}_{1.c}$.

The generalization of the matrix shape solution of the second mixed boundary value problem by the inclusion of the spherical harmonics of the 0th and 1st degree should not escape the notice here. To follow up this aim, $\underline{\underline{H}}'$ is to be replaced by $\underline{\underline{H}}$ in (212). $\underline{\underline{H}}'_{c.c}$, $\underline{\underline{H}}'_{c.s}$, $\underline{\underline{H}}'_{s.c}$ and $\underline{\underline{H}}'_{s.s}$ in (217)(218)(219)(220)(221) change over into the corresponding matrices without the prime. The important relations (221) and (222) get the following shape,

$$\underline{\underline{b}}_{c} = \underline{\underline{K}}_{c.c} \; \underline{\underline{a}}_{1.c} \quad , \tag{231}$$

with

$$\underline{\underline{K}}_{c.c} = \underline{\underline{E}} - \underline{\underline{H}}_{c.c} \; \Delta\omega^* \quad , \tag{232}$$

and

$$\underline{\underline{b}}_{c} = \underline{\underline{a}}_{2.c} + \underline{\underline{H}}_{c.s} \; \Delta\omega^* \; \underline{\underline{a}}_{1.s} \quad . \tag{233}$$

Thus, the solution of the second mixed boundary value problem of the geodesy in the space of the spherical harmonics of all degrees, $(n = 0,1,2, \ldots)$, is obtained by a transformation of (230) into (234),

$$\underline{\underline{a}}_{1.c} = (\underline{\underline{K}}_{c.c})^{-1} \; (\underline{\underline{a}}_{2.c} + \underline{\underline{H}}_{c.s} \; \Delta\omega^* \; \underline{\underline{a}}_{1.s}) \quad . \tag{234}$$

However, some discussions about the inverse $(\underline{\underline{K}}_{c.c})^{-1}$ in the relation (234) are indispensable. At first, the question arises whether the quadratic form

$$(\underline{\underline{m}}_{c})^{T} \; \underline{\underline{K}}_{c.c} \; \underline{\underline{m}}_{c} \tag{235}$$

is positive definite or not, (225). The inclusion of the harmonics of 0th and 1st degree transfers Λ' , (227), to Λ ,

$$\Lambda = 4\pi \left[\sum_{n=0}^{\infty} \bar{\eta}_n^2 - \sum_{n=0}^{\infty} \frac{2}{n+1} \bar{\eta}_n^2 \right] \quad , \tag{236}$$

or

$$\Lambda = 4\pi \left[- \bar{\eta}_0^2 + \frac{1}{3} \bar{\eta}_2^2 + \frac{1}{2} \bar{\eta}_3^2 + \frac{3}{5} \bar{\eta}_4^2 + \ldots \right] \quad , \tag{237}$$

see (104)(105)(115)(116). Λ is not positive or negative definite, as against

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to Λ'. Therefore, the matrix shape of Λ is also not positive or negative definite, it is the quadratic form (235). It follows that the determinant of the matrix $\underline{\underline{K}}_{c.c}$ is not necessarily positive definite or negative definite. However, to avoid misunderstandings, this fact does not meen that the determinant of $\underline{\underline{K}}_{c.c}$ is always equal to zero, by no means.

Despite of (237),it is possible that

$$\det \underline{\underline{K}}_{c.c} \neq 0 \quad . \tag{238}$$

(238) is the decisive condition whether the inverse $(\underline{\underline{K}}_{c.c})^{-1}$ is well-founded or not, (234). But, along the lines of (235) and (237), a prove of (238) is not possible.

Later on, a criterion about the uniqueness of the solution of this problem will be derived by the relations from (297) to (362).

4.6. The compatibility conditions of the second mixed boundary value Problem of the geodesy

In this context, the defect of the kernel function of an integral equation is in the fore. It is the number of the linear independent solutions of the homogeneous integral equation of the first kind,

$$\int_a^b K^*(s,t)\ \varphi^*(s)\ ds = 0 \quad . \tag{238a}$$

In the here discussed applications, the general relation (238a) turns to four special types of integral equations.

I. $$\iint_\omega H'\alpha \quad d\omega = 0 \quad . \tag{239}$$

(239) allows that $\alpha \neq 0$. Obviously, this fact is derived by (148) and (149), if $\mu^*(P) = 0$. Along these lines, and with (135) and (138), a condition of the following shape is obtained,

$$\sum_{n=2}^{\infty} \frac{1}{n+1} \xi_n^2 = 0 \, .$$

Thus, it is allowed to have

$$\xi_0 \neq 0 \text{ and } \xi_1 \neq 0 \, .$$

The defect of the kernel function H' for the area ω is equal to 4. The 4 independent solutions for the α function are

DOI: https://doi.org/10.2312/zipe.1986.084

$$\bar{\eta}_j \cdot Y_j(\varphi, \lambda) \quad , (j = 1,2,3,4) \quad .$$

Y_j symbolizes the well-known four surface spherical harmonics of the 0th and 1st degree and order, the detailed shape of them is given by (273) to (276). The real reason why (239) allows $\alpha \neq 0$ is found in the fact that H' does not include the Y_j functions, $(j = 1,2,3,4)$. $\bar{\eta}_j$ are arbitrary constants here. The matter is evidenced easily by the introduction of the harmonics series developments for H' and α in (239).

The second type is as follows.

$$\text{II.} \quad \iint_{\omega} H \alpha \quad d\omega = 0 \quad . \tag{240}$$

(240) requires $\alpha = 0$. The defect of the kernel function H is equal to zero for the area ω , it is proved easily by global series developments in terms of surface spherical harmonics for the functions H and α , see also (47) and (76). As it is easily found, the relation (240) gives by (136) and (139) an expression of the form

$$\overset{*}{\Theta}(\xi, \xi) = 0 = \sum_{n=0}^{\infty} \frac{1}{n+1} \bar{\xi}_n^2 \quad .$$

Thus, also $\bar{\xi}_0$ and $\bar{\xi}_1$ have to be equal to zero.

The third type has the following shape,

$$\text{III.} \quad \iint_{\omega_c} H'_{c.c} \; \alpha_c \quad d\omega = 0 \quad . \tag{241}$$

(241) leads necessarily to the relation $\alpha_c = 0$. Thus, the defect of $H'_{c.c}$ is equal to zero for the part ω_c of the globe ω . This matter is thoroughly discussed in context with the relations (142)(143).

The last type is

$$\text{IV.} \quad \iint_{\omega_c} H_{c.c} \; \alpha_c \quad d\omega = 0 \quad . \tag{242}$$

It gives no other choice than $\alpha_c = 0$. The defect of the kernel $H_{c.c}$ is equal to zero for the partial area ω_c of the globe ω , cf. (161)(162).

The fact that the type I has the defect 4 whereas the types II, III, IV have the defect zero, this circumstance is of cardinal importance. Further, it is evidenced that a continuous enlargement of ω_c to reach ω , $\omega_c \longrightarrow \omega$, is accompanied by a discontinuous jump of the defect from zero to four. This situation leads to four compatibility conditions for the mixed boundary values, to be discussed now.

Sure, it is indispensable to check whether the potential obtained as the

DOI: https://doi.org/10.2312/zipe.1986.084

It is convenient, to remember on some developments in spherical harmonics, (72) to (79),

$$\delta g = \alpha_1 = \sum_{n=0}^{\infty} \bar{a}_{1.n} \quad X_n \quad , \tag{251}$$

(251) is the spherical - harmonic development for the united boundary values

$$f \left\{ \nu_s , \nu_c \right\}$$

and

$$\nu_s$$

on the globe ω,

$$\Delta g_F = \alpha_2 = \sum_{n=0}^{\infty} \bar{a}_{2.n} \quad X_n \quad , \tag{252}$$

$$T' = \sum_{n=2}^{\infty} \bar{t}_n \quad X_n \quad . \tag{253}$$

Considering the equations (1) and (7), the constraints that must be fulfilled by the final solution potential Ξ' , (250), these constraints have the following shape, (1)(7),

$$- \frac{\partial \Xi'}{\partial r} = \nu_s \quad , \quad \text{on} \quad \omega_s \quad , \tag{254}$$

$$- \frac{\partial \Xi'}{\partial r} - \frac{2}{R} \Xi' = \nu_c \quad , \quad \text{on} \quad \omega_c \quad . \tag{255}$$

(254) and (255) are the conditions aspired and endeavoured to be fulfilled a posteriori.

But, the really obtained solution has another figure. The spatial Hotine function $H'(r,\psi)$ is, (126),[11],

$$H'(r,\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n+1} \left(\frac{R}{r} \right)^{n+1} P_n (\cos \psi) \quad . \tag{256}$$

The relations (128), (251) and (256) are introduced into (249), the spatial solution potential $\Xi'(r,\varphi,\lambda)$ is obtained,

$$\Xi'(r,\varphi,\lambda) = \frac{R}{4\pi} \iint_{\omega} H'(r,\psi) \; \delta g \; d\omega \; , \tag{257}$$

or

DOI: https://doi.org/10.2312/zipe.1986.084

solution of the 2nd mixed boundary value problem does fulfill the a priori given boundary conditions in reality and a posteriori. At first, the solution in the subspace of the harmonics of the 2nd and higher degree is to be considered. On the oceans, the gravity deviations δg are the a priori prescribed boundary conditions, they have the given functional expression ν_s ,

$$\nu_s = (\delta g)_s = \alpha_{1.s} \quad , \text{ on } \quad \omega_s \quad , \tag{243}$$

or, in vector form,

$$\underline{\underline{a}}_{1.s} \quad .$$

Along the continents, the free-air anomalies Δg_F serve as the boundary values. Here, the a priori prescribed functional expression is ν_c ,

$$\nu_c = (\Delta g_F)_c = \alpha_{2.c} \quad , \text{ on } \omega_c \quad , \tag{244}$$

or, as a vectorial representation,

$$\underline{\underline{a}}_{2.c} \quad .$$

ν_s and ν_c are arbitrarily choosen regular functions on ω_s resp. on ω_c . The gravity deviation on the continents is

$$\alpha_{1.c} \quad \text{or} \quad \underline{\underline{a}}_{1.c} \quad ,$$

it is computed in terms of ν_s and ν_c by (230),

$$(\delta g)_c = \alpha_{1.c} = \quad f\{(\delta g)_s , (\Delta g_F)_c\} \quad , \tag{245}$$

or

$$\alpha_{1.c} = f\ \{\alpha_{1.s} , \alpha_{2.c}\} \quad , \tag{246}$$

or

$$\alpha_{1.c} = f\ \{\ \nu_s , \nu_c\ \} \quad , \tag{247}$$

or
$$\underline{\underline{a}}_{1.c} = f^*\{\ \underline{\underline{a}}_{1.s} , \underline{\underline{a}}_{2.c}\} \quad . \tag{248}$$

In the geodetic reality, ν_s and ν_c are empirically given functions.

Now, the Hotine integral is applied in order to find the potential Ξ' which is the solution of the mixed boundary value problem,

$$\Xi' = \frac{R}{4\pi} \iint_{\omega} H' \ \delta g \ d\omega \ . \tag{249}$$

Regarding (249), on the oceans, δg has to be substituted by ν_s, and, on the continents, δg has to be replaced by the values computed according to (247). Thus,

$$\Xi' = \frac{R}{4\pi} \iint_{\omega_c} H' \cdot f\ \{\nu_s , \nu_c\}\ d\omega + \frac{R}{4\pi} \iint_{\omega_s} H' \nu_s \ d\omega \ . \tag{250}$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$\Xi'(r,\varphi,\lambda) = \frac{R}{4\pi} \iint_{\omega} \Big[\sum_{n=2}^{\infty} \frac{1}{n+1} \left(\frac{R}{r}\right)^{n+1} X_n(\varphi_P, \lambda_P)\; X_n(\varphi_{\overline{P}}, \lambda_{\overline{P}}) \Big] \cdot$$

$$\cdot \Big[\sum_{n=0}^{\infty} \bar{a}_{1.n}\; X_n(\varphi_{\overline{P}}, \lambda_{\overline{P}}) \Big] \; d\omega \quad . \tag{258}$$

$$\Xi'(r,\varphi,\lambda) = \sum_{n=2}^{\infty} \frac{1}{n+1}\; \bar{a}_{1.n} \frac{R^{n+2}}{r^{n+1}} \quad X_n(\varphi,\lambda) \quad . \tag{259}$$

In order to find the boundary values of $\Xi'(r,\varphi,\lambda)$ on the surface of the Earth, the operators

$$-\frac{\partial}{\partial r} \quad , \tag{260}$$

and

$$-\frac{\partial}{\partial r} - \frac{2}{R} \quad , \tag{261}$$

must be applied to $\Xi'(r,\varphi,\lambda)$. Then, the transition to the limit $r \longrightarrow R$ must be executed, (259) and (260) give on ω_s

$$-\left(\frac{\partial \Xi'}{\partial r}\right)_s = \left(\sum_{n=2}^{\infty} \bar{a}_{1.n} \; X_n\right)_s \quad , \text{ on } \omega_s \quad . \tag{262}$$

ν_s is the oceanic value of (251),

$$\nu_s = \left(\sum_{n=0}^{\infty} \bar{a}_{1.n} \; X_n\right)_s \quad , \text{ on } \omega_s \quad . \tag{263}$$

Thus,

$$-\left(\frac{\partial \Xi'}{\partial r}\right)_s = \nu_s - \left(\bar{a}_{1.0}\, X_0 + \bar{a}_{1.1}\; X_1\right)_s \quad . \tag{264}$$

(259) and (261) lead to the following relation on the continents,

$$-\left(\frac{\partial \Xi'}{\partial r} + \frac{2}{R}\,\Xi'\right)_c = \left(\sum_{n=2}^{\infty} \bar{a}_{1.n}\; X_n\right)_c - \left(\sum_{n=2}^{\infty} \frac{2}{n+1}\,\bar{a}_{1.n}\; X_n\right)_c . \tag{265}$$

Hence, because of (79),

$$-\left(\frac{\partial \Xi'}{\partial r} + \frac{2}{R}\,\Xi'\right)_c = \left(\sum_{n=2}^{\infty} \bar{a}_{2.n} \; X_n\right)_c \quad . \tag{266}$$

DOI: https://doi.org/10.2312/zipe.1986.084

ν_c is the continental value of (252),

$$\nu_c = \Big(\sum_{n=0}^{\infty} \bar{a}_{2.n} \quad X_n \Big)_c \quad , \tag{267}$$

thus,

$$- \Big(\frac{\partial \Xi'}{\partial r} + \frac{2}{R} \Xi' \Big)_c = \nu_c - \big(\bar{a}_{2.0} \; X_0 + \bar{a}_{2.1} \; X_1 \big)_c \quad . \tag{268}$$

Or, in terms of the Stokes constants of the harmonics development for δg, (251),

$$- \Big(\frac{\partial \Xi'}{\partial r} + \frac{2}{R} \Xi' \Big)_c = \nu_c + \big(\bar{a}_{1.0} \; X_0 \big)_c \quad . \tag{269}$$

Summarizing the considerations connected with the relations from (243) to (269), the second mixed boundary value problem in the subspace of the spherical harmonics of 2nd and higher degree has a certain special feature. It aspires to reach the empirically given boundary values ν_s and ν_c, (254) and (255). But in reality, this aim cannot be attained. The really reached boundary values are described by (264) and (269). There is a well-founded discrepancy between the desired and the really attained mixed boundary values, unless the a posteriori found global δg values do not include the 4 terms of the spherical harmonics of the 0th and 1st degree. The global δg values are a unification of the given boundary values on the oceans,

$$(\delta g)_s = \alpha_{1.s} = \nu_s \quad , \tag{270}$$

and of the a posteriori computed values on the continents, (247),

$$(\delta g)_c = \alpha_{1.c} = f \{ \nu_s \, , \, \nu_c \} \quad . \tag{271}$$

The vector version of the boundary values (270) and (271) is $\underline{\underline{a}}_{1.s}$ and $\underline{\underline{a}}_{1.c}$. The union of (270) and (271) leads to the δg values on the globe of the Earth ω, or to the vector $\underline{\underline{a}}_1$. In order to achieve that the results are in keeping with the intentions, the following constraints must be satisfied, (264)(269),

$$\bar{a}_{1.0} = \bar{a}_{1.1} = 0 \quad . \tag{272}$$

(272) represents the conditions for the four Stokes constants of 0th and 1st degree and order in the development for δg, (251).

The individual four normalized spherical harmonics of these degrees are now denominated by Y_j, (j = 1,2,3,4). They have the following expressions in terms of the geocentric latitude and longitude, (φ , λ), [15].

DOI: https://doi.org/10.2312/zipe.1986.084

$$Y_1 = \overline{R}_{0.0} = \overline{P}_{0.0} = 1 \quad , \tag{273}$$

$$Y_2 = \overline{R}_{1.0} = \overline{P}_{1.0} = \sqrt{3} \cdot \sin\varphi \quad , \tag{274}$$

$$Y_3 = \overline{R}_{1.1} = \overline{P}_{1.1} \cdot \cos\lambda = \sqrt{3} \cdot \cos\varphi \cdot \cos\lambda \quad , \tag{275}$$

$$Y_4 = \overline{S}_{1.1} = \overline{P}_{1.1} \sin\lambda = \sqrt{3} \cos\varphi \sin\lambda \quad ; \tag{276}$$

$$\iint_\omega Y_j^{\,2} \, d\omega = 4\pi \quad . \tag{277}$$

The relation (273) to (276) use the well-known abbreviations, [15],

$$\overline{R}_{n.m}(\varphi, \lambda) = \overline{P}_{n.m}(\varphi) \cos m\lambda \quad , \tag{278}$$

$$\overline{S}_{n.m}(\varphi, \lambda) = \overline{P}_{n.m}(\varphi) \sin m\lambda \quad , \tag{279}$$

$$\iint_\omega \overline{R}_{n.m}^{\,2} \, d\omega = \iint_\omega \overline{S}_{n.m}^{\,2} \, d\omega = 4\pi \quad . \tag{280}$$

$\overline{P}_{n.m}(\varphi)$ represents the normalized associated spherical harmonics of degree n and order m.

Considering (251) and (273) to (276), the conditions (272) change into the following shape,

$$\Psi_j = 0 = \iint_\omega \alpha_1 \, Y_j \, d\omega \quad , \tag{281}$$

$$j = 1,2,3,4 \; .$$

The division of the integration (281) into the oceanic and the continental parts leads to

$$\Psi_j = 0 = \iint_{\omega_s} \alpha_{1.s} \, Y_j \, d\omega + \iint_{\omega_c} \alpha_{1.c} \, Y_j \, d\omega \quad . \tag{282}$$

The functions $\alpha_{1.s}$ and $\alpha_{1.c}$ can be replaced in (282) by the empirically given expression ν_s and by the expression $f\{\nu_s, \nu_c\}$ which is computed in terms of the empirically given functions ν_s and ν_c, (270)(271). Hence,

$$\Psi_j = 0 = \iint_{\omega_s} \nu_s \, Y_j \, d\omega + \iint_{\omega_c} f\{\nu_s, \nu_c\} \, Y_j \, d\omega \quad , \tag{283}$$

$$j = 1,2,3,4 \; . \tag{284}$$

DOI: https://doi.org/10.2312/zipe.1986.084

Now, it is convenient to transform the four compatibility conditions $\mathcal{V}_j$, (283), into the shape they have in the matrix calculus. The functions Y_j, ν_s, ν_c transform into the concerned vectors and the kernel functions change over to certain matrices. The vector representation of Y_j is,

$$\underline{y}_j = \begin{pmatrix} \dots \\ (Y_j)_i \\ \dots \end{pmatrix} , \tag{285}$$

$$i = 1,2,3, \dots , q \quad . \tag{286}$$

For the fixation of the ideas, here again a division of the globe into equal size compartments of the number q is introduced, as formerly by the figure 1, see also (21). The extension of q to a very great number is permitted. The relations (214) and (285) lead to the vector shape of (281), it is the following scalar product,

$$\mathcal{V}_j = 0 = \underline{\underline{a}}_1 \ \underline{y}_j \quad , \tag{287}$$

$$j = 1,2,3,4 \quad .$$

Hence, the vector shape of (282) is,

$$\mathcal{V}_j = 0 = \underline{\underline{a}}_{1.s} \, (\underline{y}_j)_s + \underline{\underline{a}}_{1.c} \, (\underline{y}_j)_c \quad . \tag{288}$$

$(\underline{y}_j)_s$ is the part of $\underline{y}_j$ that refers to the oceanic compartments, and $(\underline{y}_j)_c$ is the continental part,

$$\underline{y}_j = \begin{bmatrix} (\underline{y}_j)_c \\ (\underline{y}_j)_s \end{bmatrix} , \tag{289}$$

$$\underline{\underline{a}}_1 = \begin{bmatrix} \underline{\underline{a}}_{1.c} \\ \underline{\underline{a}}_{1.s} \end{bmatrix} . \tag{290}$$

As to (288), $\underline{\underline{a}}_{1.s}$ is the expirically given vector of the gravity deviations on the oceans. $\underline{\underline{a}}_{1.c}$ is computed in terms of $\underline{\underline{a}}_{1.s}$ and $\underline{\underline{a}}_{2.c}$, the latter is the vector of the free-air anomalies on the continents, (230).

The union of (230) and (288) leads finally to the following relations in terms of the empirically given boundary values $\underline{\underline{a}}_{1.s}$ and $\underline{\underline{a}}_{2.c}$,

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$$\Psi_j = 0 = (\underline{y}_j)_s^T \underline{\underline{a}}_{1.s} + (\underline{\underline{y}}_j)_c^T (\underline{\underline{K}}'_{c.c})^{-1} [\underline{\underline{a}}_{2.c} + \underline{\underline{H}}'_{c.s} \; \underline{\underline{a}}_{1.s} \Delta\omega^*] ; \qquad (291)$$

$$j = 1,2,3,4. \qquad (292)$$

The superscript $(\ )^T$ denots the transposition.

The relation (291) describes the 4 compatibility conditions that must be fulfilled by the boundary values $\underline{\underline{a}}_{1.s}$ and $\underline{\underline{a}}_{2.c}$ of the second mixed boundary value problem of the geodesy to be solved in the subspace of the spherical harmonics of the 2nd and higher degree. Otherwise, it is not possible to find a solution of this problem that satisfies the mixed boundary conditions which are given from the beginning.

The derivations from (249) to (291) refer to certain mathematical relations in the subspace of the harmonics of the 2nd and higher degree. It will not be without interest to look for the modifications that happen if the harmonics of 0th and 1st degree are included. Following up this aim, the relation (249) changes over to

$$\Xi = \frac{R}{4\pi} \iint_{\omega} H \, \delta g \, d\omega \; . \qquad (293)$$

H is defined by (50), and (294) must be substituted for (256),

$$H(r,\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi) \; . \qquad (294)$$

(262) changes to

$$-\left(\frac{\partial \Xi}{\partial r}\right)_s = \left(\sum_{n=0}^{\infty} \bar{a}_{1.n} \; X_n\right)_s \; , \text{ on } \omega_s \; . \qquad (295)$$

(266) must be replaced by the relation

$$-\left(\frac{\partial \Xi}{\partial r} + \frac{2}{R}\,\Xi\right)_c = \left(\sum_{n=0}^{\infty} \frac{n-1}{n+1}\, \bar{a}_{1.n} \; X_n\right)_c \; . \qquad (296)$$

In (295) and (296), the harmonics of the 0th and 1st degree are not filtered off by the integral transformation (293). Obviously, this fact paralyzes here the reason that causes the compatibility conditions in case of the integral transformation of the type (249).

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4.7. A matrix criterion for the uniqueness of the second mixed boundary value problem in the space of the harmonics of 0th and higher degree

The uniqueness of the second mixed boundary value problem was proved by the relations (86) and (114) for the subspace of the harmonics of the 2nd and higher degree. The uniqueness of this problem is valid for an arbitrary run of the coast line. Now, these investigations come to be continued and expanded by the inclusion of the harmonics of the 0th and 1st degree also.

In this context, the mappings (222) and (229) by $\underline{\underline{K}}'_{c.c}$ and $(\underline{\underline{K}}'_{c.c})^{-1}$ are in the fore. The mappings by the kernel matrices $\underline{\underline{K}}_{c.c}$ and $(\underline{\underline{K}}_{c.c})^{-1}$ cannot be the base for the following deductions since it was not possible to state whether $\underline{\underline{K}}_{c.c}$ represents a closed kernel function or whether its rang defect is zero, cf. (115) to (117b), (235) and (238).

The inclusion of the harmonics of the 0th and 1st degree and order changes the potential T' into T, (46)(47).

$$T = \sum_{n=0}^{1} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^{n+1} \bar{P}_{n.m}(\varphi) \left[z^{*}_{n.m} \cos m\lambda + z^{**}_{n.m} \sin m\lambda \right] + T' . \tag{297}$$

$z^{*}_{n.m}$ and $z^{**}_{n.m}$ are the Stokes constants.

A glance on the relations (273) to (280) shows that the following representation of (297) is allowed,

$$T = \sum_{j=1}^{4} z_j \, \varrho_j(r) \, Y_j(\varphi, \lambda) + T' . \tag{298}$$

z_j are the Stokes constants, Y_j are the surface spherical harmonics of 0th and 1st degree and order. The functions $\varrho_j(r)$ depend on the geocentric distance r,

$$\varrho_1(r) = \frac{R}{r} , \tag{299}$$

$$\varrho_2(r) = \varrho_3(r) = \varrho_4(r) = \left(\frac{R}{r}\right)^2 . \tag{300}$$

The relations (46), (47), (59), (64) and (298) lead to a formula for the potential T in the space of the spherical harmonics of all the degrees and orders, (n = 0,1,2, ...), expressed in terms of the Stokes constants z_j of 0th and 1st degree and order, and expressed further in terms of the gravity deviations α'_1 , which are free of the harmonics of the 0th and 1st degree and order, since these constituents are considered to be filtered off,

$$T = \sum_{j=1}^{4} z_j \, \varrho_j(r) \, Y_j(\varphi, \lambda) + \frac{R}{4\pi} \iint_{\omega} H' \alpha'_1 \, d\omega . \tag{301}$$

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As it was stated above, the uniqueness of the second mixed boundary value problem of the geodesy in the space of the harmonics of all the degrees n=0,1,2,... is to be investigated here. This problem is governed by the constraint that the concerned boundary values have to be equal to zero, as it is well-known. This homogeneous form of the boundary conditions is , (8) (9),

$$-\frac{\partial T}{\partial r} = 0 \quad , \text{ on } \omega_s \quad , \tag{302}$$

$$-\frac{\partial T}{\partial r} - \frac{2}{R} T = 0 \quad , \text{ on } \omega_c \quad . \tag{303}$$

The operators of (302) and (303) are applied on (301),

$$-\frac{\partial T}{\partial r} = -\sum_{j=1}^{4} z_j \frac{d \varrho_j}{dr} Y_j(\varphi, \lambda) + \alpha_1' \quad , \tag{304}$$

$$-\frac{\partial T}{\partial r} - \frac{2}{R} T = -\sum_{j=1}^{4} z_j \frac{d \varrho_j}{dr} Y_j(\varphi, \lambda) + \alpha_1' - \frac{2}{R} \sum_{j=1}^{4} z_j \varrho_j Y_j(\varphi, \lambda) -$$

$$- \frac{1}{2\pi} \iint_{\omega} H' \alpha_1' \, d\omega \quad . \tag{305}$$

The four equations (302) to (305) lead to the subsequent simultaneous relations,

$$0 = -\sum_{j=1}^{4} z_j \frac{d \varrho_j}{dr} Y_j + \alpha_1' \quad , \text{ on } \omega_s \quad , \tag{306}$$

$$0 = -\sum_{j=1}^{4} z_j \left[\frac{d \varrho_j}{dr} + \frac{2}{R} \varrho_j \right] Y_j - \frac{1}{2\pi} \iint_{\omega} H' \alpha_1' \, d\omega + \alpha_1' \quad , \text{ on } \omega_c \quad . \tag{307}$$

The abbreviations

$$\Delta = \sum_{j=1}^{4} z_j \frac{d \varrho_j}{dr} Y_j \quad , \tag{308}$$

$$\Delta^* = \sum_{j=1}^{4} z_j \left[\frac{d \varrho_j}{dr} + \frac{2}{R} \varrho_j \right] Y_j \quad , \tag{309}$$

change (306) and (307) into

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$$0 = \alpha'_{1.s} - \Delta_s \quad , \quad \text{on } \omega_s \quad , \tag{310}$$

$$0 = \alpha'_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \, \alpha'_{1.c} \, d\omega - \frac{1}{2\pi} \iint_{\omega_s} H'_{c.s} \, \alpha'_{1.s} \, d\omega - \Delta^*_c \quad , \quad \text{on } \omega_c \quad . \tag{311}$$

In (311), the expression $\alpha'_{1.s}$ is replaced by Δ_s using (310); thus, an integral equation for $\alpha_{1.c}$ is obtained. Hence, by (310) and (311),

$$\alpha'_{1.s} = \Delta_s \quad , \quad \text{on } \omega_s \quad , \tag{312}$$

$$\alpha'_{1.c} - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \, \alpha'_{1.c} \, d\omega = \Delta^*_c + \frac{1}{2\pi} \iint_{\omega_s} H'_{c.s} \, \Delta_s \, d\omega , \text{ on } \omega_c \quad . \tag{313}$$

α'_1 is the global union of $\alpha'_{1.s}$ and $\alpha'_{1.c}$. The prime denotes that the harmonics of 0th and 1st degree are not included.

Here, in the investigation of the uniqueness, $\alpha'_{1.s}$ and $\alpha'_{1.c}$ are not considered as empirically given functions. They are mathematical expressions of the described properties.

$$\Omega_j = 0 = \iint_{\omega} \alpha'_1 \, Y_j \, d\omega \quad , \tag{314}$$

$$j = 1,2,3,4 \quad . \tag{315}$$

The division of (314) into the oceanic and the continental area gives,

$$\Omega_j = 0 = \iint_{\omega_s} \alpha'_{1.s} \, Y_j \, d\omega + \iint_{\omega_c} \alpha'_{1.c} \, Y_j \, d\omega \quad . \tag{316}$$

In the next step, $\alpha'_{1.s}$ (from (312)) and $\alpha'_{1.c}$ (from (313)) are introduced into the equations (316) in order to find 4 relations in terms of the 4 Stokes constants z_j, (j = 1,2,3,4). For the following deductions, it is convenient to write the mathematical expressions in the style usual in the vector and matrix calculus, cf. (163) to (205), (214) to (216). The vector shape of the global function α'_1 is

$$\underline{\underline{a}}'_1 = \begin{pmatrix} \cdots \\ (\alpha'_1)i \\ \cdots \end{pmatrix} \quad , \tag{317}$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$i = 1,2,3, \dots , q \quad . \tag{318}$$

q is the number of the equal size compartments which divide the surface ω of the globe. As in (214), the continental and the oceanic compartments are listed separately,

$$\underline{\underline{a}}_1' = \begin{bmatrix} \underline{\underline{a}}_{1.c}' \\ \\ \underline{\underline{a}}_{1.s}' \end{bmatrix} \quad . \tag{319}$$

The functions Δ and Δ^* are transferred to the vector form in a similar way, (308)(309),

$$\underline{\underline{d}} = \begin{bmatrix} \dots \\ \Delta_i \\ \dots \end{bmatrix} \quad , \tag{320}$$

$$\underline{\underline{d}} = \begin{bmatrix} \underline{\underline{d}}_c \\ \\ \underline{\underline{d}}_s \end{bmatrix} \quad , \tag{321}$$

$$\underline{\underline{d}}^* = \begin{bmatrix} \dots \\ \Delta_i^* \\ \dots \end{bmatrix} \quad , \tag{322}$$

$$\underline{\underline{d}}^* = \begin{bmatrix} \underline{\underline{d}}_c^* \\ \\ \underline{\underline{d}}_s^* \end{bmatrix} \quad , \tag{323}$$

$$i = 1,2,3, \dots , q \quad . \tag{324}$$

The vector shape of (312) is

$$0 = \underline{\underline{a}}_{1.s}' - \underline{\underline{d}}_s \quad . \tag{325}$$

The integral equation (313) changes over to, (213),

$$0 = \underline{\underline{a}}_{1.c}' - \underline{\underline{H}}_{c.c}' \; \underline{\underline{a}}_{1.c}' \, \Delta\omega^* - \underline{\underline{d}}_c^* - \underline{\underline{H}}_{c.s}' \; \underline{\underline{d}}_s \, \Delta\omega^* \quad . \tag{326}$$

Hence,

$$0 = (\underline{\underline{E}} - \underline{\underline{H}}_{c.c}' \Delta\omega^*) \, \underline{\underline{a}}_{1.c}' - \underline{\underline{d}}_c^* - \underline{\underline{H}}_{c.s}' \; \underline{\underline{d}}_s \, \Delta\omega^* \quad . \tag{327}$$

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This matrix in the braces is well-investigated above, (221).

$$0 = \underline{\underline{K}}'_{c.c}\ \underline{\underline{a}}'_{1.c} - \underline{\underline{d}}^*_c - \underline{\underline{H}}'_{c.s}\ \underline{\underline{d}}_s\ \Delta\omega^* \quad . \tag{328}$$

The inverse of $\underline{\underline{K}}'_{c.c}$ is well-founded, (229). Thus, (325) and (328) give a representation of the vector $\underline{\underline{a}}'_{1.s}$ and $\underline{\underline{a}}'_{1.c}$ in terms of the vectors $\underline{\underline{d}}_s$ and $\underline{\underline{d}}^*_c$,

$$\underline{\underline{a}}'_{1.c} = \left(\underline{\underline{K}}'_{c.c}\right)^{-1} \left(\underline{\underline{d}}^*_c + \underline{\underline{H}}'_{c.s}\ \underline{\underline{d}}_s\ \Delta\omega^*\right) \quad , \tag{329}$$

$$\underline{\underline{a}}'_{1.s} = \underline{\underline{d}}_s \quad . \tag{330}$$

The vector form of (316) is obtained by (285) (289) (329) (330), it consists of the following two scalar products,

$$\Omega_j = 0 = \underline{\underline{a}}'_{1.c}\ (\underline{\underline{Y}}_j)_c + \underline{\underline{a}}'_{1.s}\ (\underline{\underline{Y}}_j)_s \quad , \tag{331}$$

$$j = 1,2,3,4 \quad . \tag{332}$$

$\underline{\underline{a}}'_{1.c}$ and $\underline{\underline{a}}'_{1.s}$ are substituted for (329) and (330) ,

$$\Omega_j = 0 = \underline{\underline{d}}_s\ (\underline{\underline{Y}}_j)_s + \left[(\underline{\underline{K}}'_{c.c})^{-1} (\underline{\underline{d}}^*_c + \underline{\underline{H}}'_{c.s}\ \underline{\underline{d}}_s\ \Delta\omega^*) \right] (\underline{\underline{Y}}_j)_c . \tag{333}$$

In order to express the vectors $\underline{\underline{d}}_s$ and $\underline{\underline{d}}^*_c$ in terms of the 4 Stokes constants z_j, (j= 1,2,3,4), (320) is developed by (308), and (322) by (309). The components Δ_i are

$$\Delta_i = \left(\sum_{j=1}^{4} z_j \frac{d\,\varrho_j}{dr}\ Y_j \right)_i \quad . \tag{334}$$

The parameter i ,

$$i = 1,2,3,\ \ldots\ , q \ , \tag{335}$$

varies with φ and λ over the globe, from compartment to compartment. But, z_j is a constant and $d\,\varrho_j$ / dr depends on the here constant radius only. Therefore, z_j and $d\,\varrho_j$ / dr do not vary with the parameter i. Thus,

$$\Delta_i = \sum_{j=1}^{4} z_j \frac{d\,\varrho_j}{dr} (Y_j)_i \quad . \tag{336}$$

$(Y_j)_i$ is the value of the function Y_j for the compartment of the running number i. (320) and (336) give

$$\underline{\underline{d}} = \sum_{j=1}^{4} z_j \frac{d\,\varrho_j}{dr} \begin{bmatrix} \cdots \\ (Y_j)_i \\ \cdots \end{bmatrix} \quad , \tag{337}$$

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or, (285),

$$\underline{\underline{d}} = \sum_{j=1}^{4} z_j \frac{d\,\varrho_j}{dr} \underline{\underline{Y}}_j \quad . \tag{338}$$

The combination of (289), (321) and (338) leads to the following equations,

$$\underline{\underline{d}}_c = \sum_{j=1}^{4} z_j \frac{d\,\varrho_j}{dr} (\underline{\underline{Y}}_j)_c \quad , \tag{339}$$

$$\underline{\underline{d}}_s = \sum_{j=1}^{4} z_j \frac{d\,\varrho_j}{dr} (\underline{\underline{Y}}_j)_s \quad . \tag{340}$$

In the equations (339) and (340), the subscript j can be replaced by u,

$$u = 1,2,3,4 \quad . \tag{341}$$

$$\underline{\underline{d}}_c = \sum_{u=1}^{4} z_u \frac{d\,\varrho_u}{dr} (\underline{\underline{Y}}_u)_c \quad , \tag{342}$$

$$\underline{\underline{d}}_s = \sum_{u=1}^{4} z_u \frac{d\,\varrho_u}{dr} (\underline{\underline{Y}}_u)_s \quad . \tag{343}$$

The corresponding relations for $\underline{\underline{d}}_c^*$ and $\underline{\underline{d}}_s^*$, (323), are easily obtained by a comparison of (308) and (309),

$$\underline{\underline{d}}_c^* = \sum_{u=1}^{4} z_u \left[\frac{d\,\varrho_u}{dr} + \frac{2}{R} \varrho_u \right] (\underline{\underline{Y}}_u)_c \quad , \tag{344}$$

$$\underline{\underline{d}}_s^* = \sum_{u=1}^{4} z_u \left[\frac{d\,\varrho_u}{dr} + \frac{2}{R} \varrho_u \right] (\underline{\underline{Y}}_u)_s \quad . \tag{345}$$

The expressions (343) and (344) for $\underline{\underline{d}}_s$ and $\underline{\underline{d}}_c^*$ are introduced into (333),

$$\begin{aligned} \Omega_j = 0 = & \sum_{u=1}^{4} z_u \frac{d\,\varrho_u}{dr} (\underline{\underline{Y}}_u)_s (\underline{\underline{Y}}_j)_s \; + \\ & + \sum_{u=1}^{4} z_u \left[\frac{d\,\varrho_u}{dr} + \frac{2}{R} \varrho_u \right] \cdot \left[(\underline{\underline{K}}'_{o.c})^{-1} (\underline{\underline{Y}}_u)_c \right] (\underline{\underline{Y}}_j)_c \; + \\ & + \sum_{u=1}^{4} z_u \frac{d\,\varrho_u}{dr} \left[(\underline{\underline{K}}'_{o.c})^{-1} \underline{\underline{H}}'_{o.s} \; \Delta\omega^* (\underline{\underline{Y}}_u)_s \right] (\underline{\underline{Y}}_j)_c \quad . \end{aligned} \tag{346}$$

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The equation (346) can be considered to have the shape of the product of a matrix and a vector. This matrix has the following form,

$$\underline{\underline{W}}^T = \{ w_{u.j} \} \qquad , \tag{347}$$

$$u, \; j = 1,2,3,4 \; . \tag{348}$$

$$\underline{\underline{W}}^T = \begin{bmatrix} w_{1.1} & w_{1.2} & w_{1.3} & w_{1.4} \\ w_{2.1} & w_{2.2} & w_{2.3} & w_{2.4} \\ w_{3.1} & w_{3.2} & w_{3.3} & w_{3.4} \\ w_{4.1} & w_{4.2} & w_{4.3} & w_{4.4} \end{bmatrix} \qquad . \tag{349}$$

The superscript T denotes again the transposition of the matrix.

$$w_{u.j} = \frac{d\,\varrho_u}{dr} \left(\underline{\underline{y}}_j \right)_s^T \left(\underline{\underline{y}}_u \right)_s + \left[\frac{d\,\varrho_u}{dr} + \frac{2}{R} \varrho_u \right] \left(\underline{\underline{y}}_j \right)_c^T \left(\underline{\underline{K}}'_{c.c} \right)^{-1} \left(\underline{\underline{y}}_u \right)_c +$$

$$+ \frac{d\,\varrho_u}{dr} \Delta\omega^* \left(\underline{\underline{y}}_j \right)_c^T \left(\underline{\underline{K}}'_{c.c} \right)^{-1} \underline{\underline{H}}'_{c.s} \left(\underline{\underline{y}}_u \right)_s \quad . \tag{350}$$

The concerned vector has the Stokes constants z_u as the elements,

$$\underline{\underline{z}} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \qquad . \tag{351}$$

The abbreviations (350) and (351) transform the 4 equation, (346), into the following form,

$$0 = \underline{\underline{W}} \cdot \underline{\underline{z}} \quad , \tag{352}$$

or

$$\Omega_j = 0 = \sum_{u=1}^{4} w_{u.j} \, z_u \quad . \tag{353}$$

In case of the fact that the following inequation is fulfilled,

$$\det \underline{\underline{W}} = \det \underline{\underline{W}}^T = \det \{ w_{u.j} \} \neq 0 \quad , \tag{354}$$

in that case, the Stokes constants z_u are all equal to zero, as a glance on (352) and on the inversion of it does show. Thus, (352) and the criterion (354) (237) give

$$\underline{\underline{z}} = 0 \quad . \tag{355}$$

The fulfillment of (354) does mean that the rang defect of $\underline{\underline{W}}$ and $\underline{\underline{W}}^T$ is equal to zero.

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(355), (351), (308) and (309) lead to

$$\Delta = \Delta^* = 0 . \tag{356}$$

Hence, with (320) and (322),

$$\underline{\underline{d}} = \underline{\underline{d}}^* = 0 . \tag{357}$$

(357) and (329) have the following consequences,

$$\underline{\underline{a}}'_{1.c} = 0 , \tag{358}$$

$$\underline{\underline{a}}'_{1.s} = 0 . \tag{359}$$

Finally, (358) (359) (317) (355) and (301) lead to the fact that

$$\alpha'_1 = 0 , \tag{360}$$

$$z_j = o , (j = 1,2,3,4) ; \tag{361}$$

Hence, with (301),

$$T = 0 . \tag{362}$$

This relation proves the uniqueness. Therefore, the following theorem is right: The second mixed boundary value problem of the geodesy has a unique solution in the space of all the spherical harmonics of the degree n = 0,1,2, ... if the criterion (354) is fulfilled.

As regards the numerical computation of the relation (354), a detailed numerical evaluation is not intended to be given here. In case of the first mixed boundary value problem of the geodesy, a criterion similar to (354) was derived earlier, [5]. Later on, in this publication, this criterion for the first mixed boundary value problem and for its uniqueness in the space of all the harmonics of the degree n = 0,1,2, ... will undergo a detailed numerical evaluation. It will be proved that the first mixed boundary value problem has a unique solution in the space of the harmonics of all degrees and for the real geographical distribution of the continents and of the oceans.

However, returning back to the second mixed boundary value problem and to the matrix $\underline{\underline{W}}$ and the elements $w_{u.j}$ of it, (350), a short consideration about the structure of (350) seems to be convenient. The functions ϱ_u and $\frac{d\varrho_u}{dr}$, (299) (300), are constant over the globe. The vectors $\underline{y}_j$ represent certain global functions that are constant or that vary over the globe as $\cos\varphi$, $\sin\varphi$, $\cos\lambda$, $\sin\lambda$, cf. (273) to (276). Thus, these functions contain constant terms and long wave length terms only. Therefore, in the computation of the matrices $(\underline{\underline{K}}'_{c.c})^{-1}$, $\underline{\underline{H}}'_{c.s}$ in (350), it is not necessary to compute a detailed representation of these matrices, e.g. up to the order or dimension of some thousands. If these matrices $(\underline{\underline{K}}'_{c.c})^{-1}$ and $\underline{\underline{H}}'_{c.s}$ are computed basing on compartments of $1^0 \times 1^0$ square, in this case these matrices will show a very detailed structure which is not in keeping with the structure of the functions ϱ_u, $\frac{d\varrho_u}{dr}$ and

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the vectors $\underline{\underline{y}}_u$ which appear in (350). A general division of the globe into compartments of about $10^0 \times 10^0$ square or $15^0 \times 15^0$ square will suffice to do the work of computing the elements $w_{u.j}$, (350), of the matrix $\underline{\underline{W}}$ in order to test whether (354) is valid or not. This fact will be a relief to the computations about the criterion equation (354).

But, if going other ways by starting from the criterion (238) in order to investigate into whether the second mixed boundary value problem is unique in the space of all the harmonics, in this case, a lot of computation work does follow. Now, not a matrix of the dimension 4 x 4 is in the fore, as (347), but the matrix $\underline{\underline{K}}_{c.c}$ has to be investigated in all details to find whether the rang defect of it is indeed equal to zero. The dimension of this square matrix has to be extended extremely great, a lot of work would be the consequence.

4.8. The difference method

Some remarks about the numerical computation of the solution of the 2nd mixed boundary value problem seem to be advisable. The equation (230) determines the continental gravity deviations in terms of the continental free-air anomalies and in terms of the oceanic gravity deviations. Thus, (230) represents already a solution of this boundary value problem. However, it is convenient to transform (230) into a difference method since it will bring relief to the numerical computations about the inversion of the matrix of great order $\underline{\underline{K}}'_{c.c}$. This difference method will also lead to a higher flexibility in the applications, it seems to be convenient for routine computations. In this context, the full components of the vector $\underline{\underline{a}}_{1.c}$ are not intended to be determined, but the differences of these values relative to a reference component are obtained. Along these lines, the relation (230) changes into, [7][8],

$$\delta \underline{\underline{a}}_{1.c} = \delta(\underline{\underline{K}}'_{c.c})^{-1} \left\{ \underline{\underline{a}}_{2.c} + \underline{\underline{H}}'_{c.s} \, \Delta\omega^* \, \underline{\underline{a}}_{1.s} \right\} \quad (363)$$

$\delta \underline{\underline{a}}_{1.c}$ and $\delta(\underline{\underline{K}}'_{c.c})^{-1}$ denominate the concerned difference vector and the concerned difference matrix.

Some explanatory and supplementary remarks about the difference relation (363) seem to be recommended. The substitutions

$$\underline{\underline{x}} = \underline{\underline{a}}_{1.c} \quad , \quad (364)$$

$$\underline{\underline{y}} = \underline{\underline{a}}_{2.c} + \Delta\omega^* \, \underline{\underline{H}}'_{c.s} \, \underline{\underline{a}}_{1.s} \quad , \quad (365)$$

$$\underline{\underline{M}} = (\underline{\underline{K}}'_{c.c})^{-1} \quad , \quad (366)$$

change (230) into

$$\underline{\underline{x}} = \underline{\underline{M}} \; \underline{\underline{y}} \quad . \quad (367)$$

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The row vector of $\underline{\underline{x}}$ is $\underline{\underline{x}}^T$ where the superscript T denominates the transposition,

$$\underline{\underline{x}}^T = (x_1, x_2, x_3, \dots x_D, \dots , x_N) \quad . \tag{368}$$

Similarly, the row vector of $\underline{y}$ has the following shape,

$$\underline{y}^T = (y_1, y_2, y_3, \dots , y_D, \dots , y_N) \quad . \tag{369}$$

Thus, the components of the vectors $\underline{\underline{x}}$ and $\underline{y}$ are

$$x_a \ , \ (a = 1,2,3, \dots , D, \dots , N) \ , \tag{370}$$

and

$$y_b \ , \ (b = 1,2,3, \dots , D, \dots , N) \ . \tag{371}$$

The matrix $\underline{\underline{M}}$ has the elements $m_{a.b}$

$$\underline{\underline{M}} = \{ m_{a.b} \} = \{ m_{b.a} \} \ . \tag{372}$$

The matrix relation (367) can be written in the following form,

$$x_a = \sum_{b=1}^{N} m_{a.b} \ y_b \quad , \tag{373}$$

$$a,b = 1,2,3, \dots , D, \dots , N \ . \tag{374}$$

x_D is the comparison component or the reference component, $(a = D)$.

$$x_D = \sum_{b=1}^{N} m_{D.b} \ y_b \quad . \tag{375}$$

The following relations bring to bear the advantages connected with the introduction of the comparison component x_D, they are self-explanatory,

$$x_a - x_D = \delta x_a \quad , \tag{376}$$

$$m_{a.b} - m_{D.b} = \delta m_{a.b} \ , \tag{377}$$

$$x_a - x_D = \sum_{b=1}^{N} (m_{a.b} - m_{D.b}) \ y_b = \sum_{b=1}^{N} \delta m_{a.b} \ y_b \quad , \tag{378}$$

$$\delta \underline{\underline{x}} = \delta \underline{\underline{M}} \cdot \underline{y} \quad , \tag{379}$$

$$\delta \underline{\underline{x}}^T = (\dots , \delta x_a, \dots) \quad , \tag{380}$$

$$\delta \underline{\underline{M}} = \{ m_{a.b} - m_{D.b} \} = \{ \delta m_{a.b} \} \quad ; \tag{381}$$

$$\delta \underline{\underline{x}} = \delta \underline{\underline{a}}_{1.c} \quad , \tag{382}$$

$$\delta \underline{\underline{M}} = \delta (\underline{\underline{K}}'_{c.c})^{-1} \quad , \tag{383}$$

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The substitutions (382) and (383) lead back to (363), after the meaning of it was explained by the derivations from (364) to (381).

The very advantage connected with the difference method, (363), consists in the following fact. The subscript $(\ )_D$ may refer to the mean x value of the surface element $\Delta\omega^*$ of Potsdam, e.g., and the subscript $(\ )_a$ may refer to the mean x value of the surface element $\Delta\omega^*$ which does contain Budapest, e.g., [7] [8]. If it is intended, after these preliminaries, to compute $x_a - x_D = \delta x_a$, that is

$$\delta x_a = x_{Budapest} - x_{Potsdam},$$

in this case, the boundary values $\underline{\underline{a}}_{2.c}$ and $\underline{\underline{a}}_{1.s}$ for areas distant from Europe will have an effect on δx_a that is much more small than the effect of the $\underline{\underline{a}}_{2.c}$ and $\underline{\underline{a}}_{1.s}$ values of Bulgaria and the Baltic Sea, e.g.. Obviously, the greater the distance from the boundary values, the smaller their impact on δx_a. This fact is caused by the structure of $\delta(\underline{\underline{K}}'_{c.c})^{-1}$ which will be very dissimilar to the structure of $(\underline{\underline{K}}'_{c.c})^{-1}$. Thus, the greater the distance from the two test points with the components x_a and x_D the greater the admitted size of the compartments $\Delta\omega^*$. And, furthermore, the greater the compartments $\Delta\omega^*$, the smaller is the order of the matrices $\underline{\underline{K}}'_{c.c}$ and $(\underline{\underline{K}}'_{c.c})^{-1}$. And, consequently, the smaller the order of these matrices, the greater the relief to the computations. Thus, the transition from (230) to (363) will allow to save much work.

By the previous lines, the abandonment of the condition that the surface elements are constant,

$$\Delta\omega^* = \frac{\Delta\omega}{2\pi} = \text{constant} \quad , \tag{383a}$$

was discussed. Therefore, for the application of the difference method, the concerned relations (230) and (363) have to be modified. Deriving the matrix shape of (57) and (58) for a surface division into compartments of different size, a certain diagonal matrix has to be introduced. For the fixation of the ideas, a model of 101 continental compartments and of 47 oceanic compartments is introduced, cf. (167)(168). They give rise to the following diagonal matrix,

$$\underline{\underline{V}} = \begin{bmatrix} v_{1.1} & 0 & \dots & 0 & \dots & 0 \\ 0 & v_{2.2} & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v_{k.k} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & v_{q.q} \end{bmatrix} , \tag{383b}$$

$$\underline{\underline{V}} = \{ v_{i.\bar{i}} \} \quad , \tag{383c}$$

$$i, \bar{i} = 1, 2, \dots, k, \dots, q . \tag{383d}$$

$$v_{i.\bar{i}} = 0 \quad , \quad i \neq \bar{i} \quad . \tag{383e}$$

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$$v_{i.i} = \frac{1}{2\pi} \Delta\omega_i = \Delta\omega_i^* \quad . \tag{383f}$$

The separation of the continental part of $\underline{\underline{V}}$ from the oceanic part gives

$$\underline{\underline{V}}_c = \begin{bmatrix} v_{1.1} & 0 & \dots & 0 \\ 0 & v_{2.2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v_{k.k} \end{bmatrix} , \tag{383g}$$

and

$$\underline{\underline{V}}_s = \begin{bmatrix} v_{k+1.k+1} & 0 & \dots & 0 \\ 0 & v_{k+2.k+2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v_{q.q} \end{bmatrix} . \tag{383h}$$

Thus, for a division into compartments $\Delta\omega_i$ of non-constant size, the matrix shape of (56) turns to, cf. (217),

$$0 = \underline{\underline{a}}_{1.c} - \underline{\underline{H}}'_{c.c} \underline{\underline{V}}_c \underline{\underline{a}}_{1.c} - \underline{\underline{H}}'_{c.s} \underline{\underline{V}}_s \underline{\underline{a}}_{1.s} - \underline{\underline{a}}_{2.c} , \tag{383i}$$

The modified form of (219) (220) (221) (222) is

$$\underline{\underline{b}}_c^* = \underline{\underline{K}}'^*_{c.c} \quad \underline{\underline{a}}_{1.c} , \tag{383j}$$

with

$$\underline{\underline{K}}'^*_{c.c} = \underline{\underline{E}} - \underline{\underline{H}}'_{c.c} \underline{\underline{V}}_c , \tag{383k}$$

$$\underline{\underline{b}}_c^* = \underline{\underline{a}}_{2.c} + \underline{\underline{H}}'_{c.s} \underline{\underline{V}}_s \underline{\underline{a}}_{1.s} . \tag{383l}$$

Now, the transition from (383j) to the difference method is discussed. The developments are self-explanatory.

$$\underline{\underline{a}}_{1.c} = (\underline{\underline{K}}'^*_{c.c})^{-1} \quad \underline{\underline{b}}_c^* . \tag{383m}$$

Thus,

$$\delta \underline{\underline{a}}_{1.c} = \delta (\underline{\underline{K}}'^*_{c.c})^{-1} \quad \underline{\underline{b}}_c^* . \tag{383n}$$

Or, with

$$\underline{\underline{x}} = \underline{\underline{a}}_{1.c} , \tag{383o}$$

$$\underline{\underline{y}} = \underline{\underline{b}}_c^* , \tag{383p}$$

$$\underline{\underline{L}} = (\underline{\underline{K}}'^*_{c.c})^{-1} , \tag{383q}$$

follows

$$\underline{\underline{x}} = \underline{\underline{L}} \quad \underline{\underline{y}} , \tag{383r}$$

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$$\delta \underline{\underline{x}} = \delta \underline{\underline{L}} \cdot \underline{\underline{y}} \quad , \tag{383s}$$

$$\underline{\underline{L}} = \{ l_{a.b} \} \quad , \tag{383t}$$

$$x_D - x_{\bar{D}} = \sum_b (l_{D.b} - l_{\bar{D}.b}) \; y_b \quad , \tag{383u}$$

cf. (364) to (383).
The mutual distance between the couple of the two points P_D, $P_{\bar{D}}$ is choosen to be not more than about some hundred kilometers. Therefore, evaluating the impact exerted by the continental compartments on $x_D - x_{\bar{D}}$, it is very probable that this impact will be the smaller, the greater the distance is between such a compartment and the couple of points P_D, $P_{\bar{D}}$. It follows that the area $v_{i.i} = \Delta \omega_i^*$, (383f), can be choosen the greater, the greater its distance from the couple of points P_D, $P_{\bar{D}}$. The greater $v_{i.i}$, the smaller the dimension of $\underline{\underline{K}}'^*_{c.c}$. Experimental computations about the difference method seem to be recommended.

Furthermore, it is possible to do an additional step which does base on the difference method, (363). This equation, (363), allows the determination of the continental δg values relative to the a priori given offshore δg values of the boundary conditions. It is possible to develop an interpolation procedure for the determination of the continental δg values, by an interpolation between the offshore given boundary values of δg. The matrix relation (379) is a convenient starting point for a description of this method. Figure 2 shows the 5 compartments which the indices a = 1,2,3,4,5 are attributed to. The compartment of the index a = 1 is situated at the coast of the Baltic Sea, e.g.. The index a = 5 refers to a compartment at the coast of the Black Sea, e.g.. The index of the comparison compartment is choosen to be D = 3. The numerical computation of the matrix relation (379) leads to the following results,

$$\delta x_a \; , \; (a = 1,2,3,4,5) \quad . \tag{384}$$

Or,

$$\delta x_1 = x_1 - x_3 \quad , \tag{385}$$

$$\delta x_2 = x_2 - x_3 \quad , \tag{386}$$

$$\delta x_3 = 0 \quad , \tag{387}$$

$$\delta x_4 = x_4 - x_3 \quad , \tag{388}$$

$$\delta x_5 = x_5 - x_3 \quad ; \tag{389}$$

The difference of the computed values δx_1 and δx_5, (385) and (389), is

$$\delta x_5 - \delta x_1 = x_5 - x_1 = (\delta g)_5 - (\delta g)_1 \quad . \tag{390}$$

The equation (390) brings the a priori empirically given values x_1, x_5 (boundary values) and the a posteriori computed values δx_1, δx_5 (computed

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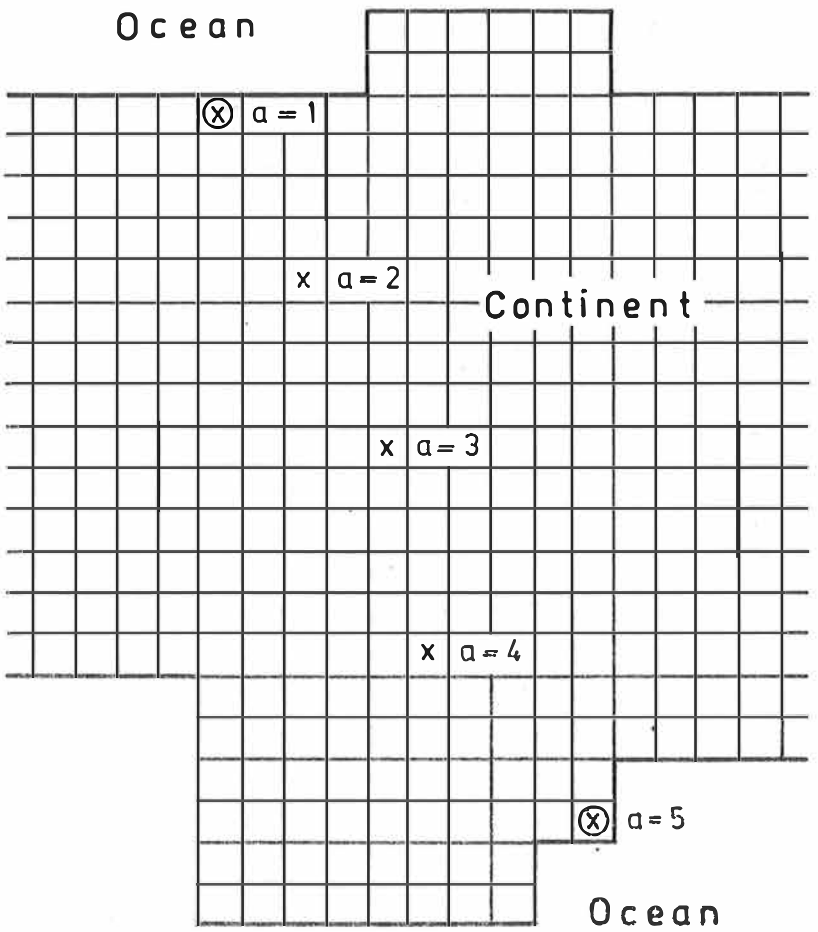

Fig. 2. A model polygon, (a = 1, ... , 5), for the application of the difference method. It is anchored in the known offshore dates using the benefits of an interpolation method.

by (363) (379)) into a mutual relation. For the terms δx_1 and δx_5, this equation, (390), is a constraint which has to undergo an adjustment according to the method of least squares, as usual. This adjustment will strengthen the obtained and computed values for δx_1 and δx_5 since the adjustment gets these values to be anchored in the offshore gravity deviations, x_1 and x_5, which belong to the empirically given boundary values.

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4.9. The transition from the gravity deviations to the perturbation potential and the accompanying smoothing effect

The gravity deviations δg and the gravity anomalies Δg_F vary by relative great amounts along the continents and the oceans from place to place. Therefore, the components of the vectors $\underline{\underline{a}}_{1.c}$ and $\delta \underline{\underline{a}}_{1.c}$ have also the property that the amounts of them vary from place to place rather considerably. Thus, the computation of the solution of the linear inhomogeneous integral equation of the second kind deals with the amounts of the gravity deviations which are to be determined and which vary by relative great amounts from place to place, (62) (67a) (230) (363). It is well-known that the determination of a smoothed system is easier to reach than the determination of a system of values varying heavily with the independent argument, it is self-explanatory.

In this context, it is interesting to see that the values of the perturbation potential T are rather smoothed. Therefore, it will be advantageous to replace the system of the δg values by the system of the T values. The T values can be introduced along the following lines. As to the T' values, the Hotine integral (46) is transformed. The fundamental differential equation of the physical geodesy

$$- \frac{\partial T}{\partial r} - \frac{2}{r} T = \Delta g_F \tag{391}$$

leads to a relation that expresses the gravity deviations,

$$\delta g = - \frac{\partial T}{\partial r} = \alpha_1 \quad , \tag{392}$$

in terms of the free-air anomalies Δg_F and the perturbation potential T,

$$\alpha_1 = \delta g = \Delta g_F + \frac{2}{R} T \quad . \tag{393}$$

(393) is valid for a spherical surface, r = R.

At first, the considerations happen in the subspace of the spherical harmonics of the 2nd and higher degree. Later, the harmonics of 0th and 1st degree are included also. The Hotine integral (46) gets here the following shape,

$$T' = \frac{R}{4\pi} \iint_{\omega} H'(\psi)\,(\delta g)'\,d\omega \quad . \tag{394}$$

It is provided that in (394) the compatibility conditions are fulfilled, (283),

$$\Psi_j = \Psi_j(\nu_c, \nu_s) = 0 \quad , \tag{395}$$

$$j = 1,2,3,4 \quad . \tag{396}$$

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It is a condition for the boundary values ν_c and ν_s, i.e. the empirically given free-air anomalies Δg_F on the continents, ω_c, and the gravity deviations δg on the oceans, ω_s. (395) represents the four conditions which must be fulfilled necessarily if the boundary conditions are to be observed a posteriori by the solution. (395) prescribes the fact that the global distribution of δg is free of the harmonics of 0th and 1st degree. Because of (76) and (79), it follows that T and Δg_F are also free of the harmonics of these degrees,

$$\bar{a}_{1.0} = \bar{a}_{1.1} = \bar{t}_0 = \bar{t}_1 = \bar{a}_{2.0} = \bar{a}_{2.1} = 0 \quad . \tag{397}$$

Therefore, (393) turns to

$$\alpha_1' = \delta g' = \delta g = (\Delta g_F)' + \frac{2}{R} T' \quad . \tag{398}$$

For test points on the continents, ω_c, the integral (394) can be brought into the following form,

$$(T')_c = \frac{R}{4\pi} \iint_{\omega_c} H'_{c.c} \; \delta g' \; d\omega + \frac{R}{4\pi} \iint_{\omega_s} H'_{c.s} \; \delta g' \; d\omega \; , \text{ on } \omega_c . \tag{399}$$

The first term on the right hand side of (399) is transformed by the introduction of (398),

$$(T')_c = \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \; T' \; d\omega + \frac{R}{4\pi} \iint_{\omega_c} H'_{c.c} \; (\Delta g_F)' \; d\omega \; +$$

$$+ \frac{R}{4\pi} \iint_{\omega_s} H'_{c.s} \; (\delta g)' \; d\omega \; , \text{ on } \omega_c \quad . \tag{400}$$

$(\Delta g_F)'$ on ω_c and $(\delta g)'$ on ω_s are the empirically given boundary values ν_c and ν_s. They did undergo a reduction or an adjustment adapting them to the 4 compatibility conditions (395) prescribed to be fulfilled by ν_c and ν_s before the beginning of the boundary value problem computations.

Thus, (400) gets the following shape which is easy to understand,

$$0 = \varepsilon_c' + T' - \frac{1}{2\pi} \iint_{\omega_c} H'_{c.c} \; T' \; d\omega \; , \text{ on } \omega_c \; , \tag{401}$$

$$\varepsilon_c' = - \frac{R}{4\pi} \iint_{\omega_c} H'_{c.c} \; \nu_c \; d\omega - \frac{R}{4\pi} \iint_{\omega_s} H'_{c.s} \; \nu_s \; d\omega \; . \tag{402}$$

The equation (401) is again a linear inhomogeneous integral equation of the

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second kind, as (56) and (57). The integral equation (401) has the same kernel function as (56) and (57). The theory of an integral equation is governed by the structure of the kernel function of it, it is here the function $H'_{c.c}$. The solution of (401) is unique as it was proved by the developments from (67a) to (86) for an analogous situation. The kernel $H'_{c.c}$ for the area ω_c has the property of being positive definite, it can be taken from the derivations (118) to (133c). $H'_{c.c}$ of (401) is also a closed kernel function, the defect of it is equal to zero, cf. (143) to (159). Furthermore, $H'_{c.c}$ can be considered as a continuous function in the area ω_c , as it is evidenced by the later developed relations (463) to (477) applying them to the function H' instead of S. The discontinuity of H' and $H'_{c.c}$ for $\psi \longrightarrow 0$ is removable.

It is possible to transform (401) into the matrix shape along the lines developed previously by (212) to (230). Conferring with the relations (163)(164)(165)(214)(215)(216)(230) and (401), the following developments are obtained,

$$\underline{\underline{a}}'_{3.c} = \frac{4\pi}{R} \begin{bmatrix} \cdots \\ (T')_i \\ \cdots \end{bmatrix} , \qquad (403)$$

$$i = 1,2,3, \ldots , k \ . \qquad (404)$$

Here, i refers to the continental compartments only.

$$\underline{\underline{e}}'_c = \begin{bmatrix} \cdots \\ (\varepsilon'_c)_i \\ \cdots \end{bmatrix} . \qquad (405)$$

(401) is transformed into

$$0 = \underline{\underline{e}}'_c + \frac{R}{4\pi} \underline{\underline{a}}'_{3.c} - \Delta\omega^* \frac{R}{4\pi} \underline{\underline{H}}'_{c.c} \underline{\underline{a}}'_{3.c} . \qquad (406)$$

Hence,

$$0 = \frac{4\pi}{R} \underline{\underline{e}}'_c + \underline{\underline{a}}'_{3.c} - \Delta\omega^* \underline{\underline{H}}'_{c.c} \underline{\underline{a}}'_{3.c} , \qquad (407)$$

$$0 = \frac{4\pi}{R} \underline{\underline{e}}'_c + (\underline{\underline{E}} - \Delta\omega^* \underline{\underline{H}}'_{c.c}) \underline{\underline{a}}'_{3.c} , \qquad (408)$$

and with (221),

$$0 = \frac{4\pi}{R} \underline{\underline{e}}'_c + \underline{\underline{K}}'_{c.c} \underline{\underline{a}}'_{3.c} , \qquad (409)$$

$$\underline{\underline{a}}'_{3.c} = -\frac{4\pi}{R} (\underline{\underline{K}}'_{c.c})^{-1} \underline{\underline{e}}'_c . \qquad (410)$$

$(\underline{\underline{K}}'_{c.c})^{-1}$ is the well-founded inverse of $\underline{\underline{K}}'_{c.c}$, (229). The defect of $\underline{\underline{K}}'_{c.c}$ was

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proved to be equal to zero.

The relation (410) is the matrix shape of the solution of the integral equation (401). It allows the computation of the T' values for the continents in terms of the empirically given boundary values ν_c and ν_s. The structure of the T' values is rather smoothed, it is much more smoothed than the structure of the Δg_F and δg values. ε'_c depends on ν_c and ν_s by certain integral transformations only, (402), which have a smoothing effect.

The combination of these continental T' values with the boundary values $(\Delta g_F)'$ on the continents, ω_c, by (398), leads to the $\delta g'$ values on the continents. These computed continental $\delta g'$ values are united with the a priori given boundary values $\delta g'$ for the oceans. Thus, the global distribution of the function $\delta g'$ is known finally. By (394), $\delta g'$ gives the global distribution of the T' values. This is the solution of the investigated boundary value problem, (401), which incolves the smoothed unknowns T' on the continents. A relief to the numerical computations is obviously the consequence since the rugged $\delta g'$ values on the continents are replaced by the smoothed T' values.

Furthermore, it is also possible to develop a difference method for the solution of (401) and for the relation (410), similar as it was carried out by (363) to (390) in case the $(\delta g)'$ values on the continents are the unknown values which are to be determined. For the application of the difference method, the relation (410) changes into

$$\delta \underline{\underline{a}}'_{3,c} = -\frac{4\pi}{R} \quad \delta (\underline{\underline{K}}'_{c,c})^{-1} \, \underline{\underline{e}}'_c \quad , \tag{411}$$

cf. (363). For the fixation of the ideas, the difference method according to (411) is applied to the model polygon of Fig. 2. By the equation (411), the difference between the T' values of the compartments with the index a = 1 and a = 3 is computed. The same is done for the compartments with the index a = 5 and a = 3. The considerations connected with the developments (384) to (390) give again rise to an equation of the following form,

$$\delta x_5 - \delta x_1 = (T')_5 - (T')_1 \; . \tag{412}$$

δx_5 and δx_1 represent here the concerned difference values of the perturbation potential T' computed by (411). $(T')_5$ and $(T')_1$ on the right hand side of (412) represent the a priori empirically given offshore dates of the satellite altimetry which can be taken from the maps of Rapp, [23] [24]. Thus, the condition equation (412) anchors the results of (411) in the offshore dates of the satellite altimetry, see also (390). Hence, the adjustment procedure connected with (412) is a help to strengthen the continental difference values of T', obtained by (411). (412) has the character of an interpolation procedure.

In order to complete the considerations about the integral equation (401), the extension to the harmonics of 0th and 1st degree should have a short discussion. Following this problem, the integral (394) changes into, (47),

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$$T = \frac{R}{4\pi} \iint_{\omega} H(\psi)\ \delta g\ d\omega \ . \tag{413}$$

The compatibility conditions are not valid in this problem, (395). The integral equation (401) takes the shape

$$0 = \varepsilon_c + T - \frac{1}{2\pi} \iint_{\omega_c} H_{c.c}\ T\ d\omega \ , \text{ on } \ \omega_c \ , \tag{414}$$

with

$$\varepsilon_c = -\frac{R}{4\pi} \iint_{\omega_c} H_{c.c}\ \nu_c\ d\omega - \frac{R}{4\pi} \iint_{\omega_s} H_{c.s}\ \nu_s\ d\omega \ . \tag{415}$$

The relation (414) is again a linear inhomogeneous integral equation of the second kind. It leads to the determination of the continental T function and to the global solution of the second mixed boundary value problem in the space of the spherical harmonics of all degrees, (n = 0,1,2, ...), i.e. in the space of the global regular functions. The kernel function is $H_{c.c}$.

A short commemorative outline of the peculiarities of this kernel function and of the integral equation (414) seem to be convenient. $H_{c.c}$ comes from (48), it appears also as kernel function of (67d). The kernel function $H_{c.c}$ is symmetrical, positive definite and closed, (119)(133d)(161)(162). The matrix shape of $H_{c.c}$ is $\underline{\underline{H}}_{c.c}$, (180). The concerned continental chief minor is det $\underline{\underline{H}}_{c.c}$, it is positive definite. The inverse $\underline{\underline{H}}_{c.c}^{-1}$ of $\underline{\underline{H}}_{c.c}$ is well-defined, (185). In the matix calculus, the solution of (414) is found by way of (231) to (234), it is similar as the deductions described by (406) to (410).

Therefore, with

$$\underline{\underline{a}}_{3.c} = \frac{4\pi}{R} \begin{bmatrix} \cdots \\ (T)_i \\ \cdots \end{bmatrix} \ , \tag{416}$$

$$i = 1,2,3,\ \ldots\ , k \ , \tag{417}$$

and, (415),

$$\underline{\underline{e}}_c = \begin{bmatrix} \cdots \\ (\varepsilon_c)_i \\ \cdots \end{bmatrix} \ , \tag{418}$$

follows, (410),

$$\underline{\underline{a}}_{3.c} = -\frac{4\pi}{R} (\underline{\underline{K}}_{c.c})^{-1}\ \underline{\underline{e}}_c \ . \tag{419}$$

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$(\underline{\underline{K}}_{c.c})^{-1}$ is the inverse of

$$\underline{\underline{K}}_{c.c} = \underline{\underline{E}} - \Delta\omega^* \underline{\underline{H}}_{c.c} \quad . \tag{420}$$

(419) is the matrix form of the solution of (414).

The question whether the solution of (414) is unique or not, this problem was already treated in context with another integral equation, cf. (231) to (234). Here, the question is in the fore whether the rang defect of $\underline{\underline{K}}_{c.c}$ is equal to zero or not. The investigations into whether the solution of an integral equation of the structure of (414) is unique did show that a certain criterion must be fulfilled, (354). In case, the criterion (354) is fulfilled, it is possible to have a unique solution of (414), and to have a well-defined inverse in (419).

At last, there is no doubt, the principles connected with the difference method, (363) to (390), (411) to (412), can be transferred also to the solution of (414) and (419).

5. The first mixed boundary value problem

5.1. The definition of the first mixed boundary value problem

The first mixed boundary value problem of the geodesy was touched already at the beginning of this publication, (7)(9a)(9b)(13)(14)(19)(20). Further, the publications/47//5/ are devoted to this problem. On the oceans, the amounts of the perturbation potential T' or T serve as the boundary values. They are determined by the methods of the satellite altimetry. The sea surface topography is neglected within the frame of the present publication, (4)(5)(6). In the following discussions about the first mixed boundary value problem, the numerical computation of a matrix criterion will be carried out as one of the main points. Even this matrix does governe the investigations into whether the first mixed boundary value problem is unique in the space of the harmonics of 0th and higher degree. This matrix criterion which is to be discussed here will be derived in close analogy to the deduction which did lead to the criterion (354) in the investigations about the second mixed boundary value problem.

The empirically determined boundary values on the oceans are here, in the first mixed boundary value problem, denominated by $\tilde{\tau}_s$, and those on the continents by $\tilde{\tau}_c$. Therefore, in regard of (9a) and (9b), the two different types of the boundary values of the first mixed boundary value problem have the following shape,

$$\tilde{\tau}_s = \tilde{\tau}_s(\varphi, \lambda) = \alpha_{3.s} = T' \ , \text{ on } \ \omega_s \ , \tag{421}$$

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and $\tilde{\tau}_c = \tilde{\tau}_c(\varphi, \lambda) = \alpha_{2.c} = \Delta g_F = -\frac{\partial T'}{\partial r} - \frac{2}{r} T'$, on ω_c . (422)

In order to give a self-contained representation of the ideas,it seems to be convenient to put an introduction into the theory of the first mixed boundary value problem at the beginning of the developments.

In inhomogeneous linear integral equation of the first kind for the solution of the first mixed boundary value problem of the geodesy has the following form, (14)(41),

$$0 = -\tilde{\tau}_s + \frac{R}{4\pi} \iint_{\omega_c} \tilde{\tau}_c S_{s.c} \, d\omega + \frac{R}{4\pi} \iint_{\omega_s} \alpha_{2.s} S_{s.s} \, d\omega \text{ , on } \omega_s. \quad (423)$$

$S_{s.s}$ is the kernel function valid for the oceanic couples of points only. The inversion of (423) gives $\alpha_{2.s}$ in terms of the empirically obtained values $\tilde{\tau}_s$ and $\tilde{\tau}_c$. Consequently, along these lines, the values $\alpha_{2.s}$ and $\alpha_{2.c}$ are finally reached. They are combined and the free-air anomalies $\Delta g_F = \alpha_2$ are determined successfully for whole the globe ω. This global function α_2 leads to the computation of the solution for T' for test points in the exterior space and on the whole globe by

$$T' = T'(r, \varphi, \lambda) = \frac{R}{4\pi} \iint_{\omega} \alpha_2 S(r, \psi) \, d\omega \quad , \quad (424)$$

$$S(r, \psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{R}{r}\right)^{n+1} P_n(\cos \psi) \quad , \quad (425)$$

$S(r, \psi)$ is the generalized Stokes function, [12][15]. Thus, in case

$$r = R \quad , \quad (426)$$

$$T' = T'(R, \varphi, \lambda) = \frac{R}{4\pi} \iint_{\omega} \alpha_2 S(\psi) \, d\omega \quad . \quad (427)$$

$S(\psi)$ is given by (15). The above relations from (423) to (427) are self-explanatory, cf. (10)(11)(14). The inhomogeneity of the integral equation (423) is now denominated by ϱ_s , it is defined for oceanic test points only,

$$\varrho_s = -\tilde{\tau}_s + \frac{R}{4\pi} \iint_{\omega_c} \tilde{\tau}_c S_{s.c} \, d\omega \quad . \quad (428)$$

Thus, the integral equation (423) changes to

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$$0 = \varrho_s + \frac{R}{4\pi} \iint_{\omega_s} \alpha_{2.s} \; S_{s.s} \; d\omega \; , \quad \text{on } \omega_s \; . \tag{429}$$

(429) is fundamental for the solution of the first mixed boundary value problem.

In case, the refinements along the lines of the Molodenskij boundary value problem are to be included, the deductions have to begin from (12) instead of (10). For the Molodenskij type of the problem, the observed boundary values are distributed along the real surface of the Earth σ, and not along the spherical globe ω. (421) and (422) change into

$$\tau_s = \alpha_{3.s} = T' \quad , \quad \text{on } \sigma_s \; , \tag{430}$$

and

$$\tau_0 = \alpha_{2.c} = \Delta g_F = - \frac{\partial T'}{\partial r} - \frac{2}{r} T' \; , \quad \text{on } \sigma_c \; . \tag{431}$$

(14) gets the following modified form, [6][8],

$$T'_s = \frac{R}{4\pi} \iint_{\omega_s} \Delta g_F \; S(\psi) \; d\omega + \frac{R}{4\pi} \iint_{\omega_c} \Delta g_F \; S(\psi) \; d\omega +$$

$$+ \frac{R}{4\pi} \iint_{\omega} C \cdot S(\psi) \; d\omega , \quad \text{on } \omega_s \; . \tag{432}$$

The relation (432) transforms (423) into

$$0 = - \tau_s + \frac{R}{4\pi} \iint_{\omega_c} \tau_c \; S_{s.c} \; d\omega + \frac{R}{4\pi} \iint_{\omega_s} \alpha_{2.s} \; S_{s.s} \; d\omega + C^*_s , \text{ on } \omega_s , \tag{433}$$

with

$$C^*_s = \frac{R}{4\pi} \iint_{\omega} C \cdot S(\psi) \; d\omega \; , \quad \text{on } \omega_s \; . \tag{434}$$

The modified inhomogeneity is, (428),

$$\varrho^*_{s.} = - \tau_s + \frac{R}{4\pi} \iint_{\omega_c} \tau_c \; S_{s.c} \; d\omega + C^*_s \; . \tag{435}$$

Thus, the integral equation of the first kind which solves the first mixed boundary value problem changes into the following form which is in keeping with the

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Molodenskij variant of the Stokes problem, (429),

$$0 = \varrho_s^* + \frac{R}{4\pi} \iint_{\omega_s} \alpha_{2.s} S_{s.s} \, d\omega \quad , \text{ on } \omega_s \quad . \tag{436}$$

The range of validity of (436) is again the oceanic area only.

As concerns the computation of C_s^* by (434), it is well-known that the plane topographical correction of the gravity, C, is equal to zero on the oceans. Therefore, computing C_s^* by (434), the share of the continental integration will dominate considerably the share of the oceanic integration,

$$C_s^* = \frac{R}{4\pi} \iint_{\omega_s} C \cdot S_{s.s} \, d\omega + \frac{R}{4\pi} \iint_{\omega_c} C \cdot S_{s.c} \, d\omega \quad , \tag{437}$$

$$\left| \iint_{\omega_c} C \cdot S_{s.c} \, d\omega \right| > \left| \iint_{\omega_s} C \cdot S_{s.s} \, d\omega \right| \quad . \tag{438}$$

In the computation of C, the integration over the topographical surroundings of the test point has to be extended to a distance of about 100 km only, it is well-known, /12/ /15/. Hence, in the first integrand on the right hand side of (437), the C values will have non-vanishing amounts only within a belt of about 100 km width offshore the coasts and accompanying the coastline.

5.2. The first mixed boundary value problem of the geodesy in the subspace of the spherical harmonics of 2nd and higher degree

As to the investigation into whether the solution of the first mixed boundary value problem is unique, this problem is governed by the homogeneous shape of the linear integral equation of the first kind, (429), it is

$$0 = \frac{R}{4\pi} \iint_{\omega_s} \alpha_{2.s} S_{s.s} \, d\omega \quad . \tag{439}$$

In case, the relation

$$\alpha_{2.s} = 0 \quad , \text{ on } \omega_s \quad , \tag{440}$$

follows necessarily from (439), it is sure that the solution is unique.

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In the scope of the proof of the uniqueness, it is well-known that the boundary values have to be equal to zero, (68)(69), (421)(422),

$$\tilde{\tau}_s = \alpha_{3.s} = T' = 0 \quad , \quad \text{on} \quad \omega_s \quad , \tag{441}$$

and

$$\tilde{\tau}_c = \alpha_{2.c} = -\frac{\partial T'}{\partial r} - \frac{2}{r} T' = 0 \text{ , on } \omega_c \text{ .} \tag{442}$$

The relations (429)(441)(442) give (439).

The subsequent derivations follow from the corresponding developments for the second mixed boundary value problem. The relation (443) and (444) serve for the same purpose in the first mixed boundary value problem as the relations (90)(91)(92)(93) do in case of the second mixed boundary value problem.

The relation (439) does lead to the definition of a global function χ^*,

$$\chi^*_s = 0 = \frac{R}{4\pi} \iint_{\omega_s} \alpha_{2.s} \, S_{s.s} \, d\omega \, , \tag{443}$$

$$\chi^*_c = \frac{R}{4\pi} \iint_{\omega_s} \alpha_{2.s} \, S_{c.s} \, d\omega \, . \tag{444}$$

Further, a global function χ is introduced also, under the influence of (443)(444),

$$\chi_s = \alpha_{2.s} \quad , \quad \text{on} \quad \omega_s \quad , \tag{445}$$

$$\chi_c = \alpha_{2.c} = 0 \quad , \quad \text{on} \quad \omega_c \quad . \tag{446}$$

Thus, (439) leads necessarily to

$$\chi^* = \frac{R}{4\pi} \iint_{\omega} \chi \cdot S(\psi) \, d\omega \, . \tag{447}$$

This is an equation of global shape. The boundary values (443) to (446) give rise to

$$\iint_{\omega} \chi \, \chi^* \, d\omega = 0 = \frac{R}{4\pi} \iint_{\omega} \chi \cdot d\omega \iint_{\omega} \chi \cdot S(\psi) \, d\omega \, . \tag{448}$$

Along the same way which did lead from (100) to (104) in the Hotine problem, the equation (448) has here the consequence

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$$0 = \sum_{n=2}^{\infty} \frac{1}{n-1} \bar{\chi}_n^2 , \tag{449}$$

with, (101),

$$\chi = \sum_{n=0}^{\infty} \bar{\chi}_n \, X_n(\varphi,\lambda) \quad . \tag{450}$$

Hence,

$$\bar{\chi}_n = 0 \quad , (n = 2,3,4, \ldots) \quad . \tag{451}$$

Before the background of (446) and (451), and accounting for the considerations which did lead to (112), it can be stated that the equations

$$\bar{\chi}_0 = \bar{\chi}_1 = 0 \tag{452}$$

reveal themselves as a consequence of (446) and (451). Indeed, if ω_c is an area of non-vanishing extent, e.g. the continents, in this case, (446) and (451) demand to find a regular function of the form

$$\bar{\chi}_0 \, X_0 + \bar{\chi}_1 \, X_1 = 0 \, , \quad \text{on} \quad \omega_c \quad . \tag{453}$$

(453) is not fulfilled unless (452) is valid, cf. /5/ /16/ p. 344. The following relations are a self-explanatory consequence,

$$\bar{\chi}_n = 0 \quad , \quad (n = 0,1,2, \ldots) \quad , \tag{454}$$

$$\chi = 0 \quad , \quad \text{on} \quad \omega \quad , \tag{455}$$

$$\chi_s = \alpha_{2.s} = \bullet \quad . \tag{456}$$

Thus, (440) is valid and the solution of (429) is unique.

The two facts that the kernel function $S_{s.s}$ is positive definite and closed, these facts follow as by-products of the above derivations about the uniqueness. $S_{s.s}$ is positive definite if

$$\Pi(\chi_s, \chi_s) = \iint_{\omega_s} \chi_s \, d\omega \iint_{\omega_s} S_{s.s} \, \chi_s \, d\omega > 0 \tag{457}$$

is valid for an arbitrary regular function χ_s on ω_s . In case of

$$\chi = \chi_s \quad , \quad \text{on} \quad \omega_s \quad , \tag{458}$$

$$\chi = 0 \quad , \quad \text{on} \quad \omega_c \quad , \tag{459}$$

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(457) leads necessarily to

$$\Pi (\chi_s, \chi_s) = \iint_{\omega} \chi \; d\omega \iint_{\omega} S(\psi) \; \chi \; d\omega \quad > 0 \quad , \qquad (460)$$

if (458) and (459) are taken into account. The operator $\Pi(\chi_s, \chi_s)$ is equal to the right hand side of (448) after the constant factor $\frac{R}{4\pi}$ is applied. Therefore, the relations (448), (449), (460) prove obviously that the inequation (457) is valid and that the kernel function $S_{s.s}$ is positive definite.

As to the question whether the kernel function $S_{s.s}$ is closed, (439) and (440) represent already the constraints that must be fulfilled in order to have this property of $S_{s.s}$. The validity of (439) and (440) for an arbitrary regular function $\alpha_{2.s}$ was already corroborated within the course of the proof of the uniqueness, (456). Thus, the kernel function is also closed, [1][2][5][8][10][22][29].

The question whether the kernel function $S(\psi)$ is continuous was already discussed earlier, (15) to (18). It was found that the discontinuity for $\psi \longrightarrow 0$ is removable. $S(\psi)$ is defined for test points covering whole the globe. $S_{s.s}$ is identic with $S(\psi)$ if the two end points of the arc with the length ψ are in the interior of the oceanic part of the surface of the Earth. Therefore, $S_{s.s}$ has also a removable discontinuity for $\psi \longrightarrow 0$, as it is obvious.

This matter can be viewed from another standpoint also, it is the standpoint that most of the geodetic evaluation methods consider only the significantly known frequencies in the spectrum of the empirically determined functions. Only the wave lengthes of even these frequencies are filtered out and considered, in order to undergo a further evaluation. In this context, the Stokes formula is to be put into the fore, (10), for the fixation of the ideas,

$$T' = \frac{R}{4\pi} \iint_{\omega} \Delta g_F \; S(\psi) \; d\omega \quad . \qquad (461)$$

The Δg_F values have not a global analytical expression which does contain all the details. These values are reached from the gravity maps by an interpolation procedure. On the strength of the compilation procedure applied in the making of these maps, they contain only details down to a certain limit wave length L. The amount of L depends on the scale of the map. Details which have a wave length that is smaller than L are not contained in the map. Thus, it is allowed to introduce a truncation of the series development for the free-air anomalies Δg_F, (74),

$$\Delta g_F = \sum_{n=2}^{M} \bar{a}_{2.n} \; X_n(\varphi, \lambda) \quad . \qquad (462)$$

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M depends on L , M = M (L). It is usual to put

$$M = \frac{180^{o}}{L^{o}} = \frac{20\ 000\ km}{L\ km} \quad . \tag{463}$$

The truncation of the series for Δg_F, (462), allows a truncation of the Stokes function also, (11),

$$\bar{S}\ (\psi) = \sum_{n=2}^{M} \frac{2n + 1}{n - 1}\ P_n\ (\cos \psi) \quad , \tag{464}$$

because the harmonics of the degree $n > M$ are cut off. The truncated series for Δg_F and S (ψ), (462)(464), change (461) into

$$T' = \frac{R}{4\pi} \iint_{\omega} \bar{S}\ (\psi) \sum_{n=2}^{M} \bar{a}_{2.n}\ X_n\ d\omega \quad , \tag{465}$$

or, instead of (465),

$$T' = \frac{R}{4\pi} \iint_{\omega} \bar{S}(\psi) \cdot \sum_{n=0}^{\infty} \bar{a}_{2.n}\ X_n \cdot d\omega \quad , \tag{466}$$

or, which is the same relation,

$$T' = \frac{R}{4\pi} \iint_{\omega} \bar{S}\ (\psi)\ \Delta g_F\ d\omega \ . \tag{467}$$

Thus, in the theory of the mixed boundary value problem, the Stokes integral (10) and (461) is allowed to be replaced by (467), since only a model of the real gravity field and of the perturbation potential is intended to be considered in geodesy. It is a model that is in keeping with the neglects in the course of the compilation procedure of the gravity maps. As it is seen from (467), this replacement expells the function S, that does approach the infinity in case $\psi \longrightarrow 0$, in favour of another function $\bar{S}$ that is finite in case of $\psi \longrightarrow 0$. $\bar{S}\ (\psi)$ is a continuous function as it is evidenced by (464).

After these preceding considerations, it is sure that a kernel function of the type $\bar{S}_{s.s}$ in the here considered integral equation (429) can be taken as symmetrical, positive definite and continuous. Therefore, the presuppositions for the validity of the theorem of Mercer are fulfilled, [10][29]. Thus, $\bar{S}_{s.s}$ has the following convergent series development,

$$\bar{S}_{s.s} = \sum_{k=1}^{\infty} \frac{\varepsilon_k\ (\varphi, \lambda)\ \varepsilon_k\ (\bar{\varphi}, \bar{\lambda})}{\mu_k} \quad . \tag{468}$$

ε_k are the orthonormalized eigenfunctions and μ_k are the eigenvalues.

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$$\mu_k > 0 \quad , (k = 1,2, \ldots) \quad , \tag{469}$$

since $\bar{S}_{s.s}$ is positive definite. This property of $\bar{S}_{s.s}$ is obviously valid, see (448)(449)(457)(460); the sum of (449) is truncated beyond of $n = M$. The development (468) can be introduced into (429), it follows

$$-\frac{4\pi}{R} \varrho_s = \iint_{\omega_s} \alpha_{2.s} \bar{S}_{s.s} \, d\omega \, . \tag{470}$$

The inversion of (470) is

$$\alpha_{2.s} = -\frac{4\pi}{R} \iint_{\omega_s} \varrho_s (\bar{S}_{s.s})^{-1} \, d\omega \quad , \tag{471}$$

with

$$(\bar{S}_{s.s})^{-1} = \sum_{k=1}^{\infty} \mu_k \, \varepsilon_k (\varphi, \lambda) \, \varepsilon_k (\bar{\varphi}, \bar{\lambda}) \quad . \tag{472}$$

Because $\bar{S}_{s.s}$ is closed - see the concern with $S_{s.s}$ - , $\alpha_{2.s}$ has the representation

$$\alpha_{2.s} = \sum_{k=1}^{\infty} \tilde{a}_k \, \varepsilon_k (\varphi, \lambda) \quad . \tag{473}$$

Hence, (470),

$$\iint_{\omega_s} \sum_{i=1}^{\infty} \tilde{a}_i \, \varepsilon_i (\varphi, \lambda) \sum_{k=1}^{\infty} \frac{\varepsilon_k (\varphi, \lambda) \, \varepsilon_k (\bar{\varphi}, \bar{\lambda})}{\mu_k} d\omega = \sum_{k=1}^{\infty} \tilde{a}_k \frac{1}{\mu_k} \varepsilon_k(\bar{\varphi}, \bar{\lambda}), \tag{474}$$

since

$$\iint_{\omega_s} \varepsilon_i \, \varepsilon_k \, d\omega = \begin{Bmatrix} 0 \text{ , if } i \neq k \\ 1 \text{ , if } i = k \end{Bmatrix} \quad . \tag{475}$$

Thus, (470),

$$-\frac{4\pi}{R} \varrho_s = \sum_{k=1}^{\infty} \tilde{a}_k \frac{1}{\mu_k} \varepsilon_k (\varphi, \lambda) \quad , \tag{476}$$

and with (471)(472),

$$-\frac{4\pi}{R} \iint_{\omega_s} \varrho_s (\bar{S}_{s.s})^{-1} \, d\omega = \sum_{k=1}^{\infty} \tilde{a}_k \, \varepsilon_k (\varphi, \lambda) \quad . \tag{477}$$

A glance on (473) shows that (477)(468)(472) lead to a corroboration of (471). Thus, it is evidenced that (472) is the right inversion of (468). Therefore, the kernel function $\bar{S}_{s.s}$ and its inversion $(\bar{S}_{s.s})^{-1}$ have well-defined analytical expressions.

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In the numerical applications with a certain compartment division of L_{km} x L_{km} squares, it is no matter whether $S_{s.s}$ or $\bar{S}_{s.s}$ is introduced. They can be exchanged since $S_{s.s}$ represents here the mean value of $S_{s.s}$ for the individual compartments. Further, for such a compartment division, it is no matter whether $S_{s.s}$ or $\bar{S}_{s.s}$ is intended to be inversed. These functions approximate their mean values for the compartments. Therefore, $(S_{s.s})^{-1}$ approximates $(\bar{S}_{s.s})^{-1}$ for the choosen compartment division also.

The inverse of the global Stokes function S is the function S^{-1}. Or, by matrix denominations, $\underline{\underline{S}}^{-1}$ is the inverse of $\underline{\underline{S}}$. S^{-1} and $\underline{\underline{S}}^{-1}$ are well-known and well-defined expressions, they are explained by the Numerov formula used in the routine work for the computation of Δg_F by T, [12][15].

Returning back to the integral equation (429), this relation is transformed into the matrix calculus, with respect to the following computations. The relations (10)(11)(19)(164)(165) reveal

$$\underline{\underline{a}}_3 = \underline{\underline{S}} \quad \underline{\underline{a}}_2 \, \Delta\omega \quad . \tag{478}$$

The introduction of (215) and (216) gives the separation into the oceanic and continental parts, (20),

$$\underline{\underline{a}}_{3.s} = \Delta\omega \, \underline{\underline{S}}_{s.s} \, \underline{\underline{a}}_{2.s} + \Delta\omega \, \underline{\underline{S}}_{s.c} \, \underline{\underline{a}}_{2.c} \quad , \tag{479}$$

$$\underline{\underline{a}}_{3.c} = \Delta\omega \, \underline{\underline{S}}_{c.s} \, \underline{\underline{a}}_{2.s} + \Delta\omega \, \underline{\underline{S}}_{c.c} \, \underline{\underline{a}}_{2.c} \quad . \tag{480}$$

(479) is equivalent to the integral equation (429). The matrix shape solution of (429) is, (42),

$$\underline{\underline{a}}_{2.s} = (\underline{\underline{S}}_{s.s})^{-1} \left(\frac{1}{\Delta\omega} \underline{\underline{a}}_{3.s} - \underline{\underline{S}}_{s.c} \, \underline{\underline{a}}_{2.c} \right) \quad . \tag{481}$$

The combination of (480) and (481) yields the T' values on the continents from the boundary values $\underline{\underline{a}}_{3.s}$ and $\underline{\underline{a}}_{2.c}$, i.e. the T values on the oceans and the Δg_F values on the continents,

$$\underline{\underline{a}}_{3.c} = \underline{\underline{S}}_{c.s} (\underline{\underline{S}}_{s.s})^{-1} (\underline{\underline{a}}_{3.s} - \Delta\omega \, \underline{\underline{S}}_{s.c} \, \underline{\underline{a}}_{2.c}) + \Delta\omega \, \underline{\underline{S}}_{c.c} \, \underline{\underline{a}}_{2.c} \quad . \tag{482}$$

Or,

$$\underline{\underline{a}}_{3.c} = \underline{\underline{S}}_{c.s}(\underline{\underline{S}}_{s.s})^{-1} \, \underline{\underline{a}}_{3.s} + \Delta\omega \left(\underline{\underline{S}}_{c.c} - \underline{\underline{S}}_{c.s}(\underline{\underline{S}}_{s.s})^{-1} \, \underline{\underline{S}}_{s.c} \right) \underline{\underline{a}}_{2.c} \quad . \tag{483}$$

The relation (483) and the given boundary values lead to the knowledge of $\underline{\underline{a}}_{3.s}$ and $\underline{\underline{a}}_{3.c}$. Hence, $\underline{\underline{a}}_3$ is known in its global shape; this means that the T' values are given on ω. Then, the Dirichlet boundary value problem leads to the T' values in the exterior space, and the solution of the first mixed boundary value problem has reached the final point, it is computed.

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The matrix $\underline{\underline{S}}_{s.s}$ is already known by (22). The determinant of $\underline{\underline{S}}_{s.s}$,

$$\det \underline{\underline{S}}_{s.s} > 0 \quad , \tag{484}$$

is the positive definite oceanic chief minor already discussed earlier, (36) to (40).

The boundary values $\tilde{\tau}_s$ and $\tilde{\tau}_c$ of the first mixed boundary value problem are not allowed to have arbitrary amounts, similarly as in case of the second mixed boundary value problem, (238a) to (296). They have to fulfill 4 well-defined compatibility conditions, otherwise, the boundary values computed a posteriori by the solution function are not the same as the a priori expirically given boundary values.

The very deep reason for the appearance of these 4 compatibility conditions is founded in the defect of the global Stokes function, cf. (238a).

Two types of integral equations are of interest in this context.

$$\text{I.} \quad \iint_{\omega} S(\psi) \cdot \alpha \cdot d\omega = 0 \quad . \tag{485}$$

(485) has the following 4 independent solutions, (239), (273) to (276),

$$\bar{\gamma}_j \; Y_j(\varphi, \lambda) \quad , \quad (j = 1,2,3,4) \quad .$$

Thus, the defect of the kernel S for the area ω is equal to 4.

The second type is the subsequent one.

$$\text{II.} \quad \iint_{\omega_s} S_{s.s} \cdot \alpha_s \cdot d\omega = 0 \quad . \tag{486}$$

The integral equation (486) for α_s demands necessarily that

$$\alpha_s = 0 \quad , \tag{487}$$

since (439) leads to (440). $S_{s.s}$ is a closed kernel function for the area ω_s, the defect of it is equal to zero.

The defect of the number 4 for the type I gives rise to the 4 compatibility conditions which have to be fulfilled by the empirically given boundary values. The computed free-air anomalies are the elements of the vector $\underline{\underline{a}}_{2.s}$, (481),

$$(\Delta g_F)_s = \alpha_{2.s} = g\left\{ (\Delta g_F)_c \, , \, T_s \right\} \quad , \tag{488}$$

or,

$$\alpha_{2.s} = g\left\{ \tilde{\tau}_c \, , \, \tilde{\tau}_s \right\} \quad , \tag{489}$$

DOI: https://doi.org/10.2312/zipe.1986.084

or

$$\underline{\underline{a}}_{2.s} = g^* \{ \underline{\underline{a}}_{2.c} , \underline{\underline{a}}_{3.s} \} \quad . \tag{490}$$

The continental free-air anomalies are given values, (422), **a priori** ,

$$(\Delta g_F)_c = \alpha_{2.c} = \tau_c \quad , \tag{491}$$

or, in vector shape,

$$\underline{\underline{a}}_{2.c} \quad . \tag{492}$$

The two functions $\alpha_{2.s}$ and $\alpha_{2.c}$ or the two vectors $\underline{\underline{a}}_{2.s}$ and $\underline{\underline{a}}_{2.c}$ compute the solution potential of the first mixed boundary value problem, (488)(491), (249), (10),

$$\Xi^* = \frac{R}{4\pi} \iint_{\omega} \Delta g_F \cdot S(\psi) \cdot d\omega \quad . \tag{493}$$

Thus,

$$\Xi^* = \frac{R}{4\pi} \iint_{\omega_s} g \{ \tau_c , \tau_s \} \cdot S(\psi) \cdot d\omega + \frac{R}{4\pi} \iint_{\omega_c} \tau_c \cdot S(\psi) \cdot d\omega \quad . \tag{494}$$

The considerations about the second mixed boundary value problem, from (249) to (271), did lead to the condition that (272) has to be fulfilled. It is self-explanatory that similar considerations about the first mixed boundary value problem will lead to constraints about the free-air anomalies and the Stokes constants of the 0th and 1st degree. The following conditions result obviously, (74),

$$\bar{a}_{2.0} = \bar{a}_{2.1} = 0 \quad . \tag{495}$$

(495) represents 4 conditions for the four Stokes constants of 0th and 1st degree and order in the spherical harmonics development for Δg_F. (495) gives, cf. (281),

$$\Psi_j^* = 0 = \iint_{\omega} \alpha_2 \cdot Y_j (\varphi , \lambda) \cdot d\omega \quad , \tag{496}$$
$$j = 1,2,3,4 \quad .$$

The division into the oceanic and the continental area ω_s and ω_c leads to, (282),

$$\Psi_j^* = 0 = \iint_{\omega_s} \alpha_{2.s} \cdot Y_j \cdot d\omega + \iint_{\omega_c} \alpha_{2.c} \cdot Y_j \cdot d\omega \quad , \tag{497}$$

and, with (488) and (491),

$$\Psi_j^* = 0 = \iint_{\omega_s} g \{ \tau_c , \tau_s \} \cdot Y_j \cdot d\omega + \iint_{\omega_c} \tau_c \cdot Y_j \cdot d\omega \quad , \tag{498}$$

$$j = 1,2,3,4 \quad . \tag{499}$$

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The vector shape of (496) is, cf. (287),

$$\Psi_j^* = 0 = \underline{\underline{a}}_2 \; \underline{\underline{y}}_j \quad , \tag{500}$$

and for (497) follows, cf. (288), /5/,

$$\Psi_j^* = 0 = \underline{\underline{a}}_{2.s} \, (\underline{\underline{y}}_j)_s + \underline{\underline{a}}_{2.c} \, (\underline{\underline{y}}_j)_c \quad . \tag{501}$$

(501) is an expression in scalar products.

(501) and (481) let find the following conditions,

$$\Psi_j^* = 0 = (\underline{\underline{y}}_j)_c^T \, \underline{\underline{a}}_{2.c} + (\underline{\underline{y}}_j)_s^T \, (\underline{\underline{S}}_{s.s})^{-1} \left(\frac{1}{\Delta\omega} \underline{\underline{a}}_{3.s} - \underline{\underline{S}}_{s.c} \, \underline{\underline{a}}_{2.c} \right) \quad , \tag{502}$$

$$j = 1,2,3,4 \quad .$$

The superscript T denominates in (502) the transposition. Hence, $(\underline{\underline{y}}_j)_c^T$ is a row vector, and $\underline{\underline{a}}_{2.c}$ is a column vector.

The relations (502) describe the 4 compatibility conditions that must be fulfilled by the boundary values $\underline{\underline{a}}_{2.c}$ and $\underline{\underline{a}}_{3.s}$ of the first mixed boundary value problem. Otherwise, it is not possible to find a solution of this boundary value problem.

After the matrix shape of the 4 compatibility conditions is found, it seems to be convenient to conclude the developments about this question by a represention of these four compatibility conditions applying the means of the functional analysis. Here, the relations of the form (497) and (498) are the starting points. The function $\alpha_{2.s}$ or $g\{\tau_c, \tau_s\}$ comes from (471). (428), (429) and (471) lead to

$$\alpha_{2.s} = -\frac{4\pi}{R} \iint_{\omega_s} (S_{s.s})^{-1} \left[-\tau_s + \frac{R}{4\pi} \iint_{\omega_c} \tau_c \, S_{s.c} \, d\omega \right] d\omega \quad , \tag{503}$$

since $(\bar{S}_{s.s})^{-1}$ can be replaced by $(S_{s.s})^{-1}$ in the here discussed applications; see above. The multiplication with the concerned spherical harmonics Y_j according to (498) gives the final form of these conditions, /5/.

$$\Psi_j^* = 0 = \iint_{\omega_c} Y_j \; \tau_c \, d\omega \; +$$

$$+ \frac{4\pi}{R} \iint_{\omega_s} Y_j \left\{ \iint_{\omega_s} (S_{s.s})^{-1} \left[\tau_s - \frac{R}{4\pi} \iint_{\omega_c} S_{s.c} \; \tau_c \, d\omega \right] d\omega \right\} d\omega \quad , \tag{504}$$

$$j = 1, 2, 3, 4. \tag{505}$$

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5.3. The first mixed boundary value problem of the geodesy in the space of the spherical harmonics of 0th and higher degree

An extension of the considerations about the Stokes kernel function $S_{s.s}$ from the subspace of the harmonics of 2nd and higher degree over the space of the harmonics of 0th and higher degree is not possible. In any case, such an extension is not possible along the lines of the derivations from (439) to (505). It stands to reason that the reasoning about the here fundamental equation (449) cannot be expanded to n = 0 and n = 1, since the following relation will be obtained,

$$0 = \sum_{n=0}^{\infty} \frac{1}{n-1} \bar{\chi}_n^2 = -\bar{\chi}_0^2 + \infty \cdot \bar{\chi}_1^2 + \bar{\chi}_2^2 + \frac{1}{2} \bar{\chi}_3^2 + \ldots \quad . \tag{506}$$

The quadratic form on the right hand side of (506) is not positive definite. The fact that (449) is positive definite was fundamental for (439) to (505). This is the reason why it is not allowed to renew the derivations from (439) to (505) in case of the inclusion of the harmonics of 0th and 1st degree. But, to avoid misunderstandings, the above considerations about (506) do not prove that the 3 relevant properties of the function $S_{s.s}$ or of the matrix $\underline{\underline{S}}_{s.s}$ (positive definite, closed, unique inversion) are no more valid in case of the extension of the developments to n = 0 and n = 1. This question remains to be open.

In case, the solution of the first mixed boundary value problem is to be found in the space of the harmonics of 0th and higher degree, i.e. in the space of the regular functions, the investigation into whether the solution is unique can be carried out along another way. This way is in close neighborhood to the developments from (297) to (362) for the second mixed boundary value problem. The Stokes integral is in the subspace of the harmonics of 2nd and higher degree, (10),

$$T' = \frac{R}{4\pi} \iint_{\omega} S\,(\psi) \cdot \alpha_2' \cdot d\omega \quad , \tag{507}$$

with

$$\alpha_2' = (\Delta g_F)' \quad . \tag{508}$$

The extension over the space of the harmonics of 0th and higher degree is, cf. (298),

$$T = \sum_{j=1}^{4} z_j \cdot \rho_j(r) \cdot Y_j\,(\varphi, \lambda) + \frac{R}{4\pi} \iint_{\omega} S\,(\psi) \cdot \alpha_2' \, d\omega \quad . \tag{509}$$

The introduction of (509) into the fundamental differential equation of the geodesy, (7), leads to

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$$- \frac{\partial T}{\partial r} - \frac{2}{R} T = - \sum_{j=1}^{4} z_j \frac{d\varrho_j}{dr} Y_j - \frac{2}{R} \sum_{j=1}^{4} z_j \varrho_j Y_j + \alpha_2' \quad . \tag{510}$$

Since the uniqueness of the solution is to be proved, the boundary value conditions take the following homogeneous form,

$$T = 0 \quad , \qquad , \quad \text{on } \omega_s \quad , \tag{511}$$

$$- \frac{\partial T}{\partial r} - \frac{2}{R} T = 0 \qquad , \quad \text{on } \omega_c \quad . \tag{512}$$

The above two equations are introduced into (509) and (510). Hence,

$$0 = \sum_{j=1}^{4} z_j \varrho_j Y_j + \frac{R}{4\pi} \iint_{\omega} S(\psi) \alpha_2' \, d\omega \quad , \text{ on } \omega_s , \tag{513}$$

$$0 = - \sum_{j=1}^{4} z_j \left[\frac{d\varrho_j}{dr} + \frac{2}{R} \varrho_j \right] Y_j + \alpha_2' \quad , \text{ on } \omega_c \quad . \tag{514}$$

With, (309),

$$\Delta^{*} = \sum_{j=1}^{4} z_j \left[\frac{d\varrho_j}{dr} + \frac{2}{R} \varrho_j \right] Y_j \quad , \tag{515}$$

and

$$\Delta^{**} = \sum_{j=1}^{4} z_j \varrho_j Y_j \quad , \tag{516}$$

the relations (513) and (514) change into

$$0 = \Delta^{**} + \frac{R}{4\pi} \iint_{\omega} S(\psi) \, \alpha_2' \, d\omega \, , \text{ on } \omega_s \quad , \tag{517}$$

$$0 = \Delta^{**} - \alpha_{2.c}' \quad , \text{ on } \omega_c \quad . \tag{518}$$

The separation into the continental and the oceanic part gives

$$0 = \Delta^{*} - \alpha_{2.c}' \quad , \text{ on } \omega_c \quad , \tag{519}$$

$$0 = \Delta^{**} + \frac{R}{4\pi} \iint_{\omega_s} S_{s.s} \alpha_{2.s}' \, d\omega + \frac{R}{4\pi} \iint_{\omega_c} S_{s.c} \Delta^{*} d\omega \, , \text{ on } \omega_s \, . \tag{520}$$

DOI: https://doi.org/10.2312/zipe.1986.084

With regard to (470) and (471), the inversion of (520) has the following shape,

$$\alpha'_{2.s} = -\frac{4\pi}{R}\iint_{\omega_s}(S_{s.s})^{-1}\left[\Delta^{**} + \frac{R}{4\pi}\iint_{\omega_c} S_{s.c}\cdot\Delta^{*}\cdot d\omega\right] d\omega \;. \tag{521}$$

Here, in (521), the form $(S_{s.s})^{-1}$ is allowed to be written instead of $(\bar{S}_{s.s})^{-1}$.

According to (507), (519) and (521), the functions $\alpha'_{2.c}$ and $\alpha'_{2.s}$ are understood to be the continental and the oceanic parts of the function α'_2 which is free of the spherical harmonics of the 0th and 1st degree. Thus, cf. (314), (496),

$$\Omega^{*}_{j} = 0 = \iint_{\omega}\alpha'_2\cdot Y_j\,(\varphi,\lambda)\cdot d\omega \;, \tag{522}$$

$$j = 1,2,3,4 \;. \tag{523}$$

The division into the functions $\alpha'_{2.c}$ and $\alpha'_{2.s}$ gives,

$$\Omega^{*}_{j} = 0 = \iint_{\omega_s}\alpha'_{2.s}\cdot Y_j\cdot d\omega + \iint_{\omega_c}\alpha'_{2.c}\cdot Y_j\cdot d\omega \;. \tag{524}$$

The vector shape of the function α'_2 is, cf. (317)(500),

$$\underline{\underline{a}}'_2 = \begin{pmatrix} \cdots \\ (\alpha'_2)_i \\ \cdots \end{pmatrix} \;, \tag{525}$$

$$i = 1,2,3,\ \ldots\ ,\ q \;. \tag{526}$$

The division into the continents and the oceans gives the following vectors, cf. (319)(501),

$$\underline{\underline{a}}'_2 = \begin{pmatrix} \underline{\underline{a}}'_{2.c} \\ \underline{\underline{a}}'_{2.s} \end{pmatrix} \;. \tag{527}$$

Further, cf. (320) to (324),

$$\underline{\underline{d}}^{**} = \begin{pmatrix} \cdots \\ \Delta^{**}_i \\ \cdots \end{pmatrix} \;, \tag{528}$$

and

DOI: https://doi.org/10.2312/zipe.1986.084

$$\underline{\underline{d}}^{**} = \begin{pmatrix} \underline{\underline{d}}_c^{**} \\ \underline{\underline{d}}_s^{**} \end{pmatrix} \quad . \tag{529}$$

The equation (519) changes into the following vector equation, cf. (322)(323),

$$0 = \underline{\underline{a}}'_{2.c} - \underline{\underline{d}}_c^{*} \quad . \tag{530}$$

The functional equation (520) turns with (213) to

$$0 = R\, \underline{\underline{S}}_{s.s}\, \underline{\underline{a}}'_{2.s}\, \Delta\omega^{*} + R\, \underline{\underline{S}}_{s.c}\, \underline{\underline{d}}_c^{*}\, \Delta\omega^{*} + 2\, \underline{\underline{d}}_s^{**} \quad , \tag{531}$$

or,

$$\underline{\underline{a}}'_{2.s} = -\,(\underline{\underline{S}}_{s.s})^{-1} \left[\, \underline{\underline{S}}_{s.c}\, \underline{\underline{d}}_c^{*} + \underline{\underline{d}}_s^{**} \frac{2}{R\,\Delta\omega^{*}} \right] \quad . \tag{532}$$

The vector shape of the constraint Ω_j^{*}, (524)(331), is by scalar products

$$\Omega_j^{*} = 0 = \underline{\underline{a}}'_{2.s}\, (\underline{\underline{y}}_j)_s + \underline{\underline{a}}'_{2.c}\, (\underline{\underline{y}}_j)_c \quad . \tag{533}$$

The relations (530) and (532) for $\underline{\underline{a}}'_{2.s}$ and $\underline{\underline{a}}'_{2.c}$ are introduced into (533). The following condition equation is obtained,

$$\Omega_j^{*} = 0 = -\,(\underline{\underline{y}}_j)_s^T\, (\underline{\underline{S}}_{s.s})^{-1} \left[\, \underline{\underline{S}}_{s.c}\, \underline{\underline{d}}_c^{*} + \underline{\underline{d}}_s^{**} \frac{2}{R\,\Delta\omega^{*}} \right] + (\underline{\underline{y}}_j)_c^T\, \underline{\underline{d}}_c^{*} \quad . \tag{534}$$

Now, the relation (534) is to be expressed in terms of the four constant coefficients

$$z_j \quad , \tag{535}$$

$$j = 1,2,3,4 \quad . \tag{536}$$

Following up this aim, each of the two vectors $\underline{\underline{d}}_c^{*}$ and $\underline{\underline{d}}_s^{**}$ of (534) have to be developed into a sum of 4 partial vectors, cf. (344)(516)(529),

$$\underline{\underline{d}}_c^{*} = \sum_{u=1}^{4} z_u \left[\frac{d\varrho_u}{dr} + \frac{2}{R}\,\varrho_u \right] (\underline{\underline{y}}_u)_c \quad , \tag{537}$$

$$\underline{\underline{d}}_s^{**} = \sum_{u=1}^{4} z_u\, \varrho_u\, (\underline{\underline{y}}_u)_s \quad . \tag{538}$$

The relations (534)(537)(538) are combined. The following expression is the result, cf. (346),

DOI: https://doi.org/10.2312/zipe.1986.084

$$\Omega_j^* = 0 = -\sum_{u=1}^{4} z_u \left[\frac{d\varrho_u}{dr} + \frac{2}{R}\varrho_u\right] (\underline{y}_j)_s^T (\underline{\underline{S}}_{s.s})^{-1} \underline{\underline{S}}_{s.c} (\underline{y}_u)_c -$$

$$- \frac{2}{R\Delta\omega^*} \sum_{u=1}^{4} z_u \varrho_u (\underline{y}_j)_s^T (\underline{\underline{S}}_{s.s})^{-1} (\underline{y}_u)_s +$$

$$+ \sum_{u=1}^{4} z_u \left[\frac{d\varrho_u}{dr} + \frac{2}{R}\varrho_u\right] (\underline{y}_j)_c^T (\underline{y}_u)_c , \tag{539}$$

$$j = 1,2,3,4 . \tag{540}$$

Along the lines of the derivations elaborated from (347) to (362) about the Hotine function, the four equations (539) give rise to the following deductions in the matrix calculus.

$$(\underline{\underline{W}}^*)^T = \{w_{u.j}^*\} , \tag{541}$$

$$u,j = 1,2,3,4 . \tag{542}$$

$$w_{u.j}^* = -\left[\frac{d\varrho_u}{dr} + \frac{2}{R}\varrho_u\right] (\underline{y}_j)_s^T (\underline{\underline{S}}_{s.s})^{-1} \underline{\underline{S}}_{s.c} (\underline{y}_u)_c -$$

$$- \frac{2}{R\Delta\omega^*} \varrho_u (\underline{y}_j)_s^T (\underline{\underline{S}}_{s.s})^{-1} (\underline{y}_u)_s +$$

$$+ \left[\frac{d\varrho_u}{dr} + \frac{2}{R}\varrho_u\right] (\underline{y}_j)_c^T (\underline{y}_u)_c , \tag{543}$$

$$u, j = 1,2,3,4 . \tag{544}$$

$$\underline{\underline{W}}^* = \begin{bmatrix} w_{1.1}^* & w_{2.1}^* & w_{3.1}^* & w_{4.1}^* \\ w_{1.2}^* & w_{2.2}^* & w_{3.2}^* & w_{4.2}^* \\ w_{1.3}^* & w_{2.3}^* & w_{3.3}^* & w_{4.3}^* \\ w_{1.4}^* & w_{2.4}^* & w_{3.4}^* & w_{4.4}^* \end{bmatrix} . \tag{545}$$

The vector $\underline{z}$, (351), and the matrix $\underline{\underline{W}}^*$ are combined to

$$\Omega_j^* = 0 = \sum_{u=1}^{4} w_{u.j}^* z_u , \tag{546}$$

or,

$$0 = \underline{\underline{W}}^* \underline{\underline{z}} . \tag{547}$$

DOI: https://doi.org/10.2312/zipe.1986.084

The following criterion is demanded to be fulfilled,

$$\det \underline{\underline{W}}^{*} = \det \left\{ w^{*}_{u.j} \right\} \neq 0 \quad . \qquad (548)$$

Thus, because of (548), the following relations are self-explanatory, (515)(516)(530)(532)(509),

$$z_j = 0 \quad , \qquad (549)$$

$$\Delta^{*} = \Delta^{**} = 0 \quad , \qquad (550)$$

$$\underline{\underline{d}}^{*} = \underline{\underline{d}}^{**} = 0 \quad , \qquad (551)$$

$$\underline{\underline{a}}'_{2.c} = 0 \quad , \qquad (552)$$

$$\underline{\underline{a}}'_{2.s} = 0 \quad , \qquad (553)$$

$$\underline{\underline{a}}'_{2} = 0 \quad , \qquad (554)$$

$$T = 0 \quad . \qquad (555)$$

The relation (555) proves the uniqueness. Therefore, the first mixed boundary value problem of the geodesy has a unique solution in the space of all the spherical harmonics of the degree n = 0,1,2, ... , if the condition (548) is fulfilled. The test by the condition (548) can be applied to every course of the coastline , the run of the coastline underlies no restriction.

The elements $w^{*}_{u.j}$ of the matrix $\underline{\underline{W}}^{*}$, (543)(545), can be expressed also by the methods of the functional analysis. The compatibility conditions of the shape (524) are the starting point of the concerned deductions. The relation (515), (516), (519) and (521) are put in the conditional equation (524) and the following expression for $w^{*}_{u.j}$ is obtained,

$$w^{*}_{u.j} = - \frac{4\pi}{R} \iint\limits_{\omega_s} Y_j \left[\iint\limits_{\omega_s} (S_{s.s})^{-1} \; \rho_u Y_u \; d\omega \right] d\omega \; -$$

$$- \iint\limits_{\omega_s} Y_j \left[\iint\limits_{\omega_s} (S_{s.s})^{-1} \left[\iint\limits_{\omega_c} S_{s.c} \left\{ \frac{d\rho_u}{dr} + \frac{2}{R} \rho_u \right\} Y_u \; d\omega \right] d\omega \right] d\omega \; +$$

$$+ \iint\limits_{\omega_c} Y_j \left\{ \frac{d\rho_u}{dr} + \frac{2}{R} \rho_u \right\} Y_u \; d\omega \; , \qquad (556)$$

$$u,j = 1,2,3,4 \quad . \qquad (557)$$

The functions ρ_u are described by (299) and (300),

$$\rho_1 = \frac{R}{r} \quad , \qquad (558)$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$\varrho_2 = \varrho_3 = \varrho_4 = \left[\frac{R}{r}\right]^2 , \qquad (559)$$

$$\frac{d\varrho_1}{dr} = -\frac{R}{r^2} , \qquad (560)$$

$$\frac{d\varrho_2}{dr} = \frac{d\varrho_3}{dr} = \frac{d\varrho_4}{dr} = -2\frac{R^2}{r^3} . \qquad (561)$$

At the surface of the globe ω , for r = R, the above relations turn to

$$\varrho_u = 1 , \quad (u = 1,2,3,4) , \qquad (562)$$

$$\frac{d\varrho_1}{dr} = -\frac{1}{R} , \qquad (563)$$

$$\frac{d\varrho_2}{dr} = \frac{d\varrho_3}{dr} = \frac{d\varrho_4}{dr} = -\frac{2}{R} , \qquad (564)$$

$$\frac{d\varrho_1}{dr} + \frac{2}{R}\varrho_1 = \frac{1}{R} , \qquad (565)$$

$$\left\{\frac{d}{dr} + \frac{2}{R}\right\}(\varrho_2, \varrho_3, \varrho_4) = 0 . \qquad (566)$$

Some peculiarities of the function $S_{s.s}$ and its inverse $(S_{s.s})^{-1}$ are discussed by the developments explained from (461) to (477). In case, the two points the kernel function $S_{s.s}$ depends on do approach **each other very close** , $\psi \longrightarrow 0$, in this case, the kernel has a singularity. But, this peculiarity is removable, (17), Thus, it should not be put into the fore. It is not of dominating importance in the here discussed problem. The values that are here required are not the local values of the elements of $\underline{\underline{S}}_{s.s}$ and $(\underline{\underline{S}}_{s.s})^{-1}$ or of the functions $S_{s.s}$ and $(S_{s.s})^{-1}$. The values here required are the mean regional values of these expressions obtained by an integration over the compartments $\Delta\omega$. If these compartments have a side length of about L , (463), in this case, just the constituents in the relevant functions which have a **wavelength** of smaller than L cannot be brought to bear. Vice versa, if the considered empirically determined boundary value functions do not enclose some constituents with **wavelengthes** just smaller than L , then the size of the compartments must not be smaller than L .

The here considered functions Y_j, (556), have long **wavelengthes** only. Therefore, in the computations according to (556), the side lengthes L of the compartments $\Delta\omega$ can be choosen rather great; thus, much work is saved in the computations.

There is no hope to find an analytical expression for $(S_{s.s})^{-1}$ since the run of the coastline is involved. But, there is no analytical expression

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for the course of the coast line. The fact that the singularity for $\psi \longrightarrow 0$ is removable, it does paralyse the importance of this singularity, and it is not essential for the here discussed problem.

This singularity of the kernel function is not of importance for the integral transformations established by the kernel, (67a)(184)(185)(429)(470)(471). Therefore, in the here discussed applications, it is of no use considering the singularity of the kernels for $\psi = 0$. However, it is very probable that a consideration of the structure of kernel $\overline{S}_{s.s}$ and of its inverse, (470)(471), will be of use for the investigation of the stability of the solution.

Now, a numerical computation is to be carried out in order to find out whether the criterion (548) for the uniqueness of the first mixed boundary value problem in the space of the harmonics of all the degrees, (n= 0,1,2, ...), is fulfilled before the background of the real geographical distribution of the oceans and continents. In this context, the numerical amounts of the matrix elements $w^{*}_{u.j}$ must be computed, (543)(556). For the execution of the numerical integrations according to the formula (556), the surface of the globe is divided into 194 compartments, 112 oceanic compartments and 82 continental ones. All the 112 oceanic compartments have the same size of $\Delta\omega = 0.067\ 758$. The 82 continental compartments have different sizes, they vary between $\Delta\omega = 0.032$ and $\Delta\omega = 0{,}068$, in adaptation to the gaps remaining after the coverage of the oceans by the 112 compartments of equal size, it is in keeping with the run of the coast line. Thus, considering the Stokes matrix (21) and the compartment division of figure 1, the running index of all the compartments covers

$$i = 1,2, \ldots , p, p+1, \ldots , q ; \qquad (567)$$

but here, (568) is now valid:

$$p = 112 \quad , \quad q = 194 \quad . \qquad (568)$$

Figure 3 visualizes this global compartment division.

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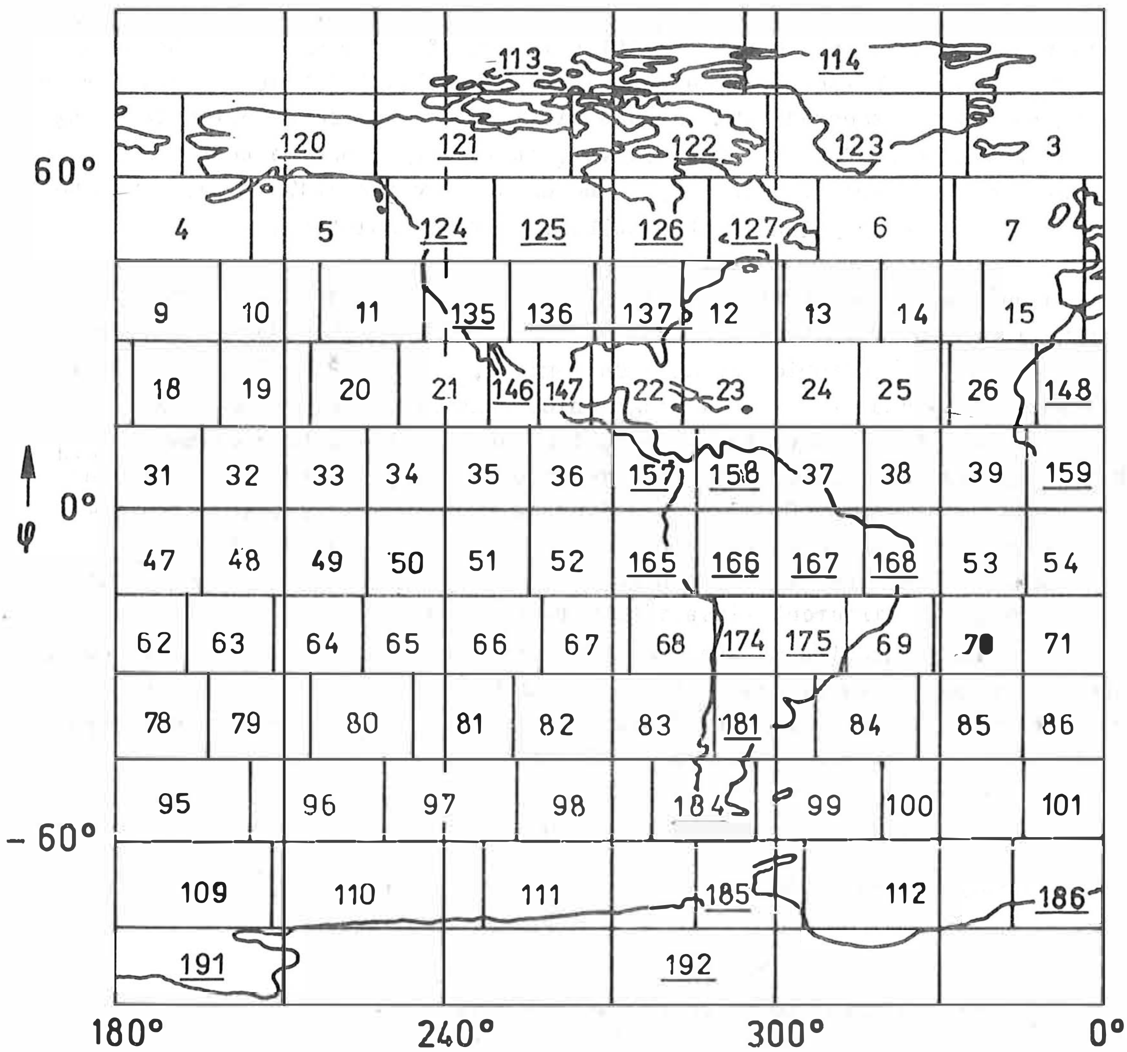
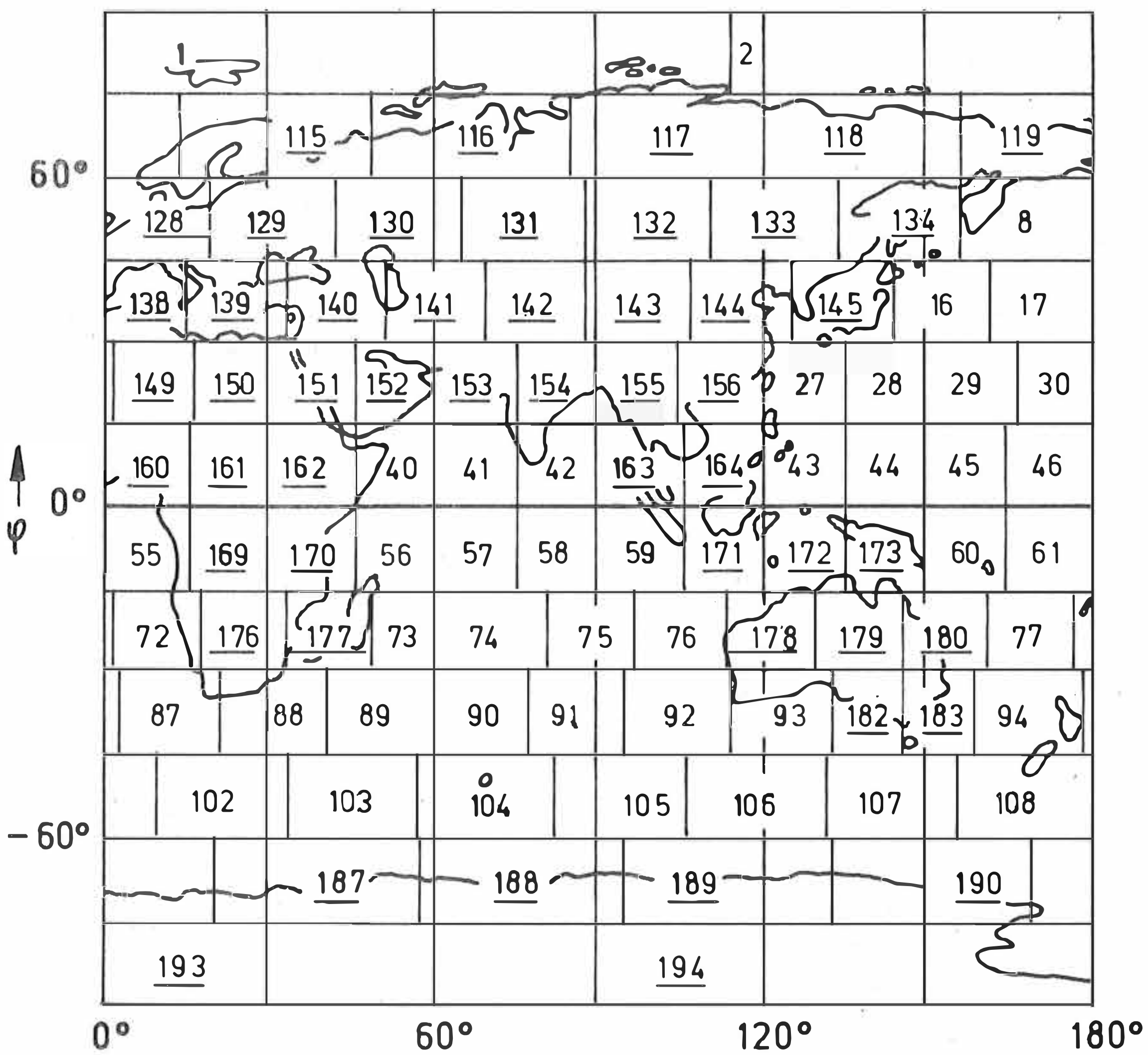


Fig. 3 : The division of the globe into 112 oceanic and 82 continental compartments for the investigation into whether the solution of the first mixed boundary value problem of the geodesy is unique in the space of all the harmonics of all the degrees and orders, (n = 0,1,2, ...) .

DOI: https://doi.org/10.2312/zipe.1986.084



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In the integration computations for the determination of the matrix elements $w^{*}_{u.j}$ according to (556), all the functions Y_j, $S_{s.s}$, $(S_{s.s})^{-1}$ and $S_{s.c}$ are introduced as functions of the type of a step function or rectangle function. Within an individual compartment, these functions are considered to have constant values, it is the amount the concerned function takes at the center of this compartment. Thus, an integration is replaced by a sum.

At first, the elements of the 112 x 112 matrix $\underline{\underline{S}}_{s.s}$ are computed by the Stokes function. The argument ψ is the spherical distance between the two concerned compartment centers. The computations are uncomplicated.

The computations for the elements in the main diagonal are to be explained by a short comment, since $S \longrightarrow \infty$ if $\psi \longrightarrow 0$. The mean value of S within such a compartment is

$$\frac{1}{\Delta\omega} \iint_{\Delta\omega} S(\psi)\, d\omega = S_{center} \quad . \tag{569}$$

With, [19],

$$S(\psi) \cong \frac{2}{\psi} + \ldots \quad . \tag{570}$$

For

$$\psi \longrightarrow 0 \tag{571}$$

follows,

$$S_{center} \cong \frac{1}{\Delta\omega} \iint_{\Delta\omega} \frac{2}{\psi} \sin\psi \; d\psi \, d\bar{\alpha} \quad , \tag{572}$$

$$S_{center} \cong \frac{2}{\Delta\omega} \iint_{\Delta\omega} d\psi \; d\bar{\alpha} \quad , \tag{573}$$

$$S_{center} \cong \frac{2}{\Delta\omega} \int_{\bar{\alpha}=0}^{2\pi} \psi(\bar{\alpha}) \; d\bar{\alpha} \quad , \tag{574}$$

$$\frac{1}{2\pi} \int_{\bar{\alpha}=0}^{2\pi} \psi(\bar{\alpha}) \; d\bar{\alpha} = \psi^{*} \quad , \tag{575}$$

The value ψ^{*} can be considered as the mean radius of the compartment of the size $\Delta\omega$. Hence, self-explanatory,

$$S_{center} \cong \frac{4\pi}{\Delta\omega} \psi^{*} \quad , \tag{576}$$

$$\Delta\omega = \pi(\psi^{*})^2 \quad , \tag{577}$$

$$\psi^{*} = \sqrt{\frac{\Delta\omega}{\pi}} \quad , \tag{578}$$

$$S_{center} = \frac{4\pi}{\Delta\omega} \sqrt{\frac{\Delta\omega}{\pi}} \quad , \tag{579}$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$S_{center} = 4 \sqrt{\frac{\pi}{\Delta\omega}} \quad . \tag{580}$$

(580) computes the main diagonal elements of the $\underline{\underline{S}}_{s.s}$ matrix or the amount of the kernel function $S_{s.s}$ attributed to the surface compartment $\Delta\omega$, if the argument ψ does go to zero, $\psi \longrightarrow 0$,within this area $\Delta\omega$.

After these explanatory lines, the matrix $\underline{\underline{W}}^*$ can be computed by (556). The following result was obtained, c is a definite constant,

$$\underline{\underline{W}}^* = c \begin{pmatrix} 4.31 & 0.24 & 0.26 & 0.24 \\ 1.84 & -0.37 & 0.01 & 0.03 \\ 1.53 & 0.01 & -0.60 & -0.06 \\ 1.43 & 0.03 & -0.06 & -0.43 \end{pmatrix} . \tag{581}$$

The inversion gives,

$$(\underline{\underline{W}}^*)^{-1} = \frac{1}{c} \begin{pmatrix} 0.14 & 0.10 & 0.06 & 0.08 \\ 0.75 & -2.17 & 0.28 & 0.23 \\ 0.33 & 0.25 & -1.55 & 0.41 \\ 0.49 & 0.15 & 0.42 & 2.12 \end{pmatrix} . \tag{582}$$

The crucial determinant is, (548),

$$\det (\underline{\underline{W}}^* \frac{1}{c}) = \det \left\{ w^*_{u.j} \frac{1}{c} \right\} = -0.66 \tag{583}$$

Together with (547), the relations (582)(583) reveal that

$$\underline{\underline{z}} = 0 \quad . \tag{584}$$

It proves the uniqueness of the first mixed boundary value problem of the geodesy in the space of the spherical harmonics of the 0th and higher degree and for the real geographical distribution of the continents and oceans, cf. (549) to (555).

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5.4. The difference method and the first mixed boundary value problem

For the second mixed boundary value problem, the derivations from (363) to (390) and the figure 2 develope a difference method for the computation of the solution. It allows to interpolate the computed continental gravity deviations between the a priori empirically given offshore boundary values of the gravity deviations. By the integral equation (401)(402), a modification of the second mixed boundary value problem is represented, it determines the T' values which are much more smoothed than the δg values.

Also this way of solution allows the application of a difference method, cf. (411)(412). It gives the possibility of computing directly the differences of the T' values between two continental test points. If the distance between these two points is not too great, the direct computation of the difference of the T' values, $\delta T'$, will not be so laborious as the direct determination of the individual T' values for the two test points. Much work is saved by the difference method since those compartments $\Delta\omega$ which are very distant from the two test points can be enlarged. It gives rise to a diminution of the dimension of the matrix $\underline{\underline{K}}'_{c.d}$, (409). A more easy computation of its inverse will be the consequence, (410).

The couples of points with the directly determined $\delta T'$ values are lined up and arranged in the broken line of a polygon which begins at one coastal point to cross a continent and to end at another coastal point. Thus, obviously, the $\delta T'$ values can be interpolated between the offshore T' values at the two end points of this polygon.

In the here discussed problem, (483), the difference method gets the following shape, in conformity with (364) to (383).

$$\delta \underline{\underline{a}}_{3.c} = \delta \underline{\underline{S}}_{c.s} (\underline{\underline{S}}_{s.s})^{-1} \underline{\underline{a}}_{3.s} + (\delta \underline{\underline{S}}_{c.c} - \delta \underline{\underline{S}}_{c.s} (\underline{\underline{S}}_{s.s})^{-1} \underline{\underline{S}}_{s.c}) \underline{\underline{a}}_{2.c} \Delta\omega . \tag{585}$$

$$\underline{\underline{x}} = \underline{\underline{a}}_{3.c} , \tag{586}$$

$$\underline{\underline{y}}_1 = \underline{\underline{a}}_{3.s} , \tag{587}$$

$$\underline{\underline{y}}_2 = \underline{\underline{a}}_{2.c} , \tag{588}$$

$$\underline{\underline{M}}_1 = \underline{\underline{S}}_{c.s} (\underline{\underline{S}}_{s.s})^{-1} , \tag{589}$$

$$\underline{\underline{M}}_2 = (\underline{\underline{S}}_{c.c} - \underline{\underline{S}}_{c.s} (\underline{\underline{S}}_{s.s})^{-1} \underline{\underline{S}}_{s.c}) \Delta\omega , \tag{590}$$

$$\underline{\underline{x}} = \underline{\underline{M}}_1 \underline{\underline{y}}_1 + \underline{\underline{M}}_2 \underline{\underline{y}}_2 , \tag{591}$$

$$\underline{\underline{M}}_1 = \{ m_{1.a.b} \} , \tag{592}$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$a, b = 1,2,3, \ldots , D , \ldots , N \quad , \tag{593}$$

$$\underline{\underline{M}}_2 = \left\{ m_{2.a.b} \right\} \quad , \tag{594}$$

$$x_a = \sum_{b=1}^{N} m_{1.a.b} \, y_{1.b} + \sum_{b=1}^{N} m_{2.a.b} \, y_{2.b} \quad , \tag{595}$$

and for a = D, (reference point),

$$x_D = \sum_{b=1}^{N} m_{1.D.b} \, y_{1.b} + \sum_{b=1}^{N} m_{2.D.b} \, y_{2.b} \quad , \tag{596}$$

$$x_a - x_D = \sum_{i=1}^{2} \sum_{b=1}^{N} (m_{i.a.b} - m_{i.D.b}) \, y_{i.b} \quad , \tag{597}$$

$$\delta M_i = \left\{ m_{i.a.b} - m_{i.D.b} \right\} \quad , \tag{598}$$

$$\delta M_i = \left\{ \delta m_{i.a.b} \right\} \quad , \tag{599}$$

$$\delta \underline{\underline{x}}^T = (\ldots , x_a - x_D , \ldots) \quad , \tag{600}$$

$$\delta \underline{\underline{x}} = \delta \underline{\underline{M}}_1 \, \underline{\underline{y}}_1 + \delta \underline{\underline{M}}_2 \, \underline{\underline{y}}_2 \quad , \tag{601}$$

cf. [7][8].

6. Numerical applications

At last, some comparative considerations about the 1st and the 2nd mixed boundary value problems seem to be appropriate. Here, some aspects of the numerical applications are in the fore now.

The influence of the sea surface topography N^{**} is to be considered at first, (5)(6). The previously discussed 2nd mixed boundary value problem has the boundary values Δg_F and δg. They are free of a hypothesis about N^{**}. The involved dates of $(g)_Q$ and N^* are obtained by measurements free of any hypothesis.

However, in contradiction to the 2nd mixed boundary value problem, the 1st mixed boundary value problem has boundary values on the oceans, being biased by the N^{**} values of the sea surface topography. There, the GN^* values are introduced instead of T, (4)(7a).

DOI: https://doi.org/10.2312/zipe.1986.084

The ooeanic boundary values have the error T^{**}, (6), in the determination of the solution of the 1st mixed boundary value problem. The T^{**} values shift the solution vector and the result reveals to be a biased vector. Hence, in case of the 1st mixed boundary value problem, the finally obtained T values on the continents are biased by the T^{**} values which appear on the oceans.

But, the here trouble effecting T^{**} values are small and, in most cases, they have a rather great distance to the continents, /20/. Therefore, it is very probable, these biases in the computed T values of a continent or of a subcontinent can be approximated by an analytical expression of constant and linear terms of the latitude and longitude differences. The unknown coeffioients of this expression can be determined by some well distributed Doppler derived values for ζ or N or T on the continents.

The numerical computations to find the solution of the 1st mixed boundary value problem can also happen along an indirect way. This roundabout way consists of two steps. The first step leads from the oceanic altimeter dates to the oceanic free-air anomalies along the lines of the inverse Stokes integral, /23/ /24/. The inverse Stokes relation permits to obtain a standard error of about $\pm$ 2 to $\pm$ 5 mgal for the average values of the gravity anomalies of the oceanic compartments of 200 km square, /9/. The second step of the indirect method is the computation of the oceanic and global surface perturbation potential T by means of the Stokes integral supplemented in the integrand by the effect of the plane topographical reduction of the gravity.

However, this indirect way is not an optimal procedure. It involves a clear and significant loss of precision. The high precision of the altimeter derived T values is not fully exhausted and it is not brought to bear completely. Indeed, a comparision of the two discussed methods show impressively that the direct method is much more effective than the indirect way of the two steps. The standard errors of the T values reveal to be much more small by the direct method of the first mixed boundary value problem than by the indirect two - step - method following the way via the intermediary stage of the free-air anomalies.

The first mixed boundary value problem diminishes the standard error of T by a factor between 0.5 and 0.1 , as a comparison of the indirect and the direct method does reveal, /7/. This enhanced precision is equivalent to a standard error of about 0.5 to 1 mgal in the free-air anomalies of the oceanic compartments of 200 km square, /9/.

Furthermore, a comparison between the first and the second mixed boundary value problem shows that the gravity deviations δg of the second mixed boundary value problem have a standard error of about $\pm$ 15 mgal on the oceans, /9/. Thus, their standard error is by an order greater than the comparition values of the first mixed boundary value problem, (0.5 to 1 mgal). Therefore, it is questionable whether the 2nd mixed boundary value problem can generally concur with the 1st mixed boundary value problem. Probably, the 2nd

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type will not be able to reach the precision of the 1st type of the mixed boundary value problem.

Furthermore, returning back to the pros and cons of the previously discussed direct method and the two - step - method, the direct way along the lines of the first mixed boundary value problem has the advantage to cancel the instabilities which bring some trouble into the inversion of the Stokes integral and, consequently, into the first step of the two - step - method also, [7]. Further, the integrations over the oceanic T values cover a circle of about 100 km radius only. Thus, they cannot bring to bear the long waves in the results of the satellite altimetry. This is a handicap for the two - step - method.

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D. Downwards continuations and the proof of the convergence of the spherical - harmonic development for a potential in the exterior of a regular surface

Contents

Page

Summary

The spherical - harmonic series development for the gravitational potential of the Earth is considered. The gravitating masses are presupposed to be enclosed by a regular surface of the Earth. The Earth is surrounded by a sphere exterior of which the Laplace differential equation is valid. The exploration of the residual term of this series development is put into the fore. The methods of the harmonic downwards continuation from the exterior of this Brillouin sphere down to the surface of the Earth show that the spherical - harmonic series development is uniform convergent in whole the exterior space of the surface of the Earth and also along this surface. The microstructure in the surface of the real Earth does not paralyse the convergence of this series development in the mass-free exterior space.

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Zusammenfassung

Die Kugelfunktionsentwicklung für das Gravitationspotential der Erde wird untersucht. Die die Erdmassen umschließende Erdoberfläche soll eine reguläre Fläche sein. Es wird eine Kugel eingeführt, die die Erdmassen enthält und in deren Außenraum die Laplace-sche Differentialgleichung gültig ist. Ausgehend vom Außenraum dieser Brillouin-Kugel wird das Restglied der betrachteten Reihenentwicklung ermittelt durch das Verfahren der harmonischen Fortsetzung bis herab zur Erdoberfläche. Es zeigt sich, daß die fragliche Kugelfunktionsentwicklung für das Gravitationspotential im gesamten massenfreien Außenraum der Erdoberfläche gleichmäßig konvergent ist. Die Konvergenz der Kugelfunktionsentwicklung für das Gravitationspotential der wirklichen Erde gilt unbeschadet durch die Existenz einer Mikrostruktur in der Gestalt der Erdoberfläche.

Резюме

Исследуется разложение сферической функции гравитационного потенциала Земли. Предполагается, что земная поверхность всей массы Земли является регулярной поверхностью. Берется содержащий земную массу шар, во внешнем пространстве которого действует дифференциальное уравнение Лапласа. Исходя из внешнего пространства этого шара, определяется остаточный член рассматриваемого разложения в ряд посредством метода гармонического продолжения вплоть до поверхности Земли. Установлено, что спорное разложение сферической функции для гравитационного потенциала Земли во всем внешнем пространстве Земли является равномерно конвергентным.

1. Introduction

The gravitational potential of the Earth W has the following well-known integral expression, [10][12][15],

$$W = f \iiint\limits_{\Phi} \frac{\varrho \, d\Phi}{l} \quad . \tag{1}$$

f is the gravitational constant, ϱ is the density of the masses of the Earth, Φ is the volume of the masses of the Earth and $d\Phi$ the volume element of it. l is the straight distance between the test point P and the variable integration point Q which does move through whole the volume of the Earth, Φ , carrying out the integration according to (1). Therefore,

DOI: https://doi.org/10.2312/zipe.1986.084

$$l = l\,(P,Q) \quad , \tag{2}$$

$$l = \overline{P,Q} \quad . \tag{3}$$

In spatial spherical coordinates, the straight distance between the two points

$$P\,(r, \vartheta, \lambda)$$

and

$$Q\,(r', \vartheta', \lambda')$$

is described by

$$l^2 = r^2 + r'^2 - 2\,r\,r' \cos\psi \quad . \tag{4}$$

r and r' are the distances from the gravity center of the Earth. ϑ and ϑ' are the polar distances. λ and λ' are the geocentric or the geographical longitudes. ψ is the angle between the radius vectors r and r'.

A series expansion of

$$\frac{1}{l} = \frac{1}{\sqrt{r^2 - 2\,r\,r' \cos\psi + r'^2}} \quad , \tag{5}$$

leads to the following well-known development in Legendre's polynomials $P_n(\cos\psi)$, [10] [12] [15],

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{(r')^n}{r^{n+1}} P_n(\cos\psi) \quad , \quad r' < r \quad . \tag{6}$$

The inequation

$$r' < r \tag{7}$$

is here of special importance. It is sure that the series development (6) is convergent in case the inequation (7) is valid. Further, it is sure that the series development (6) is divergent in case

$$r' > r \quad . \tag{8}$$

Thus, the substitution of $\frac{1}{l}$ in the integral (1) by the series development (6) leads to a uniform convergent series development for W as far as the inequation (7) is fulfilled. Or, speaking with other words, the integral transformation (1) for the convergent series development of $\frac{1}{l}$ as kernel function, ($r' < r$), leads to a convergent series development for W. However, in case of $r' > r$, the integral transformation (1) for the divergent series development of $\frac{1}{l}$ leads to a series development for W the convergence properties of which are not known from the first. It is not sure, from the first, whether this series development for W does converge or diverge in case the inequation (8) is valid. Thus, further special investigations about this problem are necessary in order to find whether the discussed series development for

DOI: https://doi.org/10.2312/zipe.1986.084

W is convergent or divergent in case the inequation (8) is valid. However, a divergent series development for $\frac{1}{l}$ does not lead inevitably to a divergent series development for W, (1). This fact must be mentioned in order to avoid an often found misunderstanding. Along the lines above, and in case the inequation (8) is valid, it is not possible to have an evidence about the convergence of the spherical - harmonic development for W.

After all, the relations (1), (6), (7) and the decomposition formula

$$P_n(\cos\psi) = \frac{1}{2n+1}\sum_{m=0}^{n}\Big[\bar{R}_{n.m}(\vartheta,\lambda)\,\bar{R}_{n.m}(\vartheta',\lambda') + \bar{S}_{n.m}(\vartheta,\lambda)\,\bar{S}_{n.m}(\vartheta',\lambda')\Big] , \tag{9}$$

give the well-known development for W, [10] [12] [15],

$$W = f\,\frac{M}{r}\left\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n \bar{P}_{n.m}(\cos\vartheta)\Big[w_{1.n.m}\cos m\lambda + w_{2.n.m}\sin m\lambda\Big]\right\} , \tag{10}$$

$$r \geqslant R_b \quad . \tag{11}$$

M is the mass of the Earth, R_b is the radius of the Brillouin sphere, $w_{1.n.m}$ and $w_{2.n.m}$ are the Stokes constants. R is the radius of the globe. $\bar{P}_{n.m}(\cos\vartheta)$ are the normalized spherical harmonics,

$$\bar{R}_{n.m}(\vartheta,\lambda) = \bar{P}_{n.m}(\cos\vartheta)\cos m\lambda , \tag{12}$$

$$\bar{S}_{n.m}(\vartheta,\lambda) = \bar{P}_{n.m}(\cos\vartheta)\sin m\lambda , \tag{13}$$

$$\frac{1}{4\pi}\iint_{\Theta^*} \bar{R}^2_{n.m}\, d\Theta^* = 1 , \tag{14}$$

$$\frac{1}{4\pi}\iint_{\Theta^*} \bar{S}^2_{n.m}\, d\Theta^* = 1 . \tag{15}$$

Θ^* symbolizes the surface of the unit sphere,

$$d\Theta^* = \cos\varphi\; d\varphi\; d\lambda , \tag{16}$$

φ is the geocentric latitude.

As is the general rule, the investigation into whether the series develop-

DOI: https://doi.org/10.2312/zipe.1986.084

ment (6) is convergent happens by the methods of the functional analysis for a function of a complex argument. The relation (5) leads to

$$\frac{1}{l} = \frac{1}{r} \frac{1}{\sqrt{1 - 2\alpha\xi + \alpha^2}} , \qquad (17)$$

with

$$\alpha = \frac{r'}{r} , \qquad (18)$$

and

$$\xi = \cos\psi = \frac{1}{2}(e^{i\psi} + e^{-i\psi}) . \qquad (19)$$

In (17), α is originally a real number, (18), But, generalizing the meaning of (17) for the consideration of the convergence, the argument domain of α can be extended to whole the Gaussian complex plane. Thus, 1/l can be considered as a function of the complex argument α . Hence, the following power series development of the analytical function 1/l in terms of rising powers of the complex argument α turns out to be possible,

$$\frac{1}{\sqrt{1 - 2\alpha\xi + \alpha^2}} = \sum_{n=0}^{\infty} P_n(\xi)\, \alpha^n . \qquad (20)$$

The singularities of (20) are found for

$$1 - 2\alpha\xi + \alpha^2 = 0 . \qquad (21)$$

The condition (21) is fulfilled if

$$\alpha = e^{\pm i\psi} . \qquad (22)$$

(22) gives

$$|\alpha| = 1 . \qquad (22a)$$

Therefore, the development (20) is convergent in the Gaussian complex plane within the circle

$$|\alpha| < 1 . \qquad (23)$$

(18) and (23) lead to

$$\alpha = \frac{r'}{r} < 1 . \qquad (24)$$

This inequation is the condition for the convergence of (6) and (20), see (7).

On the other hand,,the potential W has no singularities, (1); especially, it has not the singularities of the complex function (20). The gravitational potential of the Earth is well-known to be a limited and continuous function. This fact is valid for test points in the exterior space of the Earth and even for test points in its interior, within the masses of the Earth, [4][5][12].

DOI: https://doi.org/10.2312/zipe.1986.084

Since the singularities of (20) are so very different from those of the geopotential, it cannot be taken for granted that (10) will have the same convergence properties as (20).

The series development (10) can be written in the following abbreviation form,

$$T = \sum_{n=0}^{\infty} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n (\varphi, \lambda) \quad , \tag{25}$$

$$r \geqslant R_b \quad , \tag{26}$$

$$T = W \quad , \text{ if } \quad r \geqslant R_b \quad . \tag{27}$$

R_b is the radius of the Brillouin sphere, i.e. the smallest geocentric sphere that encloses whole the mass of the Earth.

The essential problem to be discussed here is the question whether the series development (25) is convergent not only in the exterior of the Brillouin sphere, but also beyond of it down to the test points at the surface of the Earth. This is an often discussed subject, /1//2//4//5//6//7//9//11//13//19/.

The objections against the validity of the convergence down to the surface of the Earth turned out to be not convincing, /4//5/.

2. The geodetic aspects of the analytical representation of a potential exterior of a regular surface

In the physical geodesy, according to common use, the gravitational and the gravity potential in the exterior space are represented by certain well-defined analytical standard expressions of different shape.

Molodenskij has preferred the representation by the potential of a surface distribution μ,

$$W = f \iint_{\Psi^*} \frac{\mu}{l} \, d\Psi^* \quad . \tag{28}$$

Ψ^* is the surface of the Earth. Molodenskij has derived an integral equation for the determination of μ in terms of the free-air anomalies.

In the dynamical satellite geodesy, the approximation of the gravitational potential by a truncated spherical - harmonic development is an often used method,

$$W \cong \sum_{n=0}^{A} \bar{w}_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n (\varphi, \lambda) \quad . \tag{29}$$

DOI: https://doi.org/10.2312/zipe.1986.084

The terms with the indices

$$n = A + 1 \; , \; A + 2 \; , \; \ldots \tag{30}$$

are neglected here, they are considered to be more or less within the noise of the method.

Obviously, the expression (29) is a sum and not a series development. Thus, any convergence difficulties do not arise. However, the more rapid the convergence of (25), the smaller the theoretical residual errors of the model potential given by (29). The residual term of (29) is, (25),

$$\delta W = \sum_{n=A+1}^{\infty} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) \; . \tag{31}$$

Different authors have considered certain well-defined models of the potential of the Earth and the concerned series developments of the form (25). For test points at the surface of the Earth, they have computed a great number of terms of this series development for the favoured model potential, [11][19],

$$n = 0,1,2, \; \ldots \; , A \; . \tag{32}$$

The computed amounts of the limbs of these truncated series developments give a valuable insight into the speed of the convergence of (25). However, a rigorous proof of the convergence of (25) can never be obtained by a consideration of the truncated form. But the residual term (31) is here in the fore, it requires the computation of the infinity of model-coefficients w_n, ($n =$ $A+1$, $A+2$, ...), an impossible enterprise.

The convergence of (25) is generally accepted for the exterior space of the Brillouin sphere, $r \geqslant R_b$. Figure 1 shows the situation. Φ is the space filled up by the masses of the Earth. The exterior space of the Earth is denominated by

$$\Psi = \Psi_a + \Psi_b \; . \tag{33}$$

DOI: https://doi.org/10.2312/zipe.1986.084

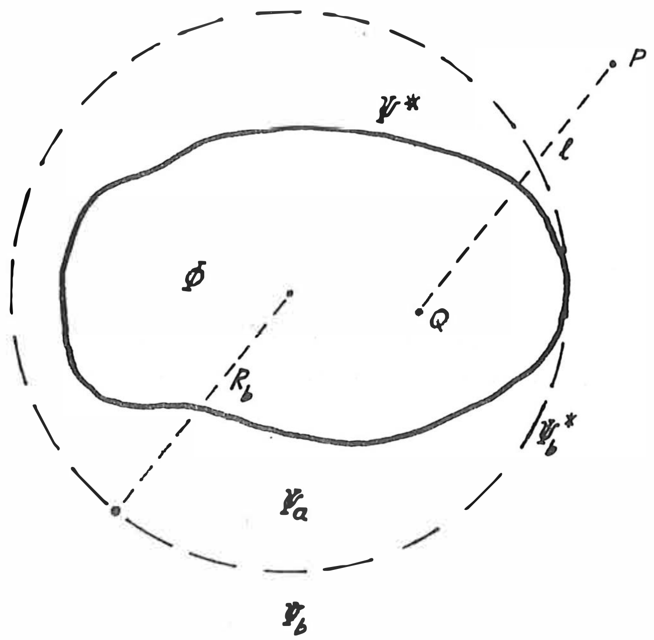

Fig. 1 : The volume of the Earth, Φ , (solid line), The exterior space of the Brillouin sphere, Ψ_b. The surface of the Earth Ψ^* and the surface of the Brillouin sphere, Ψ_b^* . Ψ_a is the space between Ψ^* and Ψ_b^* .

The closed solid line in Fig. 1 is the surface of the Earth, Ψ^*. Ψ^* is taken as a regular surface, it has a continuous radius with continuous first and second derivatives. The sphere with the radius R_b around the gravity center of the Earth is the Brillouin sphere, Ψ_b^* , (25)(26)(27). Ψ_b is the exterior space of the Brillouin sphere. Ψ_a is the space situated between Ψ_b and the surface of the Earth Ψ^*, it is the relevant space where the convergence in question, (25), is to be investigated by the subsequent considerations. Or, with other words, it is to be investigated whether the expression of (25) is identical with the geopotential W for test points situated in Ψ_a , (1). In the figure 1, P is a test point in the space Ψ_b, Q is the moving integration point and l is the straight distance between P and Q, (2)(3).

DOI: https://doi.org/10.2312/zipe.1986.084

The ensuing considerations will prove the validity of the convergence of the spherical - harmonic development for the geopotential down to the surface of the Earth, Ψ^*, (25). This is a problem by itself. Furthermore, the convergence in question of (25) in whole the space Ψ is of use for several geodetic problems. With a convergence extended to whole the space Ψ, this series development (25) is of use for the investigations about the uniqueness and about the solution of the geodetic boundary value problem, to set an example, [3][5].

The development (25) - also after an extension of the validity of it to whole the space Ψ - represents the potential of an Earth with non - time - dependent masses which are enclosed by a regular surface. This model does suffice for many geodetic applications.

3. The partition of the potential into two parts

The geopotential W is well-defined by the relation (1). In the exterior space Ψ, the function of W fulfills the Laplace differential equation,

$$\text{div grad } W = 0 \quad , \tag{34}$$

$$\Delta W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0 \quad . \tag{35}$$

x, y, z are rectangular Cartesian coordinates. W is a continuous function in the exterior space Ψ.

Furthermore, the first and the second and the higher - order derivatives of W are also continuous functions in the space Ψ, [12][15]. To set an example, the potential W has in Ψ the following first derivatives,

$$\frac{\partial W}{\partial x,y,z} = f \iiint_{\Phi} \varrho \frac{\partial \, 1/l}{\partial x,y,z} \, d\Phi \quad . \tag{36}$$

Obviously, the potentials W and T tend to zero if r tends to infinity, as can be taken from (1) and (25); thus,

$$W \longrightarrow 0 \tag{37a}$$

and

$$T \longrightarrow 0 \tag{37b}$$

if

$$r \longrightarrow \infty \quad . \tag{37c}$$

The spherical - harmonic development (25) for the geopotential W in the space Ψ_b has the character of a uniform convergent series development, [10] [12][15].

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Now, the series development for T, (25), is split - up into two parts, into the truncated part which comprises the spherical harmonics

$$\varkappa_n(\varphi,\lambda) \quad , \quad (n = 0,1,2, \ldots, B), \tag{38}$$

and into the residual part. The following expression is obtained, it is valid along the surface of the Brillouin sphere,

$$T = \sum_{n=0}^{B} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) + V_B \, , \; r = R_b \; . \tag{39}$$

Or,

$$T = U_B + V_B \; , \quad r = R_b \; , \tag{40}$$

with

$$U_B = \sum_{n=0}^{B} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) \; . \tag{41}$$

The development (25) for the potential T is known to be uniform convergent for $r = R_b$. Therefore, if r is equal to R_b, it is sure that V_B tends to zero if B tends to infinity:

$$(V_B)_{r=R_b} \longrightarrow 0 \; , \text{ if } \; B \longrightarrow \infty \; . \tag{42}$$

Thus:

Corresponding to any given positive number

$$\varepsilon_1^{\,2} > 0 \; , \tag{43}$$

however small , it shall be possible to find an integer B. such that

$$\left| V_B \right|_{r=R_b} < \varepsilon_1^{\,2} \; . \tag{44}$$

The considerations connected with the relations which cover (39) to (44) must not be restricted to test points situated on the Brillouin sphere, $r = R_b$. The development (25) is also valid for test points above the Brillouin sphere. It is valid for whole the exterior space of the Brillouin sphere, Ψ_b; see fig. 1. Hence,

$$T = \sum_{n=0}^{B} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) + V_B \, , \text{ in } \Psi_b \; . \tag{45}$$

$$V_B = \sum_{n=B+1}^{\infty} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) \, , \text{ in } \Psi_b \; . \tag{45a}$$

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$$U_B = \sum_{n=0}^{B} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi, \lambda) \quad , \quad \text{in } \Psi_b \quad . \tag{45b}$$

The relations (45) to (45b) lead to

$$T = U_B + V_B \quad , \quad \text{in } \Psi_b \quad , \tag{46}$$

or,

$$V_B = T - U_B \quad , \quad \text{in } \Psi_b \quad . \tag{47}$$

On the Brillouin sphere, $|V_B|$ obeys the inequation (44), for a sufficient great integer B.

Now, it is of interest to investigate into whether $|V_B|$ fulfills in Ψ_b an inequation which is similar to the inequation (44) valid for test points on the Brillouin sphere only, $r = R_b$. In this context, the maximum and minimum properties of a harmonic function are of importance, [12][16][17]. Because T and U_B obey the Laplace differential equation, (45)(45b),

$$\Delta T = 0 \quad , \quad \text{in } \Psi_b \quad , \tag{48}$$

and

$$\Delta U_B = 0 \quad , \quad \text{in } \Psi_b \quad , \tag{49}$$

the relation (47) has the consequence that V_B is a harmonic function in Ψ_b ,

$$\Delta V_B = \Delta T - \Delta U_B \quad , \tag{50}$$

$$\Delta V_B = 0 \quad , \quad \text{in } \Psi_b \quad . \tag{51}$$

By (47), the harmonic function V_B is a definitely given expression within the space Ψ_b . For a point at infinity, the relation (52) follows, (37a)(37b)(37c)(47),

$$V_B \longrightarrow 0 \quad , \quad \text{if } r \longrightarrow \infty \quad . \tag{52}$$

A certain well-known theorem of the potential theory states that the maximum and minimum values of a harmonic potential function are always situated on the boundary of the space for which the relevant potential is described, [12][16]. In the here discussed applications, the space Ψ_b is the area for which the potential V_B is described. Thus, the maximum value of $|V_B|$ is situated on the Brillouin sphere Ψ_b^*, $r = R_b$, see Fig. 1. The following self-explanatory inequation is valid,

$$|V_b|_{\Psi_b} < \max \; |V_B|_{\Psi_b^*} \quad . \tag{53}$$

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Because of (44), the inequation (54) follows

$$\max \mid V_B \mid_{\Psi_b^*} < \varepsilon_1^2 \quad . \tag{54}$$

The relations (53) and (54) can be combined to the following inequation,

$$\mid V_B \mid_{\Psi_b} < \varepsilon_1^2 \quad . \tag{55}$$

The inequation (55) has the following meaning, [8][20] :

Corresponding to any given positive number

$$\varepsilon_1^2 > 0 \quad , \tag{56}$$

no matter how small, an integer B can be chosen such that

$$\mid V_B \mid_{\Psi_b} < \varepsilon_1^2 \ , \ \text{in } \Psi_b \ . \tag{57}$$

All the above considerations, especially the representations (1), (41) and (47), lead to the following properties of the threee functions W, U_B and V_B:

In the space Ψ_b, the subsequent relations are valid:

In Ψ_b, the functions of W, U_B and V_B are well-defined harmonic potentials, they have well-explained expressions, (25)(27)(45)(45a)(45b). Furthermore, they have continuous expressions and continuous derivations, (36). They fulfill the Laplace differential equation,

$$\Delta W = \Delta U_B = \Delta V_B = 0 \ , \ \text{in } \Psi_b \ . \tag{58}$$

If r tends to the distance to the point at infinity

$$r \rightarrow \infty \quad , \tag{59}$$

in this case, the subsequent relations follow,

$$W \rightarrow 0 \ , \tag{60}$$

$$U_B \rightarrow 0 \ , \tag{61}$$

$$V_B \rightarrow 0 \ . \tag{62}$$

In the space Ψ_a, the subsequent relations are valid:

W and U_B are well-defined harmonic potential functions, (1)(45b). Further, they have continuous expressions and continuous derivatives. They fulfill the Laplace differential equation,

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$$\Delta W = \Delta U_B = 0 \text{ , in } \Psi_a \text{ .} \tag{63}$$

As to V_B in Ψ_a, it is true that V_B has not in advance such a series development in Ψ_a as it is provided for V_B in Ψ_b by (45a), (56) and (57). The meaning of V_B in Ψ_a is in advance well-explained by the difference of W and U_B, defined according to (1) and (45b),

$$V_B = W - U_B \text{ , in } \Psi_a \text{ .} \tag{64}$$

The relations (63) and (64) have the consequence

$$\Delta V_B = 0 \text{ , in } \Psi_a \text{ .} \tag{64a}$$

An analytical series development for V_B in Ψ_a will be obtained later, it will not be an expression in terms of the parameters of W and U_B, (1)(45b). This series will be fundamental for the proof of the convergence.

On the other hand, it is not sure in advance whether the potential function V_B can be described in Ψ_a by the expression (45a) extending the argument domain of the coordinates of the test point (r, φ, λ) in (45a) over Ψ_a and Ψ_b.

Or, to be more precise, the main aim of the following deductions is to find out whether the expression (64) of V_B fulfills an inequation of the shape (57) also for the area of Ψ_a.

These ensuing investigations make use of the fact that V_B ia a harmonic potential function, (58)(64a). Further, it is important here that the potential V_B fulfills the relations (56) and (57) for test points in the space Ψ_b. These start conditions fulfilled by V_B, (58)(64a)(56)(57), will allow to evaluate the amount of V_B in Ψ_a.

Thus, the very point of the subsequent investigations is to show even for test points in the space Ψ_a that the subsequent sentence is right:

The amount $|V_B|_{\Psi_a}$ is smaller than a given positive number ε_2^2, no matter how small, if the integer B is chosen sufficient great.

Or, with other words, the open question is the fact that it is to find out whether

$$B \longrightarrow \infty$$

leads to

$$|V_B|_{\Psi_a} < \varepsilon_2^2 \text{ .} \tag{64b}$$

Hence, if (64b) should be right, it should be possible to neglect the potential V_B even for test points situated in Ψ_a, in case the integer B is

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sufficient great. This circumstance - if it will be proved - will allow to replace W by U_B in Ψ_a with arbitrary precision.Therefore, extending the argument domain in the expression for V_B over $\Psi_a + \Psi_b$, it is to prove that

$$W = U_B \pm \varepsilon_2^2 \quad , \text{ in } \Psi_a \ , \tag{65}$$

$$U_B = \sum_{n=0}^{B} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) \ . \tag{65a}$$

B is in (65)(65a) a sufficient great integer in order to have ε_2^2 as a suf- ficient small number, (64b).

This relation, (65), is equivalent to the representation of W in Ψ by means of the following uniform convergent series development,

$$W = \sum_{n=0}^{\infty} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) \ , \text{ in } \Psi \ . \tag{66}$$

The ensuing lines intend to prove the validity of (66).

4. The downwards continuation procedure and the uniform convergence of the series development

The theorem about the harmonic continuation of a harmonic potential has the following shape, [12][16][17].

Theorem 1:

If Γ_1 and Γ_2 are two domains with common points, and if Y_1 is a harmonic potential in Γ_1 and Y_2 in Γ_2, these functions coinciding at the common points of Γ_1 and Γ_2, then they define a single function, harmonic in the domain Γ consisting of all points of Γ_1 and Γ_2. This harmonic continuation is a unique procedure.

This theorem 1 above has the following corollary.

Theorem 2:

If Y is a harmonic function in a domain Γ, and if Y vanishes at all the points of a domain Γ' in Γ, then Y vanishes at all the points of Γ .

The proof of the last theorem 2 can be derived in the following way, it is in close connection to the here discussed applications:

The potential Y is harmonic in the exterior space $\Psi = \Psi_a + \Psi_b$ of the

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surface of the Earth Ψ^*, Fig. 2,

$$\Delta Y = 0 \text{ , in } \Psi \text{ .} \tag{67}$$

In Ψ_b, the potential Y vanishes,

$$Y = 0 \text{ , in } \Psi_b \text{ .} \tag{68}$$

After these start conditions, (67) and (68), the function Y turns out to be

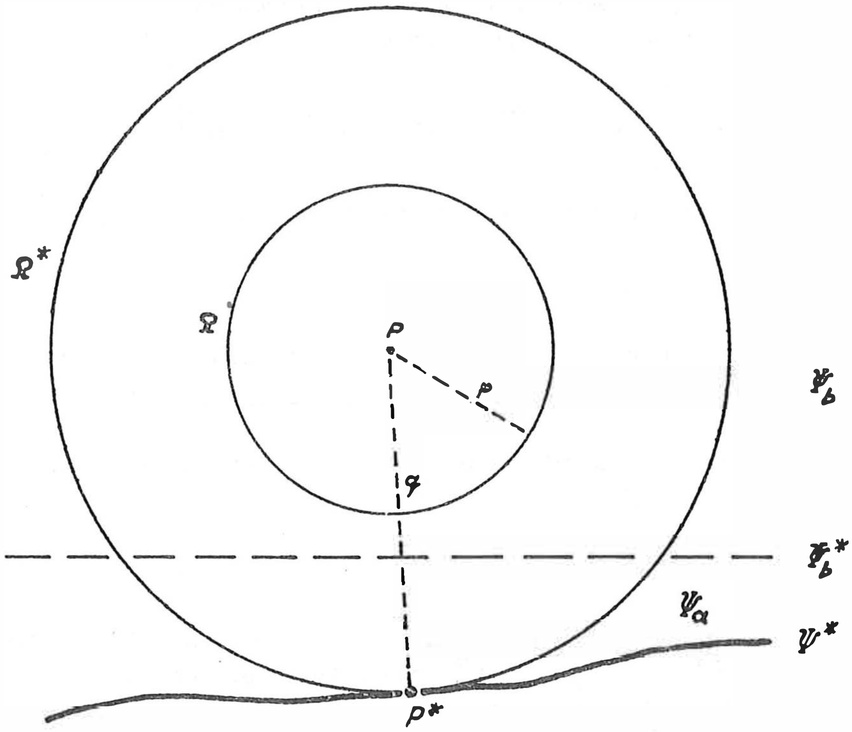

Fig. 2. The harmonic downwards continuation of a harmonic potential from the exterior Ψ_b of the Brillouin sphere Ψ_b^* down to the surface of the Earth Ψ^* and to the point P^* on Ψ^* by means of analytical developments within the sphere Ω^*. Ω^* is the surface of the sphere which has the radius q.

equal to zero also in the space Ψ_a and on the surface of the Earth Ψ^*.

In order to reach this aim, the point P in Ψ_b is surrounded by a concen-

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tric sphere with the radius p, Fig. 2. This sphere is situated completely within the space Ψ_b. Furthermore, the point P is surrounded by a second sphere with the radius q, $q > p$. This second sphere does touch the surface of the Earth Ψ^* in the point P^*. Because of (67), the potential Y has the following convergent series development within the sphere with the radius q, [10][12],

$$Y = \sum_{i=0}^{\infty} y_i \left(\frac{t}{q}\right)^i \mathcal{H}_i(\bar{\varphi}, \bar{\lambda}) \quad , \tag{69}$$

$$0 \leqq t \leqq q \quad . \tag{70}$$

The expression (69) is well-known and often proved and often applied in routine work. y_i are the Stokes constants. t is the variable radius with the point P as center, (70), Fig. 2. Some authors prefer the relation

$$0 \leqq t < q \quad , \tag{70a}$$

instead of (70). However, the interval (70) is absolutely right. A discussion about the question whether (70) or (70a) is to be applied will have no meaning in the here discussed applications because the potential Y is a continuous function. For the argument value

$$t = p < q \quad , \tag{71}$$

the expression for Y turns to, (69),

$$Y_p = \sum_{i=0}^{\infty} y_i \left(\frac{p}{q}\right)^i \mathcal{H}_i(\bar{\varphi}, \bar{\lambda}) \quad . \tag{72}$$

And for

$$t = q \quad , \tag{73}$$

the ensueing relation is obtained,

$$Y_q = \sum_{i=0}^{\infty} y_i \mathcal{H}_i(\bar{\varphi}, \bar{\lambda}) \quad . \tag{74}$$

The condition (68) leads to

$$Y_p = 0 \quad , \tag{75}$$

and furthermore, because the spherical harmonics $\mathcal{H}_i$ are orthonormalized functions,

$$y_i \left(\frac{p}{q}\right)^i = 0 \;, \; (i = 0,1,2, \ldots) \quad . \tag{76}$$

Because of (77), as it is self-explanatory,

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$$\left(\frac{p}{q}\right)^i \neq 0 \quad , \quad (i = 0,1,2, \ldots) \quad , \qquad (77)$$

(78) follows by (76)

$$y_i = 0 \ , \ (i = 0,1,2,3, \ldots) \ . \qquad (78)$$

The equations (74) and (78) have the consequence that the potential Y is equal to zero along the sphere with the radius t = q,

$$Y_q = 0 \quad , \qquad (79)$$

and further, as it is self-explanatory,

$$Y = 0 \quad , \quad (0 \leq t \leq q) \quad . \qquad (80)$$

Obviously, the potential Y has the properties described by (67) and (68), and it obeys the constraint

$$Y = 0 \quad , \quad \text{in} \quad \Psi \quad . \qquad (81)$$

Thus, the relation (81) corroborates the validity of the theorem 2.

Suppose, the point P^* cannot be reached by the surface of the sphere having the radius q and the center P, Fig. 2, in this case, another starting point P can be chosen as the center of the concerned alternative sphere. P must be situated further on within the space Ψ_b. Thus, the point P^* can be reached successfully from this new center P and by this new sphere Ω^*. Perhaps, this procedure does work by one step. Otherwise, a two - step or a multi - step procedure will provide the possibility to reach the point P^* starting from the center P. It means, that P and P^* can be connected in any case by a chain of spheres, the center of the first sphere lies in Ψ_b and the surface of the last sphere does touch the point P^*. The proof of the validity of this two - step or multi - step method happens similar as the proof of the one - step method along the lines of the deductions which cover (69) to (81). It is self-explanatory, it is in this case a two - step or a multi - step proof.

Now, returning back to the demonstration of the convergence of the spherical - harmonic series development in the space Ψ_a, (66), it is necessary to put the properties of the potential V_B, into the center of the considerations.

The equation (58) and (64a) give

$$\Delta V_B = 0 \quad , \quad \text{in} \ \Psi . \qquad (82)$$

The relations (45a), (56) and (57) show that

$$V_B \longrightarrow 0 \ , \ \text{if} \ B \longrightarrow \infty \quad , \ \text{in} \ \Psi_b \ . \qquad (83)$$

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The harmonic downwards continuation of the potential V_B down into the space Ψ_a shows obviously that the following fundamental properties of V_B are also valid, in Ψ_a, (see theorem 2),

$$V_B \longrightarrow 0 \quad , \quad \text{if} \quad B \longrightarrow \infty \quad , \text{in} \quad \Psi_a \quad . \tag{83a}$$

The properties described by (83a) derive along the following lines.

The potential Y_1 in Γ_1 is replaced by V_B in Ψ_b, and the potential Y_2 in Γ_2 is substituted by V_B in Ψ_a, (see theorem 1).

Thus, before the background of (82), the theorem 1 proves that the potential V_B in the "new" space Ψ_a can be determined in terms of the values which this potential V_B has in the "old" space Ψ_b. It happens by means of a unique procedure.

Furthermore, as to the theorem 2, the potential Y can be replaced by V_B and the domains Γ and Γ' by the spaces Ψ and Ψ_b. The potential V_B is the substitute for Y in the equations (67), (68) and (81). The transient behaviour of (83 a) is necessarily the consequence. The relations (83a) are an inevitable consequence of (82) and (83) and of both the theorem 1 and the theorem 2.

In case, the relations (82) and (83) would have other consequences than (83a), then, instead of (83a), a relation of the following type would be the result of the downwards continuation,

$$V_B = \varsigma\,(x,y,z) \quad , \text{in} \quad \Psi_a \quad ; \tag{83b}$$

$$\varsigma^2 > 0 \quad , \text{if} \quad B \longrightarrow \infty \quad , \text{in} \quad \Psi_a \quad . \tag{83c}$$

But, the properties (83b) and (83c) can never be in keeping with (82) and (83) and with the property of V_B to be a continuous function in Ψ, it is evidenced by a look on the above formulated theorem 1 and 2 about the harmonic downwards continuation. These theorems enclose also the uniqueness of the harmonic continuation procedure.

If, (83),

$$V_B \longrightarrow 0 \quad , \text{in} \quad \Psi_b \quad , \tag{83d}$$

the rules of the harmonic downwards continuation demand no other relation than

$$V_B \longrightarrow 0 \quad , \text{in} \quad \Psi_a \quad , \tag{83e}$$

it is valid on the strength of (82). The transition behaviour (83e) proves the validity of (83a). (83e) is in clear contradiction to (83b) and (83c).

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It is self-explanatory, the relations (64b) and (83a) show that the series development (66) is a uniform convergent series development for the geopotential in whole the exterior space Ψ of the Earth, [1].

However, to be more complete and more convincing, and to avoid misunderstandings, it seems to be convenient to add the detailed and explicit description of another proof of the uniform convergence of (66) in Ψ_a.

5. An inequation for the norm of the residual potential

The potential V_B is the second part of the two parts of the geopotential W. V_B is well-defined by (45a), (56), (57), (82) and (83). It is useful to give here a summary of the 4 most important properties of V_B:

1.) The following equation is valid:

$$V_B = \sum_{n=B+1}^{\infty} w_n \left(\frac{R}{r}\right)^{n+1} \mathcal{H}_n(\varphi, \lambda) \quad , \text{ in } \Psi_b \; . \tag{84}$$

2.) The following theorem is valid:
Corresponding to any given positive number

$$\varepsilon_1^2 > 0 \quad , \tag{85}$$

however small, it shall be possible to find an integer B such that

$$| V_B |_{\Psi_b} < \varepsilon_1^2 \quad , \text{ in } \Psi_b \; . \tag{86}$$

3.) The Laplace equation is valid:

$$\Delta V_B = 0 \quad , \text{ in } \Psi . \tag{87}$$

V_B is a regular function in Ψ.

4.) The following transition behaviour is valid:

$$V_B \longrightarrow 0 \quad , \text{ if } B \longrightarrow \infty \quad , \text{ in } \Psi_b \; . \tag{88}$$

Furthermore, continuing the deductions, V_B has the subsequent uniform convergent series development valid within the sphere with the center P and the radius q, Fig. 2, (69), [12][15].

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$$V_B = V_B\,(t, \bar{\varphi}, \bar{\lambda}) = \sum_{i=0}^{\infty} v_{B.i} \left(\frac{t}{q}\right)^i \varkappa_i\,(\bar{\varphi}, \bar{\lambda}) \quad , \quad \text{in } \Omega , \tag{89}$$

$$0 \leq t \leq q \quad . \tag{90}$$

$v_{B.i}$ are the constant coefficient of (89). The expressions $\varkappa_i(\bar{\varphi}, \bar{\lambda})$ symbolize the spherical harmonics. They are now understood to be orthonormalized functions according to

$$\iint_{\Theta^* \;\Theta^*} \varkappa_j \; \varkappa_k \cos\bar{\varphi} \quad d\,\bar{\lambda} \;=\; \left\{\begin{matrix} 1, & j = k \\ 0, & j \neq k \end{matrix}\right\} \quad . \tag{91}$$

Θ^* is the surface of the unit sphere.

For $t = p$, $p \leqslant q$, the equation (89) takes the following form,

$$V_{B.p} = \sum_{i=0}^{\infty} v_{B.i} \left(\frac{p}{q}\right)^i \varkappa_i\,(\bar{\varphi}, \bar{\lambda}) \quad . \tag{92}$$

The **relation** (86) gives

$$| \, V_{B.p} \, | < \varepsilon_1^{\,2} \quad , \tag{93}$$

since the sphere with the radius $t = p$ is totally situated within the space Ψ_b. For $t = q$, (94) follows from (89),

$$V_{B.q} = \sum_{i=0}^{\infty} v_{B.i} \; \varkappa_i\,(\bar{\varphi}, \bar{\lambda}) \quad . \tag{94}$$

$V_{B.p}$ and $V_{B.q}$ are bounded, continuous and regular functions of $\bar{\varphi}$ and $\bar{\lambda}$, because W and U_B are bounded, continuous and regular functions, and because V_B depends on W and U_B by, (see (1)(45b)),

$$V_B = W - U_B \quad , \text{in } \Psi . \tag{95}$$

The series developments (92) and (94) are well-known to be uniform convergent, it can be found in the **textbooks**, [10][12][15].

Besides of (93), the following relations are valid for (92), since (92) is well-known to be a uniform convergent series development.

Theorem:

Corresponding to any given positive number

$$\varepsilon_3^{\,2} > 0 \quad , \tag{96}$$

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no matter how small, it shall be possible to find an integer C such that, (92),

$$\left| \sum_{i=C+1}^{\infty} v_{B.i} \left(\frac{p}{q}\right)^{i} \varkappa_i (\bar{\varphi}, \bar{\lambda}) \right| < \varepsilon_3^2 \quad . \tag{97}$$

Furthermore, the well-known uniform convergence of (94) leads to the ensuing sentence about the residual term of it.

Theorem:

Corresponding to any positive number

$$\varepsilon_4^2 > 0 \quad , \tag{98}$$

however small, an integer D can be chosen such that

$$\left| \sum_{i=D+1}^{\infty} v_{B.i} \varkappa_i (\bar{\varphi}, \bar{\lambda}) \right| < \varepsilon_4^2 \quad . \tag{99}$$

Now, the norms of the functions $V_{B.p}$ and $V_{B.q}$ are to be derived by means of the series developments (92) and (94). The norm of the function $V_{B.p}$ is defined by the following integral,

$$\| V_{B.p} \|^2 = \iint_{\Theta^*} V_{B.p}^2 \cos \bar{\varphi} \; d\bar{\varphi} \; d\bar{\lambda} \quad . \tag{100}$$

Θ^* symbolizes the surface of the unit sphere which has an amount of 4π. The expression $V_{B.p}$ in the integral of (100) is substituted by the series development (92). Thereafter, the orthogonality relations (91) are introduced and the ensuing series development for the norm of $V_{B.p}$ is obtained,

$$\| V_{B.p} \|^2 = \sum_{i=0}^{\infty} v_{B.i}^2 \left(\frac{p}{q}\right)^{2i} \quad . \tag{101}$$

The relations (93) and (100) give

$$\| V_{B.p} \|^2 < 4\pi \cdot \varepsilon_1^4 \quad . \tag{102}$$

with

$$4\pi \cdot \varepsilon_1^4 = \varepsilon_5^2 \quad , \tag{103}$$

the equation (102) turns to

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$$\| V_{B.p} \|^2 = \sum_{i=0}^{\infty} v_{B.i}{}^2 \left(\frac{p}{q}\right)^{2i} < \varepsilon_5{}^2 \quad . \tag{104}$$

Thus, according to (56) and (57), the following statement is valid.

Theorem:

Corresponding to any given positive number

$$\varepsilon_5{}^2 > 0 \quad , \tag{105}$$

however small, it shall be possible to find an integer B such that

$$\| V_{B.p} \|^2 < \varepsilon_5{}^2 \quad . \tag{106}$$

For the norm of $V_{B.q}$, the subsequent relations derive in a similar way,

$$\| V_{B.q} \|^2 = \frac{1}{q^2} \iint_{\Omega^*} V_{B.p}{}^2 \, d\Omega^* = \sum_{i=0}^{\infty} v_{B.i}{}^2 \quad . \tag{107}$$

$$d\,\Omega^* = q^2 \cos\bar{\varphi} \; d\bar{\varphi} \; d\bar{\lambda} = q^2 \; d\Theta^* \quad . \tag{107a}$$

The above discussed norm of $V_{B.q}$, (107), is a crucial value for the following considerations. It is of cardinal importance.

The open question is whether the norm of $V_{B.q}$ does fulfill an inequation of the shape of (106), a relation which is satisfied by the other already discussed norm $\| V_{B.p} \|^2$. The norm of $V_{B.p}$ is already known to fulfill the regulations (105) and (106). But now, even the norm of $V_{B.q}$ undergoes an investigation into whether it does obey certain constraints analogous to the inequations (105) and (106) for $V_{B.p}$. In this context, it is convenient to denominate the norm of $V_{B.q}$ by the abbreviation β^2. Thus,

$$\| V_{B.q} \|^2 = \sum_{i=0}^{\infty} v_{B.i}{}^2 = \beta^2 \quad . \tag{108}$$

The following question arives:

Will it be possible and right that β tends to zero in case the integer B tends to infinity? Or, with other words, will it be possible to corroborate the following theorem ?

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Theorem :

Corresponding to any positive number

$$\varepsilon_6^2 > 0 \quad , \tag{109}$$

however small, it shall be possible to find an integer B such that

$$\beta^2 < \varepsilon_6^2 \quad . \tag{110}$$

In this context, a certain theorem from the theory of infinite series developments is of importance, [14].

Theorem 3:

A series development with positive terms is uniform convergent if the partial sums of it are limited. This is a necessary and sufficient condition.

In the above considerations, the series (101) is a series development with positive terms. Further, this series has a limited value which is equal to the limited term $\| v_{B.p} \|^2$. Thus, it is obvious, the partial sums of (101) are necessarily limited values also. They are situated between zero and $\| v_{B.p} \|^2$. Therefore, the theorem 3 proves that the right hand side of (101) is a uniform convergent series development. Hence,

$$\| v_{B.p} \|^2 = \sum_{i=0}^{E} v_{B.i}^2 \left(\frac{p}{q}\right)^{2i} + \sum_{i=E+1}^{\infty} v_{B.i}^2 \left(\frac{p}{q}\right)^{2i} \quad . \tag{111}$$

With the abbreviations

$$\eta_{5.1}^2 = \sum_{i=0}^{E} v_{B.i}^2 \left(\frac{p}{q}\right)^{2i} \quad , \tag{112}$$

and

$$\eta_{5.2}^2 = \sum_{i=E+1}^{\infty} v_{B.i}^2 \left(\frac{p}{q}\right)^{2i} \quad , \tag{113}$$

follows, (104),

$$\| v_{B.p} \|^2 = \eta_{5.1}^2 + \eta_{5.2}^2 < \varepsilon_5^2 \quad . \tag{114}$$

Similar relations are valid for $V_{B.q}$ and the norm of it. Because $V_{B.q}$ is a limited and regular function, the theorem 3 proves that the expression

$$\sum_{i=0}^{\infty} v_{B.i}^2$$

on the right hand side of (108) is a uniform convergent series development

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which has positive terms only,

$$\| v_{B.q} \|^2 = \beta^2 = \sum_{i=0}^{E} v_{B.i}^{\;2} + \sum_{i=E+1}^{\infty} v_{B.i}^{\;2} \quad . \tag{115}$$

With the abbreviations

$$\tau_{6.1}^{\;2} = \sum_{i=0}^{E} v_{B.i}^{\;2} \quad , \tag{116}$$

$$\tau_{6.2}^{\;2} = \sum_{i=E+1}^{\infty} v_{B.i}^{\;2} \quad , \tag{117}$$

the following statement is obtained.

Theorem:

Corresponding to any given positive number,

$$\varepsilon_7^{\;2} > 0 \quad , \tag{118}$$

no matter how small, it shall be possible to find an integer E such that

$$\tau_{6.2}^{\;2} < \varepsilon_7^{\;2} \quad . \tag{119}$$

Returning back to the investigations about the amount of β^2, the relations (71), (112), (113), (114) lead to

$$\varepsilon_5^{\;2} > \eta_{5.1}^{\;2} = \sum_{i=0}^{E} v_{B.i}^{\;2} \left(\frac{p}{q}\right)^{2i} \geqslant \sum_{i=0}^{E} v_{B.i}^{\;2} \left(\frac{p}{q}\right)^{2E} =$$

$$= \left(\frac{p}{q}\right)^{2E} \sum_{i=0}^{E} v_{B.i}^{\;2} \quad . \tag{120}$$

The developments (120) make use of the following relations,

$$E \geqslant i \quad , \tag{120a}$$

$$\frac{p}{q} \leqslant 1 \quad , \tag{120b}$$

$$\left(\frac{p}{q}\right)^{2E} \leqslant \left(\frac{p}{q}\right)^{2i} \quad . \tag{120c}$$

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Thus,

$$\sum_{i=0}^{E} v_{B.i}^{2} < \left(\frac{q}{p}\right)^{2E} \varepsilon_5^{2} . \tag{121}$$

The relations (115) and (121) reveal,

$$\sum_{i=0}^{E} v_{B.i}^{2} = \beta^2 - \tau_{6.2}^{2} < \left(\frac{q}{p}\right)^{2E} \varepsilon_5^{2} . \tag{122}$$

Obviously, the inequation (122) will entail the inequation (123) as consequence,

$$\beta^2 < \left\{\left(\frac{q}{p}\right)^{2E} \varepsilon_5^{2} + \tau_{6.2}^{2}\right\} . \tag{123}$$

This is the inequation for the amount of β^2, (108), which was intended to reach.

The second term $\tau_{6.2}^{2}$ on the right hand side of (123) obeys the regulations (105), (106), (118) and (119).

Sure, the harmonic downwards continuations have to meet all standards and they have to be quite up to the mark, a matter that reflects in the amount of $\tau_{6.2}^{2}$, (117). The introduction of any approximation is not allowed in the downwards continuations. The harmonic downwards continuation procedure has to be carried out by rigorous methods. At least, these approximations have to be tolerated as the upper bound of the theoretical errors of the downwards continuation procedure.

Thus, $\tau_{6.2}^{2}$ does vanish. A rigorous downwards continuation procedure enforces the amount of $\tau_{6.2}^{2}$ to tend to zero. $\tau_{6.2}^{2}$ is the residual term of the uniform convergent series development for β^2, (115)(117). Thus, it can be taken for granted, the integer E has such a great value that $\tau_{6.2}^{2}$ takes a sufficient small amount, in accordance with (118) and (119). After this condition is fulfilled, the amount of E is considered to be fixed in the further deliberations about (123).

As to the first term on the right hand side of the inequation (123), the integer B is understood to have to reach such a great value that the number ε_5^{2} is depressed down in order to reach a sufficient small value for it. This aim can be reached without any problem, it is in keeping with (101)(104)(105)(106). The terms $v_{B.i}$ depend on B, see (42)(101). They tend to zero, since $V_{B.p}$ tends to zero if B tends to the infinity, see (156c). The three parameters p, q, E in the first term on the right hand side of the inequation (123) have limited values. p, q and E are independent parameters; especially, they do not depend on the amount of B. But, the amount of ε_5^{2} depends on B, (105), (123). Consequently, the expression

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$$\left(\frac{q}{p}\right)^{2E} \qquad (124)$$

has a limited value also. Therefore, the term

$$\left(\frac{q}{p}\right)^{2E} \varepsilon_5^{\,2} \qquad (125)$$

can be considered as an arbitrary small value also, in the same way as $\varepsilon_5^{\,2}$. If B tends to the infinity, in this case, the expression (124) does not change its own value, but $\varepsilon_5^{\,2}$ does tend to zero. Thus, the expression (125) tends to zero if B tends to the infinity.

$$\left(\frac{q}{p}\right)^{2E} \varepsilon_5^{\,2} \longrightarrow 0 \quad , \text{ if } \quad B \longrightarrow \infty \quad . \qquad (125a)$$

The product of a bounded value, as (124), and an arbitrary small value, as $\varepsilon_5^{\,2}$, is obviously again equivalent to an arbitrary small amount.

Thus, since B and E are introduced as sufficient great integers in (123), the amount of β^2 can be depressed down to an arbitrary small value by the transition behaviour connected with the procedure: B $\longrightarrow \infty$. Therefore,

$$\| v_{B.q} \|^2 = \beta^2 = \beta^2 (B) < \varepsilon_8^{\,2} \quad , \qquad (126)$$

for a sufficient great value of B.

Theorem:

Corresponding to any given positive number,

$$\varepsilon_8^{\,2} > 0 \quad , \qquad (126a)$$

no matter how small, it shall be possible to find an integer B such that (126) is valid.

Furthermore, the following by-product is found. The greater the integer B the smaller the coefficients $v_{B.i}$, (84)(88)(101)(107). Consequently, the smaller the $v_{B.i}$ values, the smaller the amount of $\tau_{6.2}^{\,2}$, (117). Thus, the amount of $\tau_{6.2}^{\,2}$ will diminish with rising values of B. It will entail the possibility to diminish the amount of E, (115).

Finally, the relations (108) and (126) yield the following inequation

$$\| v_{B.q} \|^2 < \varepsilon_8^{\,2} \quad . \qquad (127)$$

The residual term V_B of the series (66) along the surface of the sphere Ω^* with the radius q is represented by $V_{B.q}$, (94)(64)(65)(65a), Fig. 2. The norm

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of this residual term along the above described sphere Ω^* is arbitrary small if B tends to the infinity, (126)(126a)(127).

Theorem:

Corresponding to any given positive number $\varepsilon_8{}^2$, (126a),

$$\varepsilon_8{}^2 > 0 \quad ,$$

no matter how small, it shall be possible to find an integer B such that the inequation (127) is right. Within the scope of this theorem, the integer E has a fixed value which is in keeping with the relations (118)(119).

6. A first proof of the uniform convergence of the spherical-harmonic series development of the gravitational potential in the exterior of the Earth

The relation (127) is fundamental for the ensuing considerations. It shows that the expression (95),

$$W = U_B + V_B \quad , \text{ along } \Omega^* , \tag{128}$$

with the sum for U_B

$$U_B = \sum_{n=0}^{B} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) \text{ , along } \Omega^*, \tag{129}$$

and with the residual term V_B according to (89) does converge in the mean along the surface of the sphere Ω^*, if $B \longrightarrow \infty$, Fig. 2.

The function V_B is continuous along the surface of the sphere Ω. The function V_B has continuous derivatives of the first and higher order within the sphere Ω. Even these properties are valid primarily for W and U_B within the sphere Ω, they are passed on to V_B by the regulation (128).

This property of V_B to be a regular function within Ψ is now combined with the inequation (127) for the norm of it. Along these lines, the sentence will be obtained that the series development (66) for W is uniform convergent along Ω^*. Or, with other words, since W and U_B are continuous functions, in the exterior space, the **relation (127) envolves the fact that the absolute value** of the continuous function $V_{B.q}$ at the point P^* on the surface of the Earth Ψ has the following inequation, Fig. 2, which is to be proved later on,

$$\left| V_{B.q} \right|_{P^*} < \varepsilon_9^2 \quad . \tag{130}$$

Hence, it is now intended to prove the following theorem.

Theorem:

Corresponding to any given positive number

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$$\varepsilon_{9}^{2} > 0 \quad , \tag{131}$$

no matter how small, it shall be possible to find an integer B such that (130) is valid.

Without greater difficulties, the statement connected with the inequations (130) and (131) can be corroborated by some short lines. The integral (100), which defines the norm, leads to, (see Fig. 2, (107)(107a)),

$$\| V_{B.q} \|^{2} = \frac{1}{q} \iint_{\Omega^*} V_{B.q}^{2} \, d\Omega^{*} \quad , \tag{132}$$

$$d\Omega^{*} = q^{2} \cos \bar{\varphi} \; d\bar{\varphi} \; d\bar{\lambda} = q^{2} \; d\Theta^{*} \quad . \tag{133}$$

The Dirichlet integral (132) can be approximated by a sum, it is self-explanatory,

$$\| V_{B.q} \|^{2} = \frac{1}{q^{2}} \sum_{j=0}^{G} (V_{B.q}^{2})_{j} \; (\Delta\Omega^{*})_{j} \quad . \tag{134}$$

Obviously, the parameters G, q and $(\Delta\Omega^{*})_{j}$ have limited and positive amounts, all these three parameters have never vanishing values. Principally, in any case, it is possible to have a division into surface elements of equal size,

$$(\Delta \Omega^{*})_{j} = \Delta \Omega^{*} = \text{constant} \; . \tag{135}$$

$(\Delta\Omega^{*})_{j}$ and $\Delta\Omega^{*}$ will never vanish. The three relations (127), (134) and (135) reveal

$$\varepsilon_{8}^{2} > \frac{1}{q^{2}} \Delta\Omega^{*} \sum_{j=0}^{G} (V_{B.q}^{2})_{j} \quad , \tag{136}$$

(e.g.: q = 20 km, $\Delta\Omega^{*}$ = 1 meter x 1 meter).

The inequation (136) will entail a relation for the absolute value of the potential value $V_{B.q}$ at the point P^{*} on the surface of the Earth Ψ^{*}, Fig.2. It is self-explanatory, a partial sum of positive terms is always smaller than whole the sum of all the positive terms. Therefore, (136),

$$\frac{1}{q^{2}} \Delta\Omega^{*} \sum_{j=0}^{G} (V_{B.q}^{2})_{j} > \frac{1}{q^{2}} \Delta\Omega^{*} (V_{B.q}^{2})_{P^{*}} \quad . \tag{136a}$$

The above written relations (136) and (136a) give

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$$\frac{1}{q^2} \Delta \Omega^* (V_{B.q}{}^2)_{P^*} < \varepsilon_8{}^2 \quad . \tag{137}$$

Consequently, and because $\Delta \Omega^*$ will never vanish,

$$(V_{B.q}{}^2)_{P^*} < \frac{q^2}{\Delta \Omega^*} \varepsilon_8{}^2 \quad . \tag{138}$$

Principally, the **choice** of the position of the point P^* at the surface of the Earth is not subject to a restriction. The point P^* is any arbitrarily chosen point on Ψ^*. Therefore, the inequation (138) can be brought into a more general frame. Because q and $\Delta \Omega^*$ have positive, limited and non-vanishing values, which are independent of B, and because $\varepsilon_8{}^2$ is an arbitrary small value, which tends to zero if B tends to the infinity, — **for these basic** conditions, it can be stated that the following inequation is obviously valid,

$$(V_B{}^2)_{\Psi^*} < \varepsilon_{10}{}^2 \quad . \tag{139}$$

The symbol $(V_B{}^2)_{\Psi^*}$ denotes the value of $V_B{}^2$ at an arbitrarily choosen point P^* on the surface of the Earth Ψ^*. Hence, the absolute amount of V_B at the surface Ψ^* is arbitrary small if B is sufficient great. Or, with other words, the developments on the right hand side of (45b) and of (65a) are of full value a representation of the geopotential W at the surface of the Earth Ψ^* if B is a sufficient great integer.

Theorem:

Corresponding to any given positive number

$$\varepsilon_{10}{}^2 > 0 \quad , \tag{140}$$

however small, it shall be possible to find an integer B such that the inequation (139) is valid, or the inequation

$$\left| (V_B)_{\Psi^*} \right| < \left| \varepsilon_{10} \right| \quad , \tag{140a}$$

for

$$\left| \varepsilon_{10} \right| > 0 \quad . \tag{140b}$$

Summarising, the above relations show that the series development (66) is valid as a uniform convergent series for test points situated at the surface of the Earth Ψ^*, (129), (139) to (140b),

$$W = \sum_{n=0}^{\infty} w_n \left(\frac{R}{\varrho}\right)^{n+1} \varkappa_n(\varphi, \lambda) \quad , \text{ on } \Psi^* . \tag{141}$$

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The function $\varrho = \varrho(\varphi, \lambda)$ is here the height dependent geocentric radius of the surface of the Earth. The function on the right hand side of (141) is an expression in terms of the arguments φ, λ .

Now, the extension of the validity of (141) into the exterior space remains to be considered. A series theorem of Abel allows the extension of the validity of (141) upwards into the space Ψ_a above the surface of the Earth and beyond it into the space Ψ_b, where the validity of it is uncontested. This theorem has the following text, [14].

Theorem 4:

A uniform convergent series development is permitted to be multiplied limb by limb by monotone and limited factors, without any detriment for the uniform convergence. Or, the **series** development

$$\sum_{i=0}^{\infty} a_i \, b_i \tag{142}$$

is uniform convergent if the series

$$\sum_{i=0}^{\infty} a_i \tag{143}$$

is uniform convergent and if the sequence of numbers

$$\{ b_j \} \quad , \tag{144}$$

or

$$b_0, \, b_1, \, b_2, \, b_3, \, \ldots \tag{145}$$

is monotone decreasing and limited to the left and to the right,

$$b_{i+1} < b_i \quad , \tag{146}$$

$$K_u < b_i \leqslant K_o \quad , \tag{147}$$

$$i = 0,1,2, \, \ldots \quad . \tag{147a}$$

K_u is the lower bound and K_o is the upper bound.

In the here discussed applications, the following substitutes are introduced, (141)(142)(143),

$$a_i = w_i \left(\frac{R}{\varrho}\right)^{i+1} \varkappa_i(\varphi, \lambda) \quad , \tag{148}$$

$$b_i = \left[\frac{\varrho}{r}\right]^{i+1} \quad , \; (i = 0,1,2, \, \ldots) \quad , \tag{149}$$

$$\frac{\varrho}{r} \leqslant 1 \quad , \tag{150}$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$0 < b_i \leqslant 1 \quad , \tag{151}$$

$$K_u = 0 \quad , \quad K_o = 1 \quad . \tag{152}$$

(144) is here a null sequence, (149)(150). These substitutions, (148) to (152), are applied to (142). Without any complication, it follows that the series development

$$W = \sum_{n=0}^{\infty} w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) \quad , \text{ in } \Psi \ , \tag{153}$$

is uniform convergent on the surface of the Earth and in whole the exterior space of it.

The step from the convergence at the surface Ψ^* to that in the space Ψ can be achieved also by the convergence theorem of Weierstrass (Harnack's first theorem of convergence), [12]. This step leads from (141) to (153):

An infinite sequence $\{\sigma_n\}$ of certain regular and harmonic functions is given in the space Λ and on the surface of it, Λ^*. This sequence takes on Λ^* the character of a uniform convergent series development which determines the boundary values of a potential L . If these above given conditions are fulfilled, this sequence $\{\sigma_n\}$ describes the potential L in the space Λ by a uniform convergent series development.

As to the here discussed example, the development (141) represents the convergent series of the boundary values on the surface. And, (153) is the convergent spatial series of the potential, it is a consequence of (141) and of the Weierstrass theorem. Therefore, the Weierstrass sentence corroborates the validity of (153); the limbs of the sequence are equal to

$$\sigma_n = w_n \left(\frac{R}{r}\right)^{n+1} \varkappa_n(\varphi,\lambda) \quad .$$

7. A second proof of the uniform convergence of the spherical-harmonic series development of the gravitational potential in the exterior of the Earth

After all these derivations in the last chapter, it seems to be convenient to have a further independent proof of the series convergence, following another way which is shorter and free of the roundabout way via the norms. The essence of this second proof of the validity of (66) can be shown by the following deliberations.

The residual term of (92) is smaller than ε_3^2 , (97), it can be neglected for a sufficient great integer C. Thus, the residual term (97) of the development (92) has an insignificant amount, if C is sufficient great. The absolute value of the expression $V_{B.p}$ can be considered to be smaller than the arbitra-

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ry small value $\varepsilon_1^{\,2}$, (93). Further, the residual term of $V_{B.p}$, (92), can be treated as an arbitrary small value, (see the lines above). The truncated expression of (92) is introduced,

$$\left| \sum_{i=0}^{C} v_{B.i} \left(\frac{p}{q}\right)^{i} \varkappa_i (\bar{\varphi}, \bar{\lambda}) \right| = N^2 \quad . \tag{154}$$

Obviously, it follows that N^2 must necessarily be arbitrary small if B is sufficient great. N^2 deviates from $|V_{B.p}|$ by an amount of smaller than $\pm\ \varepsilon_3^{\,2}$, (97). Thus, N^2 deviates from an amount of smaller than $\varepsilon_1^{\,2}$ by an amount that is smaller than $\pm\ \varepsilon_3^{\,2}$, (93).

The substitutions, (92), $a + b = V_{B.p}$, and, (154), $|a| = N^2$, and further on, (97), $|b| < \varepsilon_3^{\,2}$, lead to, (93), $|a + b| < \varepsilon_1^{\,2}$, and $|a| < \varepsilon_{11}^{\,2} < (\varepsilon_1^{\,2} + \varepsilon_3^{\,2})$.

Hence, it is self-explanatory, N^2 is smaller than an arbitrary small value, if B is sufficient great,

$$N^2 < \varepsilon_{11}^{\,2} \quad . \tag{154a}$$

Theorem:

Corresponding to any positive number

$$\varepsilon_{11}^{\,2} > 0 \quad , \tag{154b}$$

however small, an integer B can be chosen such that the inequation (154a) is valid.

The left hand side of (154) is a sum which is linear in the $v_{B.i}$ values. The number of the terms of this sum is equal to C + 1. The expressions

$$v_{B.i} \left(\frac{p}{q}\right)^{i} \quad , (i = 0, 1, 2, \ldots , C) \quad , \tag{155}$$

are the coefficients of the spherical-harmonic series development of $V_{B.p}$ which is valid along the sphere with the radius p, (92). According to the definition of $V_{B.p}$, and because the series of the form (66) for W does converge in the exterior of the Brillouin sphere, it is clear that if B tends to infinity, in this case, the expression $V_{B.p}$ follows to have to tend to zero at all points of this sphere with the radius p. Consequently, the $v_{B.i}$ values will have to tend to zero also, simultaneously with $V_{B.p}$,

$$v_{B.i} \longrightarrow 0 \text{ , if } B \longrightarrow \infty \quad , \tag{156}$$

$$i = 0, 1, 2, \ldots \quad . \tag{156a}$$

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All the individual terms of (92) have to go to zero if $V_{B.p}$ tends to zero.

Of course, the relations (91) and (92) give self-explanatory

$$\iint_{\Theta^*} V_{B.p}\, \varkappa_i\, (\bar{\varphi}, \bar{\lambda}) \cos \bar{\varphi}\, d\bar{\varphi}\, d\bar{\lambda} = v_{B.i} \left(\frac{p}{q}\right)^i \iint_{\Theta^*} \varkappa_i^2 \cos \bar{\varphi}\, d\bar{\varphi}\, d\bar{\lambda} =$$

$$= v_{B.i} \left(\frac{p}{q}\right)^i \quad . \tag{156b}$$

Hence,

$$v_{B.i} = \left(\frac{q}{p}\right)^i \iint_{\Theta^*} V_{B.p}\, \varkappa_i\, (\bar{\varphi}, \bar{\lambda}) \cos \bar{\varphi}\, d\bar{\varphi}\, d\bar{\lambda} \quad , \tag{156c}$$

$$i = 0,1,2,3, \ldots \quad . \tag{156d}$$

The relation (156c) is valid for every integer of the sequence (156d). Thus, it is valid also for, (154),

$$i = 0,1,2, \ldots , C \quad , \tag{156e}$$

and, (99), (157),

$$i = 0,1,2, \ldots , D \quad . \tag{156f}$$

The amount of $V_{B.p}$ tends to zero if B tends to infinity. Hence, the equation (156c) does lead to (156), (see also (42)(44)(55)(56)(57)).

Now, the expression (94) for $V_{B.p}$ is to be considered. The absolute value of the residual term of this expression is smaller than ε_4^2 , a value which is arbitrary small, (99). The equation (94) takes the following shape,

$$\left| V_{B.q} - \sum_{i=0}^{D} v_{B.i}\, \varkappa_i\, (\bar{\varphi}, \bar{\lambda}) \right| < \varepsilon_4^2 \quad . \tag{157}$$

The second term on the left hand side of (157) can be interpreted as a sum representing $V_{B.q}$. It is linear in the $v_{B.i}$ values. It is not an infinite series development.

Now, in the equation (157), the integer B is considered to become greater and greater. It does tend to infinity, (see (156)). If B tends to infinity, the transition behaviour of the absolute value of $V_{B.q}$ will demonstrate that this value will tend to zero. Or, to be more precise, if B tends to infinity, $|V_{B.q}|$ will prove to tend to a value that is smaller than ε_4^2 . ε_4^2 is an arbitrary small value, (157). This transition behaviour is described by

$$V_{B.q} \longrightarrow 0 \quad , \text{ if } B \longrightarrow \infty \quad . \tag{158}$$

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The proof of (158) is uncomplicated: A sum of a fixed and limited number of terms does vanish if the individual terms of this sum tend to zero. The second term on the left hand side of (157) is such a sum if B tends to infinity, (see (156)). Further, the left hand side of (154) is also such a sum. The above lines are self-explanatory.

Summarizing:

If B tends to infinity, the second term on the left hand side of (157) tends to zero, (156). Thus, obviously, the inequation (157) cannot be fulfilled if B tends to infinity, unless $V_{B.q}$ tends to zero simultaneously. Therefore, (158) is right.

The relation (158) has certain consequences. Obviously, the following transition behaviour is valid,

$$\left| V_{B.q} \right|_{P^*} \longrightarrow 0 \quad \text{, if B is tending to the infinity} \qquad (159)$$

and, because P^* is any arbitrary point on the surface Ψ^*, further,

$$\left| V_{B.q} \right|_{\Psi^*} \longrightarrow 0 \quad \text{, if B is tending to the infinity.} \qquad (160)$$

The relation (160) corroborates the uniform convergence of the spherical-harmonic series development of the potential W along the regular surface Ψ^*, (141).

The relations (158)(159)(160) corroborate also the validity of the theorem (131)(130).

The validity of the crucial relation (153) is found again by (160). And, the combination of it with Abel's test for the convergence, (see theorem 4), or with the theorem of Weierstrass, (Harnack's first theorem of convergence), [12], leads to the convergence in Ψ.

The uniform convergence according to (160) can be obtained also applying the Schwarz inequality,

$$\underline{\underline{a}} \cdot \underline{\underline{b}} \leqslant \left| \underline{\underline{a}} \right| \cdot \left| \underline{\underline{b}} \right| \quad , \qquad (160a)$$

to the second term on the left hand side of (157). The following relation is found,

$$\sum_{i=0}^{D} v_{B.i} \; \varkappa_i \, (\bar{\varphi}, \bar{\lambda}) \leqslant \sqrt{\sum_{i=0}^{D} v_{B.i}^{\;2}} \; \sqrt{\sum_{i=0}^{D} \varkappa_i^{\;2} \, (\bar{\varphi}, \bar{\lambda})} \quad . \qquad (160b)$$

The amount of the square of the spherical harmonics of the suffixes $i = 0,1,2, \ldots , D$ has a limited upper bound. Further, the integer D has a

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limited amount. The combination of (115)(127)(160b) and (157) corroborates the uniform convergence, (160).

8. The uniqueness of the result, an ellipsoidal model Earth, particular regional and time dependent supplements

As regards the uniqueness of the representation of the geopotential W in the exterior space Ψ by the equation (153), this problem was already touched earlier in context with the theorem 1 and 2. The development (153) and the uniform convergence of it was never in question for the space Ψ_b exterior of the Brillouin sphere. The very uniqueness problem here to be discussed deals with the potential W in Ψ_a obtained by the downwards continuations. The question is as follows: Is the potential described by the development (153) even the one single solution of the problem which consists in the finding of W in Ψ_a, before the background of the given W values in Ψ_b and before the background of the Laplace equation valid in Ψ ?

Introducing the hypothesis that the potential W in Ψ is not unique, the ensuing self-explanatory relations derive for a hypothetical harmonic potential X and for the difference Z of the two potentials W and X ,

$$\Delta X = 0 \quad , \text{ in } \Psi \quad , \tag{161}$$

$$X = W \quad , \text{ in } \Psi_b \, , \tag{162}$$

$$X \neq W \quad , \text{ in } \Psi_a \, ; \tag{163}$$

$$\Delta \left\{ \left(\frac{R}{r}\right)^{n+1} \varkappa_n (\varphi, \lambda) \right\} = 0 \quad , \text{ in } \Psi \, , \tag{164}$$

$$\Delta W = 0 \quad , \text{ in } \Psi \quad , \tag{165}$$

$$Z = W - X \, , \text{ in } \Psi \, , \tag{166}$$

$$\Delta Z = \Delta W - \Delta X = 0 \quad , \text{ in } \Psi \, , \tag{167}$$

$$Z = 0 \quad , \text{ in } \Psi_b \, ; \tag{168}$$

the theorem 2 about the harmonic downwards continuation demands by (167) and (168) the validity of the following equation

$$Z = 0 \quad , \text{ in } \Psi_a \, . \tag{169}$$

Thus, the validity of

$$Z = W - X \neq 0 \quad , \text{ in } \Psi_a \, , \tag{169a}$$

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is not right, (see (163)). The relations (167)(168)(169) demonstrate that (163) and (169a) conflict with the theorem 2 about the harmonic downwards continuation, unless

$$X = W \quad , \text{ in } \Psi_a \quad . \tag{170}$$

Therefore, the development (153) represents the unique potential W in the exterior space Ψ. Another potential cannot be obtained for W.

At some places in the literature, the convergence of (153) in Ψ_a is discussed for an ellipsoid as the model Earth, [18][19]. At those other places, it is argued that in case of (8) the introduction of the series development (6) in (1) would lead to a divergence or to a questionable convergence of the series (153) in the space between the Brillouin sphere (enclosing the ellipsoid) and the surface of the ellipsoid, i.e. the space Ψ_a that does come into existence in the special case of this ellipsoid model Earth. Some furtherances from the theory of the confocal ellipsoids allow the concession of a shrinking of this special Ψ_a space. But this shrinking procedure has not the consequence that Ψ_a does vanish in this special case. Thus, it is argued that certain areas of divergence or of dubious convergence will possibly remain in the exterior of the ellipsoid.

This argumentation is not right, it does not contain a proof of the divergence. The series (6) is divergent in case of (8), But, if this divergent series is introduced for 1/l in (1), the thus obtained series for W is not necessarily divergent also. There is no proof of the divergence, it is a conjecture only, [18][19].

These considerations about the divergence of (153) in the exterior of an ellipsoidal model Earth are in conflict with the argumentations expressed by the lines between the equations (8) and (9). These considerations about an ellipsoidal model Earth are also in contradiction to the above derivations which begin with the equation (1) and which end with the equation (170). The ellipsoid considerations, [18][19], are not valid.

Another question can be brought into the discussion. No doubt, the above treated non-time-dependent model Earth with the potential W is not rigorously the real Earth. The next step to better the model Earth will be the addition of the tide potential, W_1.

Further details appearing by certain special structures within a limited surface region can be accounted for approximatively by the addition of the potential of a manifold of discrete point masses choosen in a convenient way, to set an example. This potential caused by regional sources is denoted by W_2.

Thus, along these lines and in order to do a further step beyond (66) and (153), the following refined stable model Earth potential, (171), can be taken as a refined substitute for the gravitational potential of the real

DOI: https://doi.org/10.2312/zipe.1986.084

Earth, it will fulfill all the geodetic requirements,

$$W' = W + W_1 + W_2 \quad . \tag{171}$$

The above convergence proofs, (130)(131)(158)(159)(160), refer to a model Earth with a regular surface. The transition to the real Earth leads to the surface of the real Earth which is not absolute regular in an absolutely rigorous consideration, because of the vegetation, cultivation, buildings, etc.; or, with other words, because of the micro-structure of the real surface of the Earth.

However, by no means, it is not right to argue that this micro-structure in the Earth surface will cause that the convergence of the spherical-harmonic development for the gravitational potential of the real (not time dependent) Earth will break down in whole the exterior space of the Earth.

Of cause, it is possible to introduce a regular surface that does envelope all the gravitating mass particles of the Earth, the micro-structure included. The enveloping by this regular mathematical surface is as close as possible. The above convergence proofs show that the convergence of the spherical-harmonic development for the gravitational potential of the real (not time dependent) Earth is notwithstanding valid also further on in the exterior space of this enveloping surface, irrespective of the existence of any micro-structures below the envelope. The micro-structure does not paralyse the considered convergence in the exterior of this envelope.

By no means, there is not a general break-down of the convergence property if a surface micro-structure does exist. The convergence property is a stable one.

9. Conclusions

The gravitating sources within a regular surface generate a harmonic potential in the exterior of this surface. The spherical-harmonic series development of this potential is considered. At first, this series development for test points exterior of a sphere enclosing all the gravitating masses is in the fore, (Brillouin-sphere). The uniform convergence of this series development in the exterior space of the Brillouin-sphere is generally accepted. Further on, this crucial series development exterior of the Brillouin sphere is divided into two parts. The first part comprises the truncated series development. The whole potential W has the series

$$W = \sum_{i=0}^{\infty} \omega_i \quad . \tag{172}$$

The truncated series (172) is identical with the sum

DOI: https://doi.org/10.2312/zipe.1986.084

$$W^{o} = \omega_0 + \omega_1 + \dots + \omega_B \quad . \tag{173}$$

This sum comprises the main part of the series (172). The second part is the residual potential of (172),

$$W^{oo} = \sum_{i=B+1}^{\infty} \omega_i \quad . \tag{174}$$

The equations (172)(173)(174) yield,

$$W = W^{o} + W^{oo} \quad . \tag{175}$$

This partition of W according to (175) is not a fixed one. The parameter B of this division is variable. The greater the index B the smaller the residual term W^{oo} in the exterior of the Brillouin sphere.

Now, the two parts of the potential W undergo the procedure of the harmonic downwards continuations, beginning from the exterior of the Brillouin-sphere and proceeding down to the regular surface of the gravitating mass; i.e., in the here discussed applications, the surface of the Earth. The harmonic downwards continuation of W^{o} is uncomplicated, since W^{o} is a sum of terms which are all harmonic in whole the exterior of the Earth. But, the procedure of the downwards continuation of the second part has an entirely different character. It necessitates certain special considerations. Starting from different standpoints, it is proved that the second part W^{oo} tends to zero at the surface of the Earth if the first part W^{o} extends wider and wider, $B \longrightarrow \infty$. This property of W^{oo} is valid also for whole the exterior of the Earth.

This result is equivalent with the statement that the spherical-harmonic series development for the gravitational potential is uniform convergent at the regular surface of the Earth and in whole the exterior space of the Earth.

The micro-structure of the **real** surface of the Earth does not paralyse the convergence in the exterior space.

DOI: https://doi.org/10.2312/zipe.1986.084

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E. The global embedding term in the space-time relation between the geodetic measurements and the geological masses

Contents

Summary

Within the frame of the investigation of the recent crustal movements, the space-time change of the local values of the gravity and of the height is considered, as so as the radial derivation of the potential of the space-time changes of the geological masses. A relation for the local interdependence of these three amounts is derived. This relation is completed by the regard of the global embedding term. This term is proportional to the global potential of the recent crustal movements, it is rather small, and it can be neglected in most applications. The global embedding term cannot be determined precisely by the geodetic measurements, it shares to the white and coloured noise of the method.

Zusammenfassung

Die Kräfte in den oberen Schichten der Erde, die die rezenten Erdkrustenbewegungen bewirken, verursachen auch geologische Massenänderungen. Sie sind ferner die Quelle von Veränderungen der Höhen und der Schwere an der Erdoberfläche. Ausgehend von der Fundamentalgleichung der physikalischen Geodäsie wird eine mathematische Gleichung aufgestellt, die die zeitlichen Veränderungen der folgenden 3 Oberflächenwerte zueinander in Beziehung setzt: Die radiale Ableitung des Gravitationspotentials der geologischen Massen, die Schwere und die Höhe. Ferner tritt ein Ausdruck hinzu, der proportional zum globalen Potential der rezenten Krustenbewegungen ist. Dieser Ausdruck kann aus geodätischen Messungen

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nicht bestimmt werden. Eine einfache Abschätzung zeigt, daß er relativ klein ist und in den meisten Fällen vernachlässigt werden kann. Er trägt zu den zufälligen und sytematischen Fehlern der Methode bei.

Резюме

Силы в верхних слоях земли, приводящие земную кору в новое движение, обуславливают также и геологические изменения массы. Они также являются источником изменений высот и силы тяжести на земной поверхности. Исходя из фундаментального уравнения геодезии, строится математическое уравнение, связывающее временные изменения трёх следующих величин поверхности: радиальную производную гравитационного потенциала геологических масс, силу тяжести и высоту. В последующем добавляется выражение, пропорциональное глобальному потенциалу современных движений земной коры. Это выражение невозможно установить геодезическими измерениями. Простая оценка показывает, что оно относительно мало и в большинстве случаев может не учитываться. Оно влияет на случайные и систематические ошибки метода.

1. Introduction

In geodesy, the pure geometrical measurements L depend on the spatial position $\underline{\underline{x}}$ of the observation stations only,

$$L = L(\underline{\underline{x}}) \quad . \tag{1}$$

The recent crustal movements will change the spatial position

$$\underline{\underline{x}} = \underline{\underline{x}}(t) \quad , \tag{2}$$

a change of L in the course of the time t is the consequence,

$$\frac{dL}{dt} = \frac{dL}{d\underline{\underline{x}}} \frac{d\underline{\underline{x}}}{dt} \quad . \tag{3}$$

DOI: https://doi.org/10.2312/zipe.1986.084

For instance, the relations (1)(2)(3) are valid for the distances between two stations on the surface of the Earth.

But, in the physical geodesy, the measurements l depend not only on the spatial position of the station. They depend also on the gravity potential.

$$l = l(\underline{\underline{x}}, t) \quad . \tag{4}$$

Thus, with (2) and (3),

$$\frac{dl}{dt} = \frac{\partial l}{\partial \underline{\underline{x}}} \frac{d\underline{\underline{x}}}{dt} + \frac{\partial l}{\partial t} \quad . \tag{5}$$

The physical measurements l represent here the intensity of the gravity, for instance.

Further, the results of the geometric levellings depend on the shape of the equipotential surfaces of the gravity potential, i.e. the level surfaces. Of course, the recent crustal movements give rise to a vertical shift of the level surfaces, a corresponding shift of the results of the geometrical levellings is the consequence.

2. The formula of Strang van Hees

The next step is a consideration of the potential of the mass changes which are caused by the recent crustal movements of the Earth. This approach views at the problem from a global standpoint. The gravity measurements g and the normal heights h are well-introduced in geodesy. They are understood here as the geodetic measurements which are influenced by the recent crustal movements.

The here discussed problem favours the system of the normal heights, it will be explained later.

These geodetic measurements g and h are understood that they are surface values; they can be taken as certain measurements, attached to the points at the surface of the Earth.

The full gravity potential is denominated by W, and the standard potential by U. The perturbation potential T has the equation (6) , as it is well-known.

$$T = W - U \quad . \tag{6}$$

The situation before the appearence of the phenomenon of the recent crustal movements is marked by the subscript $(\ )_1$, i.e. the old situation. So, the new situation with the impact, caused by the recent crustal move-

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ments, is marked by the subscript $(\ )_2$. Hence, the perturbation potential T of the old situation is,

$$T_1 = W_1 - U \quad , \tag{7}$$

or,

$$T_1(\underline{\underline{x}}) = W_1(\underline{\underline{x}}) - U(\underline{\underline{x}}) \quad , \tag{7a}$$

and for the new situation,

$$T_2 = W_2 - U \quad , \tag{8}$$

or,

$$T_2(\underline{\underline{x}}) = W_2(\underline{\underline{x}}) - U(\underline{\underline{x}}) \quad . \tag{8a}$$

The difference of (7) and (8) is the impact of the recent crustal movements,

$$\delta T = T_2 - T_1 \quad , \tag{9}$$

$$\delta W = W_2 - W_1 \quad . \tag{10}$$

Thus,

$$\delta T = \delta W \quad . \tag{11}$$

It is to be stressed that the spatial test point (for which the potential values T_1, W_1 and T_2, W_2 are valid) is understood to be a fixed-point in the space, (7a)(8a). This mathematical fixed-point should not be mistaken for the center of a physical particle participating at the recent crustal movements. Thus, to be more precise, the relations (7) to (11) can be written in the following shape,

$$T_1(\underline{\underline{x}}) = W_1(\underline{\underline{x}}) - U(\underline{\underline{x}}) \quad , \tag{12}$$

$$T_2(\underline{\underline{x}}) = W_2(\underline{\underline{x}}) - U(\underline{\underline{x}}) \quad , \tag{13}$$

$$\delta T(\underline{\underline{x}}) = T_2(\underline{\underline{x}}) - T_1(\underline{\underline{x}}) \quad , \tag{14}$$

$$\delta W(x) = W_2(\underline{\underline{x}}) - W_1(\underline{\underline{x}}) \quad , \tag{15}$$

$$\delta T(\underline{\underline{x}}) = \delta W(\underline{\underline{x}}) \quad . \tag{16}$$

However, **in case, $\underline{\underline{x}}'$ is not a fixed-point but** a surface particle influenced by the recent crustal movements, the following reasoning is useful. $\underline{\underline{x}}'_1$ is the position of a surface particle in the old situation, $\underline{\underline{x}}'_2$ is the position of the same surface particle in the new situation. The relation (14) gives for the change of T at $\underline{\underline{x}}'_2$,

DOI: https://doi.org/10.2312/zipe.1986.084

$$T_2(\underline{\underline{x}}_2') - T_1(\underline{\underline{x}}_2') = \delta T(\underline{\underline{x}}_2') = T_2(\underline{\underline{x}}_2') - T_1(\underline{\underline{x}}_1') -$$

$$- \frac{dT_1}{d\underline{\underline{x}}} (\underline{\underline{x}}_2' - \underline{\underline{x}}_1') . \qquad (17)$$

The difference vector

$$\underline{\underline{x}}_2' - \underline{\underline{x}}_1' , \qquad (18)$$

represents the spatial movement of the considered surface particle.

A more thorough discussion of geodetic deformation problems, as (17), is contained in:
Grafarend, E.: Six lectures on geodesy and global geodynamics, in Mitteilungen d. geodät. Institute d. Techn. Univ. Graz, Folge 41 (1982) S. 531-685, eds. Moritz, H. and Sünkel, H., Graz.

The fundamental differential equation of the physical geodesy is helpful in the evaluation of the vertical component of the derivative of T_1 appearing in the last term of the relations (17),

$$- \frac{\partial T}{\partial r} - \frac{2}{r} T = \Delta g_F . \qquad (19)$$

r is the geocentric radius, Δg_F is the free-air anomaly.

The vertical component of the last term of (17) takes the following shape by the introduction of (19), it is the crucial term of (17),

$$- \frac{dT_1}{d\underline{\underline{x}}} (\underline{\underline{x}}_2' - \underline{\underline{x}}_1') = - \frac{\partial T}{\partial h} dh = \Delta g_F \, dh + \frac{2}{r} T \, dh . \qquad (20)$$

Since vertical movements are considered, only the vertical shift dh does reflect in (20). A change of the geopotential by the amount of (20) can take its rise from a vertical position shift in the field of the geopotential W by the following amount,

$$- \frac{1}{G} \frac{\partial T}{\partial h} dh = \frac{1}{G} \Delta g_F \, dh + \frac{2}{RG} T \, dh . \qquad (21)$$

G is the global mean value of the gravity, R is the mean radius of the Earth. The amounts on the right hand side of (21) can be understood to have the following orders,

$$G = 10^3 \; [\mathrm{cm\ sec}^{-2}] , \qquad (22a)$$

$$\Delta g_F = 0,1 \; [\mathrm{cm\ sec}^{-2}] , \qquad (22b)$$

$$dh = 10 \; [\mathrm{cm}] , \qquad (22c)$$

$$R = 6 \cdot 10^8 \; [\mathrm{cm}] , \qquad (22d)$$

DOI: https://doi.org/10.2312/zipe.1986.084

$$\frac{1}{G} T = 10^4 \quad [\text{cm}] \; . \tag{22e}$$

The above listed constants, (22a) to (22e), lead to the following amount for the vertical shift of the level surface of the geopotential W caused by the crucial term (20) in the relation (17),

$$- \frac{1}{G} \frac{\partial T}{\partial h} dh = 10^{-3} \text{ cm} + 0.3 \cdot 10^{-3} \text{ cm} \; . \tag{23}$$

The two terms on the right hand side of (23) have negligible amounts. A vertical shift of 10^{-3} cm cannot be determined by geometric levellings. A vertical shift by 10^{-3} cm will entail a negligible gravity change of 0.3×10^{-2} μgal, if the standard free-air vertical gradient of the gravity is applied,

$$\frac{\partial g}{\partial h} = - \frac{2G}{R} h = - 0.3 \times h \quad [\text{mgal}] \; , \tag{24}$$

h has in (24) the dimension of the meter.

Thus, the relations (23) and (24) show that the last term of (17) can be neglected, i.e. (20).

The subsequent equation is obtained, it is valid for the here discussed applications,

$$\delta T (\underline{\underline{x}}_2') = T_2(\underline{\underline{x}}_2') - T_1(\underline{\underline{x}}_1') \; , \tag{25}$$

(see also the relations (73) to (77)).

The perturbation potential T derives from the free-air anomaly Δg_F by

$$T = \frac{R}{4\pi} \iint_p \left[\Delta g_F + C + C_1 \right] S_T(\psi) \; dp \; , \tag{26}$$

(see chapter A and B). C is the plane topographic reduction of the gravity, C_1 is equal to

$$C_1 = (h_P - h_Q) \; \frac{\partial \Delta g \text{ Bouguer}}{\partial h} \; . \tag{27}$$

C_1 can be neglected in view of the present state of the gravity nets. p symbolizes the unit sphere, dp is the surface element of it,

$$dp = \cos \varphi \; d\varphi \; d\lambda \; , \tag{28}$$

φ is the geographical latitude and λ the longitude. $S_T(\psi)$ is the well-known Stokes function, (see chapter A, B, C). ψ is the spherical distance between the fixed test point P and the variable point Q.which does run over the globe in the course of the integration. h_P and h_Q are the topographic heights of the concer-

DOI: https://doi.org/10.2312/zipe.1986.084

ned points.

The differential quotient on the right hand side of (27) is the vertical gradient of the Bouguer anomalies of the gravity. The amount of C_1 will reach about 1 mgal only in gravimetrically very disturbed regions with high Bouguer anomalies.

The relation (26) for the epoch

$$t = t_i \ , \quad (i = 1,2) \ , \tag{28a}$$

is

$$T_i = \frac{R}{4\pi} \iint_p \left[(\Delta g_F)_i + C + C_1 \right] S_T(\psi) \ dp \ . \tag{29}$$

In (29), the recent crustal movements will influence only the perturbation potential T and the free-air anomalies. Perhaps, enormous earthquakes will influence the C and C_1 values. Hence, the relation (29) turns to

$$T_i = \frac{R}{4\pi} \iint_p \left[(\Delta g_F)_i + c_i \right] S_T(\psi) \ dp \ , \tag{30}$$

$$c_i = (C + C_1)_i \ . \tag{31}$$

The change between the two epochs

$$t = t_1 \ \text{ and } \ t = t_2 \ , \tag{32}$$

$$(i = 1,2),$$

results as follows, (12) to (16), (25),

$$\delta T = \frac{R}{4\pi} \iint_p \left[\delta \Delta g_F + \delta c \right] S_T(\psi) \ dp \ . \tag{33}$$

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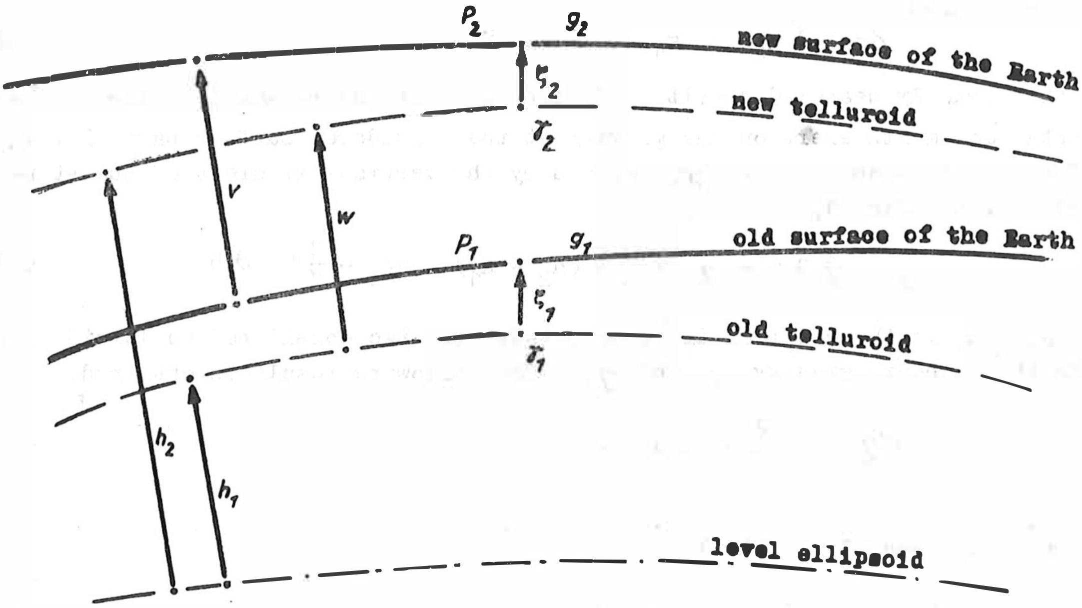

Fig. 1 : The vertical shifts of the surface of the Earth and of the telluroid caused by the recent crustal movements.

According to the way along which the relation (33) was derived, δT is the change of the T value for a surface particle. However, the relation (25) allows to consider $\delta T = \delta T(\underline{x})$ as a spatial potential function. Considering the function δT from this standpoint, the relation (33) is replaced now by the fundamental differential equation of the physical geodesy, specialized on the surface points. The relation (9), (10) and (19) yield, Fig. 1,

$$\delta \Delta g_F = (\Delta g_F)_2 - (\Delta g_F)_1 \quad , \tag{34}$$

with

$$(\Delta g_F)_2 = g_2 - \gamma_2 = -\frac{\partial T_2}{\partial r} - \frac{2}{r} T_2 \quad , \tag{35}$$

and

$$(\Delta g_F)_1 = g_1 - \gamma_1 = -\frac{\partial T_1}{\partial r} - \frac{2}{r} T_1 \quad . \tag{36}$$

g_1 is the old gravity at the point P_1 at the old surface of the Earth and g_2 is the new gravity for the same particle but on the new surface. γ_1 is the old standard gravity at the point on the old telluroid, vertical below P_1. γ_2 is the new standard gravity at the point on the new telluroid and vertical below P_2.

DOI: https://doi.org/10.2312/zipe.1986.084

The relations (25)(34)(35)(36) lead to the subsequent equations

$$\delta \Delta g_F = - \frac{\partial \delta T}{\partial r} - \frac{2}{r} \delta T = (g - \gamma)_2 - (g - \gamma)_1 = g_2 - g_1 - (\gamma_2 - \gamma_1) \quad . \qquad (37)$$

The expression

$$\delta g = g_2 - g_1 \qquad (38)$$

is the directly measured gravity difference. It is the impact that the recent crustal movements exert on the gravity at the considered surface particle. The difference between γ_2 and γ_1 derives by the vertical gradient of the standard gravity, Fig. 1,

$$\gamma_2 = \gamma_1 - \frac{2}{R} G w = \gamma_1 - \frac{2}{R} G (h_2 - h_1) = \gamma_1 - \frac{2}{R} G \ \delta h \quad . \qquad (39)$$

$w = h_2 - h_1$ is the vertical distance between the two considered telluroid points with the standard gravity γ_2 and γ_1. The following result is obtained,

$$\delta \gamma = - \frac{2}{R} G \cdot \delta h \quad . \qquad (40)$$

The relations (37)(38)(40) give

$$\delta \Delta g_F = \delta g + \frac{2}{R} G \cdot \delta h \quad . \qquad (41)$$

This is the searched expression for the change of the free-air anomalies in terms of the corresponding changes of the gravity and of the normal heights obtained by levellings. The relation (41) is introduced into (33), and the following integral is obtained, [1][5],

$$\delta T = \frac{R}{4\pi} \iint_P \left[\delta g + \frac{2}{R} G \cdot \delta h + \delta c \right] S_T(\psi) \ dp \quad . \qquad (42)$$

This is the expression for the potential of the recent crustal movements δT in terms of the measured values δg and δh. It is the formula of Strang van Hees and others, [1][2][5].

The height anomalies ζ can be expressed by the perturbation potential T at the surface of the Earth,

$$T_1 = G \cdot \zeta_1 \quad , \qquad (43)$$

$$T_2 = G \cdot \zeta_2 \quad . \qquad (44)$$

Hence,

$$\delta \zeta = \frac{1}{G} \cdot \delta T \quad . \qquad (45)$$

For the vertical shift v of the surface of the Earth, the following relations are obtained, (see figure 1),

DOI: https://doi.org/10.2312/zipe.1986.084

$$h_1 + \zeta_1 + v = h_2 + \zeta_2 \quad , \tag{46}$$

$$\delta h = h_2 - h_1 = v - (\zeta_2 - \zeta_1) = v - \delta\zeta \quad , \tag{47}$$

$$\delta h = v - \delta\zeta \quad . \tag{48}$$

Thus,

$$v = \frac{1}{G} \delta T + \delta h \, . \tag{49}$$

This equation (49) and (42) reveal,

$$v = \frac{R}{4 \pi G} \iint_{p} \left[\delta g + \frac{2}{R} G \cdot \delta h + \delta c \right] S_T(\psi) \, dp + \delta h \quad . \tag{50}$$

This is the expression for the recent crustal movements in terms of the measured values δg and δh.

The formula (50) is mathematically right, it was corroborated by different authors. However, unfortunately, this formula cannot be applied in geodesy and geophysics since the integration covers whole the Earth. The knowledge of the δg values and of the δh values all over the Earth is an indispensable prerequisite of (50), and the δg values must have a precision of about $\pm$ 1 μgal globally, and the δh values about $\pm$ 1 mm. There is no hope that these values can be measured with the needed accuracy of $\pm$ 1 μgal or $\pm$ 1 mm along the surface of the oceans, never. Therefore, a global consideration of the problem is never possible by (50).

The following lines will show, that it is possible to bring the relation — that connects δg, δh and δT — into a regional shape avoiding the global version (5o).

To make the situation clear, in this new relation which is to be developed now, the potential δT will not appear directly. But, the radial derivative of δT,

$$\frac{\partial \delta T}{\partial r} \quad , \tag{51}$$

will be the expression which introduces the potential δT.

3. The transformation of the global formula for the potential of the recent crustal movements into a regional shape

For the deliberations about the transition from the global **form** to the regional shape, the potential of the recent crustal movements is now denominated by D instead of δT, for the sake of abbreviation,

$$\delta T \longrightarrow D = D(\underline{x}) \quad . \tag{52}$$

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The fundamental differential equation of the physical geodesy for the spatial potential D is as follows, (37), [1],

$$- \frac{\partial D}{\partial r} - \frac{2}{r} D = \delta \Delta g_F \quad . \tag{53}$$

The considerations connected with the equation (25) manifest that the function D is a spatial potential that depends on the spatial position $\underline{x}$ only, in sufficient approximation, at the least.

A comparison of (41) and (53) shows the validity of the subsequent relation, [1],

$$\frac{\partial D}{\partial r} + \frac{2}{r} D = - \delta g - \frac{2}{R} G \cdot \delta h \quad . \tag{54}$$

This equation (54) is of fundamental importance. It is already the searched relation with a regional or local character connecting the potential D and the measured values δg and δh.

At first sight, the relation (54) seems to have also a global aspect, since D does appear directly in the second term on the left hand side of (54). But, the function D can not be determined by the global integration along the lines of (42) and (55), it is an integration procedure that can never be possible in geodesy and geophysics, as discussed above, since it happens by

$$D = \frac{R}{4\pi} \iint\limits_{p} (\delta g + \frac{2}{R} G \cdot \delta h + \delta c) \, S_T(\psi) \, dp \quad . \tag{55}$$

But, a more thorough examination of (54) will bring the fact to the light that the second term on the left hand side of (54) is much more small then the first term on the left hand side of (54) and, further on, much more small than the term on the right hand side of this equation, at least for the most probable phenomena of the recent crustal movements.

In order to prove this speciality of the relation (54), the potential D is represented by a spherical-harmonic development, the convergence of it is secured at the surface of the Earth and in the whole exterior space of the Earth, [1], (see chapter D), the convergence is sure for the here introduced spherical model also,

$$D = \sum_{n=0}^{\infty} \left(\frac{1}{r} \right)^{n+1} D_n \cdot Y_n (\varphi, \lambda) \quad . \tag{56}$$

r is the geocentric radius, D_n are the Stokes constants and $Y_n(\varphi, \lambda)$ stands for the spherical harmonics of degree n and of the order m = 0,1,2, ... , n.

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The introduction of the potential D, (56), into the fundamental differential equation of the geodesy gives

$$\frac{\partial D}{\partial r} + \frac{2}{r} D = - \sum_{n=0}^{\infty} (n - 1) \left(\frac{1}{r}\right)^{n+2} D_n \cdot Y_n (\varphi, \lambda) \quad . \tag{57}$$

It is very probable that in most cases the individual phenomena of the recent crustal movements will have a horizontal extension of not more than some hundred kilometer only. A spatial extension up to the size of much more than 2 000 x 2 000 km square will be very seldom.

All the constituents in the recent crustal movements which have a wave length of smaller than A = 2 000 km can be represented by certain spherical-harmonic expressions of the order

$$n > \frac{20\ 000\ \text{km}}{A_{\text{km}}} \quad , \tag{58}$$

or, if taken in degrees of arc,

$$n > \frac{180^{\circ}}{A^{\circ}} \quad . \tag{59}$$

The here chosen value of

$$A = 2\ 000\ \text{km} \quad , \quad \text{or} \quad A^{\circ} = 18^{\circ} \quad , \tag{60}$$

leads to

$$n > 10 \quad . \tag{61}$$

A comparison of the spherical-harmonic developments, (56),

$$\frac{\partial D}{\partial r} = - \sum_{n=0}^{\infty} (n + 1) \left(\frac{1}{r}\right)^{n+2} D_n \cdot Y_n (\varphi, \lambda) \quad , \tag{62}$$

and

$$\frac{2}{r} D = 2 \sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^{n+2} D_n \cdot Y_n (\varphi, \lambda) \quad , \tag{63}$$

reveals, that, in case of

$$n > 10 \quad , \tag{64}$$

and if

$$r = \varrho \quad , \tag{65}$$

the following inequation is valid

$$\left| (n+1) \left(\frac{1}{\varrho}\right)^{n+2} D_n \cdot Y_n (\varphi, \lambda) \right| \gg \left| 2 \left(\frac{1}{\varrho}\right)^{n+2} D_n \cdot Y_n (\varphi, \lambda) \right| . \tag{66}$$

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ϱ is here the radius of the Earth. The inequation (66) is right because of

$$(n+1) \gg 2 \quad , \quad \text{for} \quad n > 10 \quad . \tag{67}$$

A short discussion of the convergence properties of (62) and (63) **seems to be** recommended. The series (56) is convergent in the exterior of the Earth, [1], (see chapter D). Since

$$r \frac{\partial D}{\partial r} = - \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{r}\right)^{n+1} D_n \circ Y_n \tag{67a}$$

is harmonic and continuous as D, (56), it follows that (67a) is convergent in the exterior of the Earth, as (56). Further on, if the series

$$\sum_{n=1}^{\infty} a_n = s \tag{67b}$$

is convergent, and if (67b) is multiplied with the constant c, it is sure that the series

$$\sum_{n=1}^{\infty} c\, a_n = c\, s \tag{67c}$$

is convergent also. Thus, the multiplication of (67a) with 1/r reveals that (62) is convergent in the exterior space of the Earth. Replacing c by 2/r, (67c) and (56) demonstrate that the relation (63) is convergent in the exterior of the Earth.

Returning back to (62) and (63), the introduction of (64) gives

$$\frac{\partial D}{\partial r} = - \sum_{n=10}^{\infty} (n+1) \left(\frac{1}{r}\right)^{n+2} D_n \cdot Y_n(\varphi, \lambda) \quad , \tag{67d}$$

and

$$\frac{2}{r} D = 2 \sum_{n=10}^{\infty} \left(\frac{1}{r}\right)^{n+2} D_n \cdot Y_n(\varphi, \lambda) \quad . \tag{67e}$$

Thus, (54)(67d)(67e)(66) will entail

$$\left| \frac{\partial D}{\partial r} \right| \gg \left| \frac{2}{r} D \right| \quad , \tag{68}$$

$$\left| \delta g + \frac{2}{R} G \cdot \delta h \right| \gg \left| \frac{2}{r} D \right| \quad . \tag{69}$$

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The inequations (68) and (69) prove with high probability that the term

$$\frac{2}{r} D \qquad (70)$$

in the relation (54) has a relative small value, among the terms of (54). The relation (54) seems to be governed by the terms

$$\frac{\partial D}{\partial r} \qquad (71)$$

and

$$\delta g + \frac{2}{R} G \cdot \delta h \quad . \qquad (72)$$

Before the background that the potential D cannot be determined by (55) or by another mean, it is advantageous that the term (70) can be neglected probably in most applications of (54).

This fact will diminish the importance of the second term on the left hand side of (54).

The global term,(70), seems to be rather unimportant for the applications of the equation (54). The terms of local character, (71) and (72), are the terms of dominating influence in the equation (54). Therefore, it is justified to state that (54) is an equation of regional or local character. However, in the subsequent developments, the terms (70) will be taken into account completely, it is in order to preserve the possibility to evaluate the impact it has on the relation (54).

As a supplementary remark, a certain question appearing in a more rigorous consideration of the derivations which lead from (19) to (41) and (53) should not be overlooked.

The expression for $\delta \Delta g_F$ in (41) was obtained by (37). In this formula, $(g - \gamma)_2$ refers to the surface particle in the new position, and $(g - \gamma)_1$ is attached to the same particle in the old position. The differential relation (53) for the potential D has its origin in the equations (35) and (36), it is the difference of them. For a rigorous derivation of (53) by (35) and (36), all these three equations refer to the same spatial point $\underline{\underline{x}}$. Therefore, the expression $(g - \gamma)_2$ in (35) and $(g - \gamma)_1$ in (36) must belong exactly to the same spatial position. But, the measured values of the gravity g_2 and g_1 refer to spatial points inevitably separated by the recent crustal movements, - (the vertical component of it is here in the fore). $(g - \gamma)_2$ and $(g - \gamma)_1$ must be transformed by a spatial shift to reach a system of $(g - \gamma)$ - values which refer to one and the same surface. Of course, after this spatial shift, the values $(g - \gamma)_2$ and $(g - \gamma)_1$ are no more generally attached to the same surface mass particle.

For a fixation of the ideas, the relevant expression, (19),

DOI: https://doi.org/10.2312/zipe.1986.084

$$g - \gamma , \tag{73}$$

$$\frac{\partial T}{\partial r} , \tag{74}$$

and

$$\frac{2}{r} T \tag{75}$$

are now supposed to undergo a vertical shift by 10 cm, e.g., in order to have them all at one and the same surface.

Thus, it is to be investigated into whether the impact can be neglected which such a vertical shift does exert on the relevant expressions (73)(74)(75).

With (23) and (24), such a shift of 10 cm changes T by about $G\cdot(10^{-3}$ cm) and the amount of $\frac{2}{R}$ T, (75), changes by about $0.3 \cdot 10^{-2}$ μgal, an absolutly negligible amount if considering the equation (75).

The impact that such a point shift of 10 cm takes on the term (74) can be evaluated utilizing the Laplace differential equation. The following approximative derivations are self-explanatory,

$$\frac{\partial^2 T}{\partial z^2} \delta h = - \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \delta h = - G \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) \delta h =$$

$$= G \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right) \delta h \quad . \tag{75a}$$

The relations (75a) are well-known from the derivations about the solution of the geodetic boundary value problem, (see chapter B). x, y, z are orthogonal Cartesian coordinates, the z-axis is vertical, ζ is the height anomaly, ξ and η are the deflections of the vertical. A representative amount is

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} = 2'' \; [\mathrm{km}^{-1}] \quad . \tag{76}$$

Hence, with the above approximations, the impact of the considered vertical shift of 10 cm on (74) is equal to

$$\frac{\partial^2 T}{\partial r^2} \delta h = \frac{\partial^2 T}{\partial z^2} \delta h \lessapprox 1 \,\mu\mathrm{gal} \quad . \tag{77}$$

It is a negligible amount as opposed to the impact by the free-air gradient which contributes 30 μgal for δh = 10 cm, (24).

As to $g - \gamma = \Delta g_F$, (73), a transformation of the relation (19) by a spatial shift must necessarily have the same effect on both sides of (19). Otherwise, this equation will loose its validity. On the left hand side of (53), the recent crustal movements of 10 cm will have an impact of not more than 1 μgal,

DOI: https://doi.org/10.2312/zipe.1986.084

according to (23)(24)(77). It is sure that the impact on Δg_F on the right hand side of (19) will have the same amount.

Thus, the relation (53) is right for all the here discussed applications.

4. The global embedding term

The potential of the recent crustal movements D that fulfills the equation (54) consists of two parts. The first partial potential D_g is caused by the time changes of the geological masses in the interior of the Earth, i.e. the drift of the masses. The souces of the second partial potential D_b are the masses of the surface layer which are enclosed in the space between the old and the new surface of the Earth,

$$D = D_g + D_b \; . \tag{78}$$

Thus, the potential of the time dependent mass changes caused by certain geological phenomena has this equation,

$$D_g = D - D_b \; . \tag{79}$$

The surface layer potential D_b has the following form, see figure 1,

$$D_b = f\varrho \iint_q \frac{1}{e} v \cdot dq \; . \tag{80}$$

q is the surface of the Earth, f is the gravitational constant, ϱ is the density of the masses of the Earth at the surface of it,

$$\varrho = 2.65 \; [g \, cm^{-3}] \; . \tag{81}$$

e is the straight distance **of the** test point and the point running over the surface of the Earth within the integration procedure of (80). ψ is the spherical distance between these two points, (84).

The radial derivative of D_g has the following relation according to (79),

$$\frac{\partial}{\partial r} D_g = \frac{\partial}{\partial r} D - \frac{\partial}{\partial r} D_b \; . \tag{82}$$

The jump relations for the derivatives of the potential of a surface layer lead with (80) to

$$\frac{\partial}{\partial r} D_b = -2\pi f\varrho v + f\varrho \iint_q v \frac{\partial {}^1/_e}{\partial r} dq \; . \tag{83}$$

The surface of the Earth approximates a sphere with the radius R. Therefore, the straight distance e has the following relations, they are self-explanatory,

DOI: https://doi.org/10.2312/zipe.1986.084

$$e = 2 R \sin \frac{\psi}{2} , \qquad (84)$$

$$\frac{\partial 1/e}{\partial r} = - \frac{1}{2 R e} . \qquad (85)$$

The combination of (83)(48) and (85) yields

$$\frac{\partial}{\partial r} D_b = - 2 \pi f \varrho \ (\delta\zeta + \delta h) - f \varrho \frac{1}{2R} \iint_q \frac{1}{e} \, v \, dq . \qquad (86)$$

The integral in the relation (86) can be expressed by D_b, (80),

$$\frac{\partial}{\partial r} D_b = - 2 \pi f \varrho \ (\delta\zeta + \delta h) - \frac{1}{2R} D_b . \qquad (87)$$

The radial derivative of D derives from (54),

$$\frac{\partial}{\partial r} D = - (\delta g + \frac{2}{R} G \cdot \delta h) - \frac{2}{R} D . \qquad (88)$$

The expressions (87) and (88) are introduced into (82),

$$\frac{\partial}{\partial r} D_g = - (\delta g + \frac{2}{R} G \cdot \delta h) + 2 \pi f \varrho \ (\delta\zeta + \delta h) - \frac{2}{R} D + \frac{1}{2R} D_b . \qquad (89)$$

Some transformations of (89) give

$$\frac{\partial}{\partial r} D_g = - \delta g - (\frac{2}{R} G - 2 \pi f \varrho) \delta h + 2 \pi f \varrho \cdot \delta\zeta -$$

$$- \frac{2}{R} D + \frac{1}{2R} D_b . \qquad (90)$$

(45) and (52) leads to

$$\delta\zeta = \frac{1}{G} D . \qquad (91)$$

In the formula (90) for the radial derivative of the potential of the geological mass shifts, the change of the height anomalies shall be replaced by the full potential of the recent crustal movements D, (91).

$$\frac{\partial}{\partial r} D_g = - \delta g - (\frac{2}{R} G - 2 \pi f \varrho) \delta h + (2 \pi f \varrho \frac{1}{G} - \frac{2}{R}) D + \frac{1}{2R} D_b . \qquad (92)$$

It is convenient to express the mean gravity G by the mean density of the body of the Earth, ϱ_m. The concerned developments are self-explanatory,

DOI: https://doi.org/10.2312/zipe.1986.084

$$G = f\,\frac{M}{R^2} = f\,\varrho_m\,\frac{4}{3}\,\pi\,R^3\,\frac{1}{R^2} = \frac{4}{3}\,\pi\,f\,\varrho_m\,R \quad . \tag{93}$$

With Ledersteger, ϱ_m has the following amount,

$$\varrho_m = 5.517\ [g\ cm^{-3}] \quad . \tag{94}$$

Thus,

$$2\,\pi\,f\,\varrho\,\frac{1}{G} = \frac{3}{2}\,\frac{\varrho}{\varrho_m}\cdot\frac{1}{R} \quad , \tag{95}$$

and

$$\frac{2}{R}\,G = \frac{8}{3}\,\pi\,f\,\varrho_m \quad . \tag{96}$$

The equations (92), (95) and (96) combine to

$$\frac{\partial}{\partial r}\,D_g = -\,\delta g - \left(\frac{8}{3}\,\pi\,f\,\varrho_m - 2\,\pi\,f\,\varrho\right)\delta h + \left(\frac{3}{2}\,\frac{\varrho}{\varrho_m} - 2\right)\frac{1}{R}\,D + \frac{1}{2R}\,D_b \quad . \tag{97}$$

The expression

$$\left(\frac{2}{R}\,G - 2\,\pi\,f\,\varrho\right)\delta h = \left(\frac{8}{3}\,\pi\,f\,\varrho_m + 2\,\pi\,f\,\varrho\right)\delta h = B\cdot\delta h \tag{98}$$

is the Bouguer reduction of the gravity for a plane plate with the thickness δh.

$$B = 0.3086 - 0.1119 = 0.1967\ [mgal\ /\ m] \quad . \tag{99}$$

The introduction of (98) and (99) in (97) gives

$$\frac{\partial}{\partial r}\,D_g = -\,\delta g - B\cdot\delta h + \left(\frac{3}{2}\,\frac{\varrho}{\varrho_m} - 2\right)\frac{1}{R}\,D + \frac{1}{2R}\,D_b \quad . \tag{100}$$

With

$$K_1 = -\,\delta g - B\cdot\delta h \quad , \tag{101}$$

and

$$K_2 = \left(\frac{3}{2}\,\frac{\varrho}{\varrho_m} - 2\right)\frac{1}{R}\,D + \frac{1}{2R}\,D_b \quad , \tag{102}$$

follows

$$\frac{\partial}{\partial r}\,D_g = K_1 + K_2 \quad . \tag{103}$$

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K_1 is a value of local character, since δg comes from local gravity measurements and because δh is the local shift of the levelling results.

But, the expression K_2 has a global character since the potential of all the global recent crustal movement phenomena is involved. It was already stated that the computation of D by (55) for the formula of K_2 is never possible because the δg and δh values on the oceans have no chance to become known by geodetic measurements.

Therefore, the potential D must be considered in (102) and (103) as an unknown value which contributes to the white and coloured noise of the following equation, (see (103)). Thus,

$$\frac{\partial}{\partial r} D_g \cong K_1 = - \delta g - B \cdot \delta h \quad . \tag{104}$$

K_2 is the global embedding term of (103). In (103), the interdependence of

$$\frac{\partial}{\partial r} D_g \; , \; \delta g \; , \; \delta h \tag{105}$$

is of local character, the global term K_2 is of second order only.

If, for a moment, the hypothesis

$$D \cong D_b \tag{106}$$

is introduced, the term K_2 will get the following shape,

$$K_2 = \frac{3}{2} \left(\frac{\varrho}{\varrho_m} - 1 \right) \frac{1}{R} D \quad . \tag{107}$$

Thus, the global term K_2 is equal to zero for a homogeneous sphere, $\varrho = \varrho_m$.

The potential of the geological mass shifts, D_g, can be approximated by the potentials of a manifold of point masses m_i concentrated at the points Q_i in a certain depth below the surface of the Earth. Hence, the potential D_g at the surface points P has the expression

$$D_g(P) = f \sum_i m_i \frac{1}{e(P,Q_i)} \quad . \tag{108}$$

Neglecting the white and coloured noise that comes into being by the K_2 term, the relations (104) and (108) give

$$K_1 = f \sum_i m_i \frac{\partial}{\partial r_P} \left\{ \frac{1}{e(P,Q_i)} \right\} = - \delta g - B \cdot \delta h \quad . \tag{109}$$

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The inversion of (109) gives the space-time variation of the geological masses m_i which are the values to be determined in terms of the empirical expression $-\delta g - B \cdot \delta h$.

The following facts can be pointed out:

In the investigation of the recent crustal movements, considering the determination of the time variations of the geopotential in terms of the space-time variations of the gravity and of the levellings, this task is a difficult problem, if it is treated in full universality. Consequently, it is also difficult to determine the space-time variating geological masses directly from the D potential values which are only defective known by (55). —— The gravity methods of the geophysical prospecting can be applied for the solution of (109).

However, if the radial derivative of the D potential is introduced, and if even the constituents of this potential which have a wavelength of more than 2 000 km are considered to be negligible small, in this case, it will be possible in practice to determine the space-time variating geological masses of the individual recent crustal movement phenomena.

A phenomenon free of mass shifts or a phenomenon of mass shifts free of density alterations will entail the following equation

$$D_g = 0 \quad . \tag{110}$$

In this case, the relations (101) and (104) lead to the following interdependence of δg and δh,

$$\delta g = 0.1967 \quad \delta h \; ; \tag{111}$$

δg is here measured in mgal and δh in meters.

With (111), the plane Bouguer reduction is obtained. The relation (111) was obtained empirically in the mean by the work of Kiviniemi and Groten in Finland, [3][4], and Torge and Kanngieser in Northern Iceland, [6]. These authors found in the mean

$$B = 0.2 \quad [\text{mgal} / \text{m}] \quad . \tag{112}$$

Thus, the empirically investigated recent crustal movements are caused by an influx of new masses which have about the same density as the old masses. Any deviations from the equation (111) will be the convincing evidence of an influx of masses which have another density than the old masses.

The above described theoretical investigations, (103), show the limits of the simple Bouguer plate model for the interdependence of δg and δh, (111). The linear relation (111) is a rough approximation only. It must be supplemented by the impact of the space-time variating geological mass shifts and by the impact of the K_2 term in order to obtain the more rigorous relation (103).

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5. Numerical estimations about the global embedding term

In order to complete the above considerations, a short evaluation of the amount of the K_2 term is to be sketched. The hypothesis (106) and the relation $2\varrho \leqq \varrho_m$ lead to, (107)(80),

$$K_2 = -\frac{3}{4}\frac{1}{R} D_b = -\frac{3}{4}\frac{1}{R} f \varrho \iint\limits_{q} \frac{1}{e} v \; dq \quad . \qquad (113)$$

It is plausible that an enormous volcanic eruption in the vicinity of the test points will have a relative great impact on K_2. For instance, a height shift of 0.3 km may happen over a circle of 10 km radius. If the test point has a distance of 20 km from the center point of this volcanic eruption, the effect on K_2 will amount to about 10 μgal.

Another hypothetic example is a vertical shift of the ocean bottom by 10 cm over a circle of 1 000 km radius, perhaps during a decennium. Here, the K_2 term according to (113) undergoes a change of about 0.4 μgal at a test point 2 000 km distant from the center of this phenomenon. At first sight, the existence of 10 of such phenomena of this kind may possibly give rise to a gravity change of about 4 μgal. Such a value cannot be neglected in (103), it does reach the amount of the observed values of δg, [3][4]. It would disturb the geological interpretations by (103).

But, a gravity shift by this amount is rather improbable. As Vening-Meinesz did show for a certain model, an area of upheaval will be surrounded by a belt of depression of the crust, because the law of conservation of the masses must be observed, [7]. Thus, a regional compensation effect is indicated, it will lower down the amount of K_2.

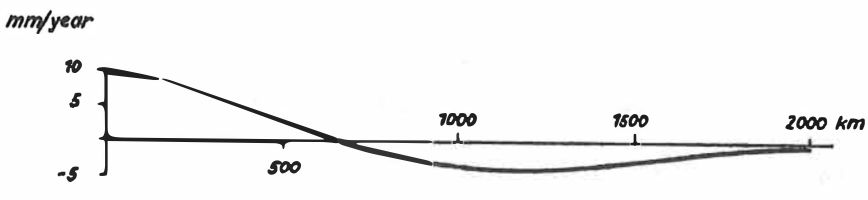

Fig. 2: The upheaval velocity for Fennoscandia according to the Vening-Meinesz model, [7], [annual upheaval].

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