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GEODETIC BOUNDARY VALUE PROBLEMS

IV

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A. A closed solution for the geodetic boundary value problem in terms of isostatic gravity anomalies

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Summary

For the computation of the perturbation potential and the height anomalies at the Earth's surface in terms of the isostatic gravity anomalies, a new and refined expression is developed, (97) (68). The theoretical error of the final solution for the height anomalies will not be greater than about 1 cm. Looking back to the traditional isostatic theory, the main progress is the fact that two amendment terms have to be added to the traditional solution; they can be computed easily.

Zusammenfassung

Für die Berechnung des Störpotentials und der Höhenanomalien an der Erdoberfläche aus isostatischen Schwereanomalien wird eine neue und genaue Formel angegeben, (97) (68). Der theoretische Fehler bei der Finallösung für die Berechnung der Höhenanomalien ist nicht grösser als etwa ± 1 cm. Der wesentliche Fortschritt im Vergleich mit der traditionellen isostatischen Theorie besteht darin, dass zu der traditionellen Lösung 2 Zusatzglieder hinzuaddiert werden müssen. Diese sind leicht zu berechnen.

Аннотация

Для вычисления возмущающего потенциала и высотных аномалий на поверхности Земли из изостатических гравитационных аномалий приводится новая и точная формула, (97) (68). Теоретическая погрешность заключительного решения при вычислении высотных аномалий составляет не более ± 1 см. Существенный прогресс по сравнению с традиционной изостатической теорией заключается в том, что к традиционному решению должны быть прибавлены 2 дополнительных члена, которые легко вычислить.

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1. Introduction

1.1. The general solution

For the general solution of the geodetic boundary problem, convenient also for high mountain test points, the following formula was found, [5] eq. (267),

$$(1) \quad \{T\} = \frac{1}{4\pi R} \iint\limits_{v} \left[\Delta g_T + C + C_1(M)\right] S(p)\, dv + \{\Omega(M)\} .$$

Here, T is the perturbation potential at the surface of the Earth u; the parentheses { } describe the fact that the share of the surface spherical harmonics of the degree $n=0$ and $n=1$ has to be split off. R is the radius of the mean globe v in ocean level, Fig. 1. Δg_T is the free-air anomaly of the perturbation potential T computed for the points at the surface of the Earth u,

$$(2) \quad \Delta g_T = - \partial T/\partial r - (2/r)\, T .$$

r is here the geocentric radius of u. C is the plane topographic reduction of the gravity. $C_1(M)$ is about the vertical gradient of the Bouguer-anomalies, [5] eq. (291) and (292), (see also chapter C of this publication),

$$(3) \quad C_1(M) = - Z \frac{\partial}{\partial H}\left(\Delta g_{Bouguer}\right) ,$$

or

$$(4) \quad C_1(M) = -Z \frac{R^2}{2\pi} \iint\limits_{l} \frac{(\Delta g_{Bouguer})_Y - (\Delta g_{Bouguer})_Q}{e_{00}^3}\, dl .$$

Further, in (1), S(p) is the Stokes function depending on the spherical distance p from the moving surface point Q to the fixed test point P, Fig. 1. $\Omega(M)$ has the following expression, [5] eq. (268),

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$$(5) \quad \Omega(M) = \Omega_1(M) + M\frac{H_P}{R} + [B]'' + \frac{1}{4\pi R}\iint_v C_2 \cdot S(p) \cdot dv.$$

As to the first term of the right hand side of (5), the formula for $\Omega_1(M)$ is given by the equation (224) of [5]. Concerning the development for $\Omega_1(M)$, please, confer to the equations (87) through (96) of the publication in hand, also. In the second term on the right hand side of (5), Fig. 1, H_P is the height of the test point P above the sphere v with the radius R, and, further, M can be approximated here in this term by, (see [5] eq. (271)),

$$(6) \quad M \cong T - f\vartheta \iint_v H_Q \cdot \frac{1}{e_0} \cdot dv.$$

In (5), the third term is $[B]''$. It is defined by eq. (248a) of [5], being the difference of the potential of the visible mountain masses B at the test point P, on the one hand, and the potential of these masses condensed at the globe v and computed for the point P* perpendicular below P, on the other hand, Fig. 1. It can be computed precisely by the formulae of eq. (82) through (88) of the chapter B of [3], and by the equation (68) of the chapter B of [4], too. The quantity of $[B]''$ turned out to be very small, [3] page 36.
In the last term on the right hand side of (5), the expression C_2 is described by eq. (266) of [5], (accounting for eq. (240) of [5]).
The formula (1) for $\{T\}$ refers to the T values along the surface of the Earth u. Thus, in (1), it is essential that the parentheses $\{\}$ demand that the surface spherical harmonics of degree n=0 and n=1 (contained in the T values distributed along the surface of the Earth u) are eliminated. After these terms are eliminated, we are confronted with the fact that many geodetic applications need not the elimination of these above terms, but, instead of them, the elimination of the <u>spatial</u> spherical harmonics of degree n=0 and n=1 in the <u>spatial</u> three-dimensional spherical harmonics representation of the T values in the exterior of the body of the

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Earth is required.
If we have freed the $\{T\}$ values of (1) from the surface spherical harmonics of n=0 and n=1, it is afterwards a little work only modifying these $\{T\}$ values in order to reach the situation where even the spatial spherical harmonics of n=0 and n=1 are eliminated, finally. The concerned mathematical transformations can be found in [5], chapter 6, eq. (115a) through (141v). The numerical quantities effecting this transition procedure will be rather small, probably, since in good approximation, the surface of the Earth is equal to a sphere.

1.2. The lowland solution

The above relation (1) solves the problem for all cases, also for test points situated in high mountains. By far in most cases, the test point P is situated in the lowland, in low mountain ranges, or on the oceans. Considering this lowland constraint, (7), the formula (1) simplifies enormously. The lowland condition is, Fig. 1,

$$(7)\quad [Z/e_0]^2 = [(H_Q - H_P)/e_0]^2 = x^2 \ll 1 ,$$

[5], eq. (225i). $H_Q - H_P$ is the height difference with regard to the test point P, e_0 is the distance from P. As to the meaning of the various symbols here applied, this meaning can be taken from Fig.1. Hence, the lowland variant of (1) has the following shape, (see [5], eq. (272); [6]),

$$(8)\quad \{T\} = \frac{1}{4\pi R} \iint_v \left[\Delta g_T + C + C_1(M)\right] \cdot S(p) \cdot dv + \left\{\Omega^*(M)\right\} ;$$

the formula for $\Omega^*(M)$ is with [5], eq. (273),

$$(9)\quad \Omega^*(M) = \Omega_1^*(M) + M \cdot \frac{H_P}{R} + [B]'' + \frac{1}{4\pi R} \iint_v C_2 \cdot S(p) \cdot dv .$$

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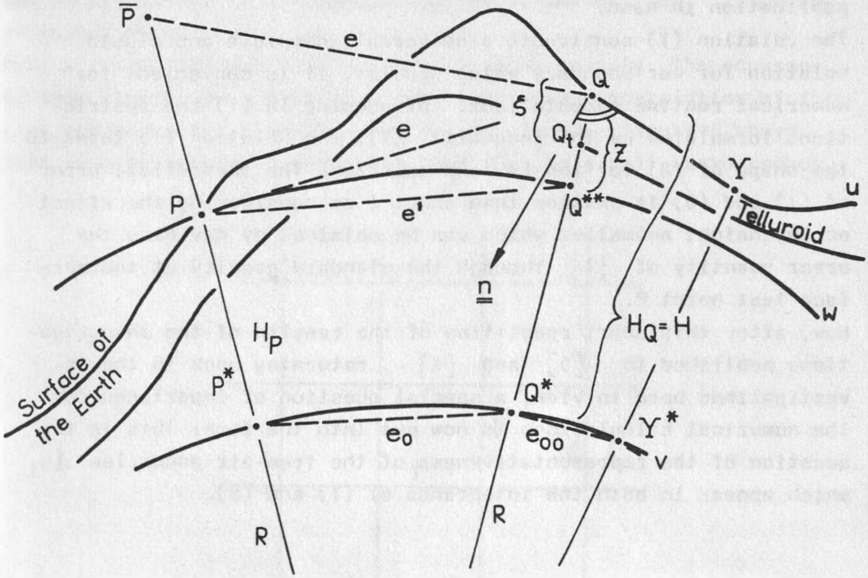

Fig. 1.

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As to the first term on the right hand side of (9), the detailed formula for the computation of $\Omega_1^*(M)$ was described by [5] , eq. (226) through (227c). See also eq. (98) through (103) of the publication in hand.
The relation (1) constructs a universal, complete and closed solution for our boundary value problem, it is convenient for numerical routine computations. Introducing in (1) the restrictions formulated by the inequality (7), the solution (1) turns to the shape of (8) for the lowland solution. The theoretical error of (1) and (8) is smaller than about 1 cm considering the effect on the height anomalies which can be obtained by deviding the error quantity of $\{T\}$ through the standard gravity at the surface test point P.
Now, after this short repetition of the results of the investigations published in [5] and [6] , returning back to the investigations here in view, a special question of importance for the numerical calculations is now put into the fore: This is the question of the representativeness of the free-air anomalies Δg_T which appear in both the integrands of (1) and (8).

1.3. The representativeness of the free-air anomalies

In the formulas (1) and (8), the integral

$$(10) \quad X = (1/(4\pi R)) \iint_v \Delta g_T \cdot S(p) \cdot dv$$

does appear. The more smoothed the free-air anomalies Δg_T the easier the computation of the integral (10) for X, this matter is obvious. Or, speaking with other words, the better the representativeness of the free-air anomalies Δg_T the easier the numerical evaluation of the integral (10) in order to find the X value. Along the oceans and in the lowlands, the free-air anomalies have a rather good representativeness, it is well-known. But, in hilly and mountainous areas, this good representativeness of the free-air anomalies is lost. In the mountains, these anomalies show a clear

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linear correlation with the topographical heights, H. Thus, the first impression may come up that the hilly and mountainous areas demand a relative dense net of gravity stations, counteracting the bad representativity of these anomalies in these areas, - a very expensive affair.
But, a remedy against this handicap is found easily. The scource of this remedy comes even from the clear linear correlation of the free-air anomalies with the heights, H, already discussed above. This is a well-known correlation, and this is a well-known remedy.

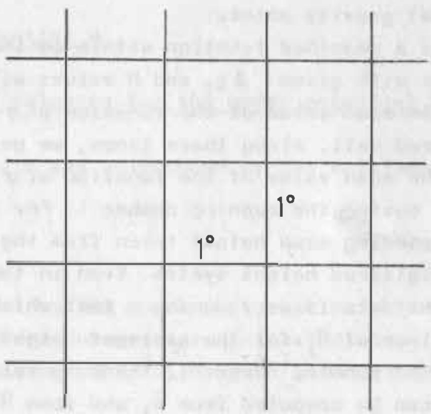

Fig. 2.

In this context, a square grid is laid over the mountainous area considered. The grid cells may have a side length of $1^{o} \times 1^{o}$, perhaps, (see Fig. 2). For the interior of such a cell, we have the well-known relation

$$(11) \quad \Delta g_T = a(\varphi , \lambda) + b \cdot H \quad .$$

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Within a cell of $1^o \times 1^o$ size, b is taken as a constant; in most cases, we have $b \cong 0.1$. The free-air anomalies Δg_T and the function $a(\varphi, \lambda)$ depending on the latitude and longitude are measured in 10^{-3} cm sec^{-2}, (mgal). The heights H are taken in meters. The essence of the relation (11) lies in the fact that $a(\varphi, \lambda)$ is a rather smoothed function within an individual cell, $a(\varphi, \lambda)$ can be computed from the free-air anomalies and the heights by

$$(12) \quad a(\varphi, \lambda) = \Delta g_T - b \cdot H \quad ,$$

for the individual gravity points.
Since $a(\varphi, \lambda)$ is a smoothed function within an individual cell, some few stations with given Δg_T and H values will suffice for finding a reliable mean value of the function $a(\varphi, \lambda)$ averaged over the considered cell. Along these lines, we can find the mean value $\bar{a}_i$ being the mean value of the function $a(\varphi, \lambda)$ for the considered cell, having the running number i. For the same cell, $\bar{H}_i$ is the corresponding mean height taken from the topographical maps or from a digitized height system. Even in the mountains, this net of height data is very dense, a fact which allows finding reliable values of $\bar{H}_i$ for the averaged heights. Hence, (11), for the cell of the running number i, the mean value of the gravity anomaly can be computed from $\bar{a}_i$ and from $\bar{H}_i$ by the formula (13),

$$(13) \quad (\overline{\Delta g_T})_i = \bar{a}_i + b_i \cdot \bar{H}_i \quad .$$

The value of b_i for the cell of the running number i is determined in such a way that the amounts of $a(\varphi, \lambda)$ within this cell have no more any correlation with the heights H.
The relation (13) can be inserted into (10). The integration of (10) can be transformed into a summation. Along these lines, the relation (10) turns to

$$(14) \quad X = (1/(4\pi R)) \cdot \Delta v \sum_i \bar{a}_i \cdot (S(p))_i +$$
$$+ (1/(4\pi R)) \cdot \Delta v \sum_i b_i \cdot \bar{H}_i \cdot (S(p))_i \quad .$$

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$\bar{a}_i$ comes from an averaging over the smoothed values $a(\varphi, \lambda)$, the amount of $\bar{H}_i$ is precisely computed from the topographical maps. Thus, finally, in (14) there is no more any trouble with an averaging over too few free-air anomalies of bad representativity in the mountains.
But now, in the subsequent investigations, we follow another way which leads to a second remedy against the bad representativeness of the free-air anomalies in hilly and mountainous areas: This is the way which uses isostatic anomalies of the gravity.

2. The model potential M

2.1. The universal solution for the model potential M

The model potential M was introduced by the relation

(15) $M = T - B$,

[5] eq. (145). T is the usual perturbation potential and B is the potential of the mountain masses situated above sea level (the mass density being $\vartheta = 2670 \text{ kg m}^{-3}$, some authors prefer 2650 kg m^{-3}).
In the exterior of the body of the Earth, the spatial function for M fulfills the Laplace differential relation, as the function for T and B do.

(15a) $$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} + \frac{\partial^2 M}{\partial z^2} = 0 ;$$

x, y, z: Spatial Cartesian
co - ordinates.
We have, [5] eq. from (148) through (152),

(16) $$\Delta g_M = - \partial M / \partial r - (2/r) M ,$$

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$$(17) \quad \Delta g_M = \Delta g_T - \Delta g_B \quad ,$$

$$(18) \quad \Delta g_T = -\partial T/\partial r - (2/r)\cdot T = (g)_Q - (\gamma)_{Q_t} ,$$

$$(19) \quad \Delta g_B = -\partial B/\partial r - (2/r)B \quad ,$$

$$(20) \quad \Delta g_M = -\partial T/\partial r + \partial B/\partial r - (2/r)\cdot(T-B).$$

The above 5 lines are self-explanatory. In (18), $(g)_Q$ is the observed gravity at the surface point Q, and $(\gamma)_{Q_t}$ is the standard gravity at the telluroid point Q_t perpendicular below Q, Fig. 1. The distance between Q and Q_t is the height anomaly ζ. Our model potential M fulfills the following relation, [5] eq. (223),

$$(21) \quad \{M\} = \frac{1}{4\pi R'} \iint\limits_{w} \left[\Delta g_M + C_1(M)\right]\cdot S(p)\cdot dw + \{\Omega_1(M)\} \quad .$$

R' is the geocentric radius of the test point P,

$$(22) \quad R' = R + H_P \quad .$$

w is the ball of the radius R', (see Fig. 1). Δg_M is described by (16), and $C_1(M)$ is given by (3) and (4). As to $\Omega_1(M)$, the reader is asked to refer to eq. (87) and (88) of the publication in hand and to [5] eq. from (224) through (225h).

2.2. The lowland solution for the model potential M

By the lowland condition (7), the formula (21) turns to its lowland variant, [5] eq. (231),

$$(23) \quad \{M\} = \frac{1}{4\pi R'} \iint\limits_{w} \left[\Delta g_M + C_1(M)\right]\cdot S(p)\cdot dw + \{\Omega_1^*(M)\} \quad .$$

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As to the meaning of $\Omega_1^*(M)$, the reader is asked to refer to eq. (98) and (99) of the publication in hand, and to [5], eq. (230) and (226) through (227c). The formula for $\Omega_1^*(M)$ is much more short and much more easy to compute than that for $\Omega_1(M)$.

3. The perturbation potential T

3.1. The universal solution for the perturbation potential T

In the formula (21), the model potential M can be substituted by the perturbation potential T, because both of them obey the Laplace differential equation, and because both of them have about the same structure and amounts of about the same order. Hence, T has the universal formula

$$(24) \quad \{T\} = \frac{1}{4\pi R'} \iint_w \left[\Delta g_T + C_1(T)\right] \cdot S(p) \cdot dw + \{\Omega_1(T)\} .$$

In (24), M was replaced by T. Δg_T comes from (18). $C_1(T)$ is explained by (3) (4) of the publication in hand, or, better, by [5] eq. from (278) through (284), and by [5] eq. (290). There, in [5], we found in good approximation, replacing M by T, $(Z=H_Q - H_p)$,

$$(24a) \quad C_1(T) \cong Z \cdot \frac{\partial^2 T}{\partial z^2} ,$$

neglecting $C_{1.b}(T)$, [5] eq. (290); see also eq. (69)(84) of the publication in hand. In (24), $\Omega_1(T)$ is found by (87) (88) of the publication in hand, or by [5] eq. (224) and (225).

3.2. The lowland solution for the perturbation potential T

In that way that transforms from (21) to (24), the lowland solution for T follows from (23). We have

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(25) $$\{T\} = \frac{1}{4\pi R^T} \iint_w \left[\Delta g_T + C_1(T)\right] S(p)\, dw + \left\{\Omega_1^*(T)\right\} .$$

$\Omega_1^*(T)$ is found by [5] eq. (230) (227), substituting M by T.

4. The potential B_{iso} of the Airy-Heiskanen isostatic system

4.1. The potential B_{iso} in terms of the masses

Now, we have to think back to the traditional isostatic system of Airy-Heiskanen. In this context, first of all, it seems to be advisable to recapitulate the main ideas inherent in this system. In the center of our retrospect lies the isostatic reduction of the gravity values transporting them from the surface of the Earth u, down to the geoid.

This topographic - isostatic reduction removes the gravitational effect exerted on the surface gravity value g by the mountain masses and by their roots, and by the oceans, and by their anti-roots. Further, this topographic - isostatic reduction involves also the free-air reduction of the gravity which accompanies a vertical shift of the point Q from the surface u down to the point Q′ on the geoid, Fig. 3, Fig. 1,

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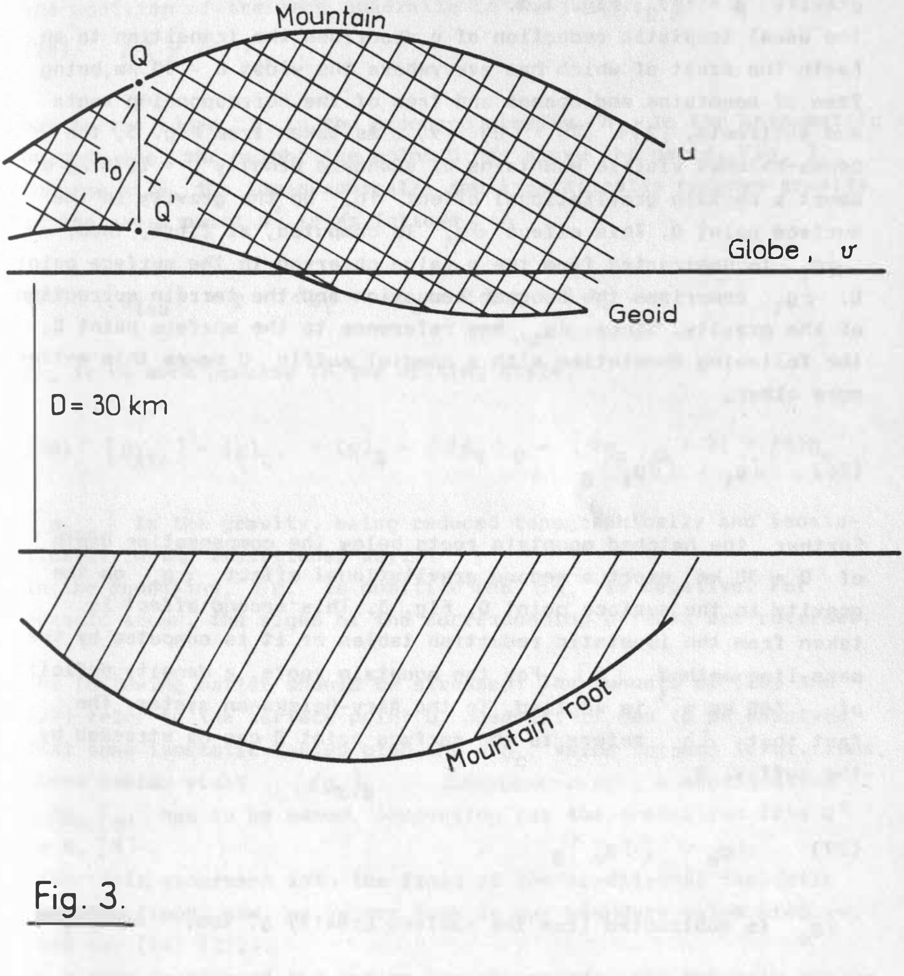

Fig. 3.

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The running point Q on the surface of the Earth u has the observed gravity $g = (g)_Q$; Fig. 1,3.
The usual isostatic reduction of g describes the transition to an Earth the crust of which has everywhere the widst D = 30 km, being free of mountains and oceans and free of the corresponding roots and antiroots, [2] [7] [8] [9] . As taken from Fig. 3, the cross-hatched visible mountains of standard density $\vartheta = 2670\ \mathrm{kg\ m^{-3}}$ exert a certain gravitational effect δg_t on the gravity in the surface point Q. This effect δg_t is computed, at first. Than, δg_t is subtracted from the g value observed in the surface point Q. δg_t comprises the Bouguer reduction and the terrain correction of the gravity. Since δg_t has reference to the surface point Q, the following denotation with a special suffix Q makes this matter more clear,

(26) $$\delta g_t = (\delta g_t)_Q \quad .$$

Further, the hatched mountain roots below the compensation depth of D = 30 km exert a second gravitational effect δg_c on the gravity in the surface point Q, Fig. 3. This second effect is taken from the isostatic reduction tables or it is computed by the mass-line method, [8] . For the mountain roots, a density deficit of $-600\ \mathrm{kg\ m^{-3}}$ is applied, in the Airy-Heiskanen system. The fact that δg_c refers to the surface point Q can be stressed by the suffix Q,

(27) $$\delta g_c = (\delta g_c)_Q \quad .$$

δg_c is subtracted from the surface gravity g, too, - as δg_t.

Finally, as the third step, the point Q is subsided downwards in vertical direction, down to the point Q' on the geoid, Fig. 3. The accompanying gravity change is approximated by the standard value of the free-air reduction, according to the instructions which can be found in the text books on isostasy, [8] . Hence,

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the amount of (as to (28), more precise considerations require the addition of the term quadratic in the height h_o)

$$(28) \quad 2\cdot(\gamma/R)\cdot h_o$$

has to the added. γ is the standard gravity. h_o is the orthometric height, i.e. the height the point Q has above the geoid, Fig. 3. Consequently, the topographically and isostatically reduced gravity at the geoid point Q' is as follows,

$$(29) \quad [g_{iso}] = g - \delta g_t - \delta g_c + 2\cdot(\gamma/R)\cdot h_o .$$

Or, to be more precise in the writing style,

$$(30) \quad [g_{iso}] = (g)_{Q'} = (g)_Q - (\delta g_t)_Q - (\delta g_c)_Q + 2(\gamma/R)h_o .$$

$[g_{iso}]$ is the gravity, being reduced topographically and isostatically in the traditional way.
In the mountains, δg_t is positive and δg_c is negative. For oceanic areas, the signs of the corresponding effects are reversed.

The following matter should be stressed: The amounts of (26) and (27) refer to the surface point Q. However, it has to be observed that some isostatic tables give the δg_c value for sea level. Thus, these tables yield $(\delta g_c)_{Q'}$. Supplementary, a modification of $(\delta g_c)_{Q'}$ has to be added, accounting for the transition from Q' to Q, [8].
After this excursion into the field of the traditional isostatic considerations, now, we return back to our boundary value problem, (see eq. (24) (25)).
As a main feature of the coming investigations, the harmonic potential B_{iso} is considered in the exterior of the Earth and on the surface of the Earth u. It is the gravitational potential generated by the following 4 scources, Fig. 4, [8]:

1. The mass surplus of the visible mountains, having the density surplus $\delta\vartheta_1$, and filling the volume V_1.
2. The mass defect of the oceans, having the density defect $\delta\vartheta_2$, and filling the volume V_2.

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3. The mass defect of the continental mountain roots, having the density defect $\delta\vartheta_3$ and filling the volume V_3.
4. The mass surplus of the oceanic antiroots, having the density surplus of $\delta\vartheta_4$ and covering the volume V_4.

These densities here implied have the subsequent values:

(31) $\delta\vartheta_1 = +\ 2670\ \mathrm{kg\ m^{-3}}$,

(32) $\delta\vartheta_2 = -\ (2670 - 1027)\ \mathrm{kg\ m^{-3}}$,

(33) $\delta\vartheta_3 = -\ (3270 - 2670)\ \mathrm{kg\ m^{-3}}$,

(34) $\delta\vartheta_4 = \ (3270 - 2670)\ \mathrm{kg\ m^{-3}}$.

The volume V_1 has the running point J_1 in its interior. The analogous property is valid for V_2 and J_2, V_3 and J_3, and V_4 and J_4; Fig. 4.
In the coming derivations, the volume element is expressed by

(35) $dV = r^2 \cdot \sin p \cdot dr \cdot dp \cdot dA.$

The mass element around the running point J_j, $(j = 1, 2, 3, 4)$, has the following expression, Fig. 4,

(36) $dm_j = \delta\vartheta_j \cdot dV, \ (j = 1, 2, 3, 4).$

In sequence, the suffix j of V_j and J_j, $(j = 1, 2, 3, 4)$, is also assigned to the corresponding expressions $\delta\vartheta_j$ of (31) (32) (33) (34), one after the other.
Hence, in the exterior point $\bar{Q}$, the potential B_{iso}, now in the fore, is computed in the subsequent way, (37), Fig. 4,

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$$(37) \quad (B_{iso})_{\bar{Q}} = f \sum_{j=1}^{4} \delta\vartheta_j \iiint_{V_j} \frac{1}{\bar{e}\,(\bar{Q}, J_j)}\, dV.$$

$\bar{e}\,(\bar{Q}, J_j)$ is the distance between the exterior point $\bar{Q}$ and the running point J_j with the mass element dm_j situated in the volume V_j, (36); Fig. 4. V_j is the volume having the density $\delta\vartheta_j$.

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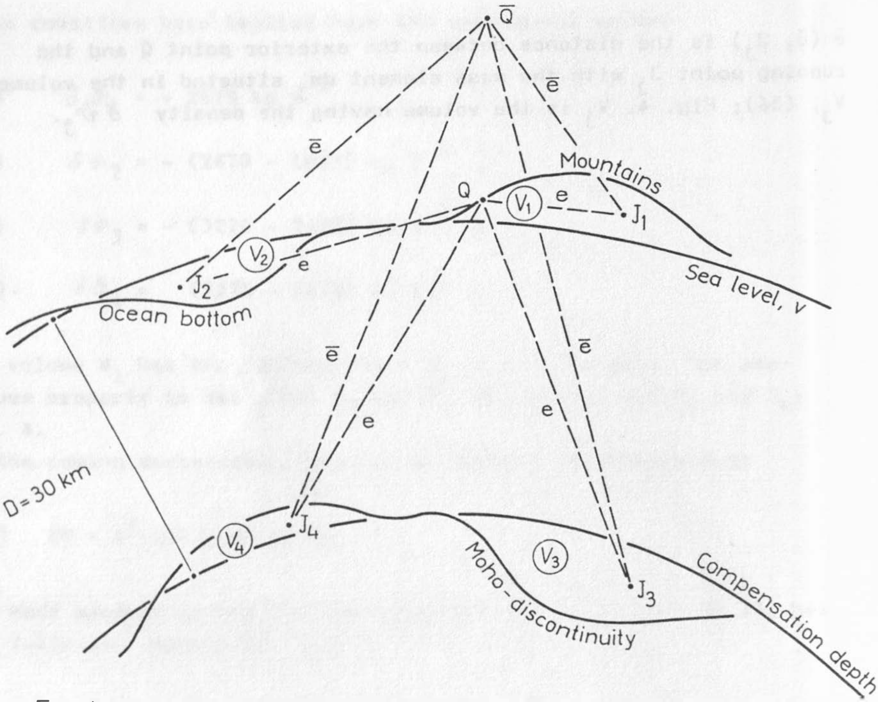

Fig. 4.

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Now, the exterior point $\bar{Q}$ subsides down to the surface of the Earth u reaching the point Q, Fig. 4. In this procedure, the distance $\bar{e}\,(\bar{Q}, J_j)$ turns to the distance $e\,(Q, J_j)$, (j= 1, 2, 3, 4); (see Fig. 4). Consequently, (37) changes to (38),

$$(38)\qquad (B_{iso})_Q = f \cdot \sum_{j=1}^{4} \delta\vartheta_j \cdot \iiint_{V_j} \frac{1}{e\,(Q,\,J_j)} \cdot dV.$$

For our investigations, we need the potential B_{iso}, just as the radial derivative of it

$$(39)\qquad \partial B_{iso} \,/\, \partial r \quad ,$$

and, just as the gravity anomaly, (19),

$$(40)\qquad \Delta g_{B_{iso}} = -\ \partial B_{iso} \,/\, \partial r\ -\ (2/r)\cdot B_{iso}.$$

All these 3 values have to be computed for points on the surface of the Earth u having the radius r, Fig. 1,

$$(41)\qquad r = R + H_Q \quad .$$

(37) and (38) give

$$(42)\qquad \frac{\partial B_{iso}}{\partial r} = f \cdot \sum_{j=1}^{4} \delta\vartheta_j \cdot \iiint_{V_j} \left\{ \frac{\partial \dfrac{1}{\bar{e}\,(\bar{Q},\,J_j)}}{\partial r} \right\}_{\bar{Q}=Q} \cdot dV,$$

or, abbreviating,

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(42a) $$\frac{\partial}{\partial r} B_{iso} = f \cdot \sum_{j=1}^{4} \delta \vartheta_j \cdot \iiint_{V_j} \frac{\partial \frac{1}{e(Q, J_j)}}{\partial r} \cdot dV .$$

Further,

(43) $$\Delta g_{B_{iso}} = - f \cdot \sum_{j=1}^{4} \delta \vartheta_j \cdot \iiint_{V_j} \frac{\partial \frac{1}{e(Q, J_j)}}{\partial r} \cdot dV -$$

$$- f \cdot \frac{2}{r} \cdot \sum_{j=1}^{4} \delta \vartheta_j \cdot \iiint_{V_j} \frac{1}{e(Q, J_j)} \cdot dV.$$

Here, in (42) (42a) (43), the derivative

(44) $$\frac{\partial}{\partial r} \left[1/e(Q, J_j) \right]$$

is reached by the radial derivation of

(45) $$1/\bar{e}(\bar{Q}, J_j)$$

and by the subsequent transition from the exterior point $\bar{Q}$ to the point Q situated on the surface of the Earth u, (42).

Comparing (26) and (42a), the relation (46) is obtained,

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$$(46)\qquad \delta g_t = -f\cdot\sum_{j=1}^{2}\delta\vartheta_j\cdot\iiint_{V_j}\frac{\partial\frac{1}{e(Q,J_j)}}{\partial r}\cdot dV\ .$$

A look on (27) and (42a) shows that (47) is right,

$$(47)\qquad \delta g_c = -f\cdot\sum_{j=3}^{4}\delta\vartheta_j\cdot\iiint_{V_j}\frac{\partial\frac{1}{e(Q,J_j)}}{\partial r}\cdot dV\ .$$

Hence, from (42a) (46) (47),

$$(48)\qquad \left(\frac{\partial}{\partial r}B_{iso}\right)_Q = -\left(\delta g_t\right)_Q - \left(\delta g_c\right)_Q\ .$$

The values on the right and left hand side of (48) refer to the surface points.

4.2. The universal solution for the potential B_{iso}

Some properties of the potential B_{iso} can be confronted with some properties of the perturbation potential T of (24) and (25): In the exterior of the body of the Earth, B_{iso} is harmonic as T. B_{iso} and T are continuous functions. Further, in the exterior and on the surface u of the Earth, B_{iso} has about the same order as the perturbation potential T.

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Hence, in (24), B_{iso} can be used as a substitute for T. In doing so, (24) changes to (49),

$$(49) \quad \{B_{iso}\} = \frac{1}{4\pi R'} \cdot \iint\limits_{w} \left[\Delta g_{B_{iso}} + C_1(B_{iso})\right] \cdot S(p) \cdot dw + \{\Omega_1(B_{iso})\} .$$

$\Delta g_{B_{iso}}$ is explained by (40), and even these values of (40) are understood that they are distributed on the Earth´s surface u, Fig. 1.

$C_1(B_{iso})$ of (49) needs no separate detailed discussion, since, later on, this term disappears. It is merged in the term $C_1(T-B_{iso}) = C_1(I)$, (see eq. (52)). $C_1(B_{iso})$ and $C_1(T)$ are combined into the one term $C_1(I)$. Certain, this term $C_1(I)$ and the numerical calculation of it is thoroughly discussed by the equations from (51) through (86), later on.

$\Omega_1(B_{iso})$ comes from the equation (224) of the former publication [5], replacing M by B_{iso}.

4.3. The lowland solution for the potential B_{iso}

The lowland equation for the potential B_{iso} is derived from (25), in a similar way as (49) was obtained from (24). Consequently, we have

$$(50) \quad \{B_{iso}\} = \frac{1}{4\pi R'} \cdot \iint\limits_{w} \left[\Delta g_{B_{iso}} + C_1(B_{iso})\right] \cdot S(p) \cdot dw + \{\Omega^*_1(B_{iso})\} .$$

$\Omega^*_1(B_{iso})$ derives from the equation (230) of [5], replacing M by B_{iso}. In (50), the lowland condition (7) is effective. The question how to find $\Delta g_{B_{iso}}$ and $C_1(B_{iso})$ was already discussed in connection with eq. (49).

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5. The superposition of the two potentials T and B_{iso}

5.1. The superposition of the universal solutions for T and B_{iso}

Considering the fact that the free-air anomalies have not very smoothed values in the mountains and in the Mittelgebirge, we leave now the free-air anomalies in order to reach a system of anomalies which has smoothed values. But now, we do not prefer the smoothing procedure connected with the equations from (10) through (14). Instead of it, now, we change over to the isostatic anomalies of the gravity. Within this procedure, the relation (49) is subtracted from the relation (24). Thus, the equation (51) is found, [1],

$$(51) \quad \{T\} - \{B_{iso}\} = \frac{1}{4\pi R'} \cdot \iint_{w} \left[\Delta g_T - \Delta g_{B_{iso}} + C_1(I)\right] \cdot S(p) \cdot dw + \{\Omega_1(I)\} .$$

In (51), the subsequent relation (52) is inserted,

$$(52) \quad I = T - B_{iso} .$$

With (3) (4), and considering the relations (269) (278) (279) (280) of [5] (replacing M by T, $C_1(I)$ is found to be linear in I. Thus,

$$(53) \quad C_1(T) - C_1(B_{iso}) = C_1(T - B_{iso}) = C_1(I) .$$

Further, accounting for (224) and (225) of [5], $\Omega_1(I)$ is linear in I, too.

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Hence,

$$(54) \qquad \Omega_1(T) - \Omega_1(B_{iso}) = \Omega_1(T - B_{iso}) = \Omega_1(I) \quad .$$

The relations (53) and (54) were respected in the derivation of (51).

The essential of the equation (51) is the fact that the Δg_T values, being rugged in the mountains, are now replaced by the values of

$$(55) \qquad \Delta g_T - \Delta g_{B_{iso}} = \Delta g_I \quad .$$

These values of (55) are very smoothed, also in the mountains. By the superposition, (52), we came away from rugged gravity anomalies. The anomalies of (55) are in close vicinity to the isostatic anomalies of the gravity, this matter is discussed in chapter 6, later.

Thus,

$$(56) \qquad \Delta g_T - \Delta g_{B_{iso}} = \Delta g_I \cong [\Delta g_{iso}] \quad .$$

$[\Delta g_{iso}]$ are the traditional isostatic anomalies of the gravity. The relation (56) and the precise shape of it are also discussed later, in paragraph 6, from page 31 through page 35.

5.2. The superposition of the lowland solution for T and B_{iso}

From (25) and (50) , the relation (57) for the lowland solution follows,

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(57) $$\{T\} - \{B_{iso}\} = \frac{1}{4\pi R^2} \cdot \iint_{w} \left[\Delta g_T - \Delta g_{B_{iso}} + C_1(I)\right] \cdot S(p) \cdot dw + $$

$$+ \left\{ \Omega_1^*(I) \right\} .$$

$\Omega_1^*(I)$ derives from the equations (230) (227) of [5]. These relation of [5] are linear in I. See also the equations (98) (99) of the publication in hand. Thus, (52) (53) (54),

(58) $$\Omega_1^*(T) - \Omega_1^*(B_{iso}) = \Omega_1^*(T-B_{iso}) = \Omega_1^*(I).$$

The relation (57) has the essential property that the smoothed anomalies $\Delta g_T - \Delta g_{B_{iso}}$ appear, instead of the Δg_T anomalies which are rugged in the mountains.

6. The isostatic gravity anomalies

Now, the relations (55) and (56) are in the fore. The free-air anomaly Δg_T, appearing in these formulas, can be computed by the observed gravity at the surface point Q, $g = (g)_Q$, and by the standard gravity $(\gamma)_{Q_t}$ at the telluroid point Q_t perpendicular below of Q, (18).

(59) $$\Delta g_T = (g)_Q - (\gamma)_{Q_t} .$$

(59) is an often used elementary formula, (18); it can be brought into the following shape,

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(60) $\Delta g_T = (g)_Q - \gamma_o + 2(\gamma/r)h_n$.

γ_o is the standard gravity at the level ellipsoid, and h_n is the normal height of the point Q_t above the level ellipsoid (mean Earth ellipsoid). (40) and (48) yield (at the surface point Q)

(61) $\Delta g_{B_{iso}} = \delta g_t + \delta g_c - (2/r)\, B_{iso}$.

The relation (61) is understood that it refers to the points Q situated on the surface of the Earth u, Fig. 1.
In this elaboration in hand, we define the isostatic anomaly in the following way, in view of (68),

(62) $\Delta g_{iso} = (g)_Q - \delta g_t - \delta g_c + 2(\gamma/r)\, h_n - \gamma_o$.

The term quadratic in h_n can be added to $2(\gamma/r)\, h_n$, in (62). However, the traditional isostatic anomaly is as given by (63); (see (29) (30)), (see also [8]).

(63) $[\Delta g_{iso}] = [\, g_{iso}] - \gamma_o =$

$= (g)_Q - \delta g_t - \delta g_c + 2(\gamma/R)\, h_o - \gamma_o$.

The term quadratic in h_o can be added to $2(\gamma/r)\, h_o$, in (63). The difference between (62) and (63) comes from the difference between the normal and the orthometric heights,(30) (62).With (6o) and (61), the crucial anomaly (given by (64))

(64) $\Delta g_T - \Delta g_{B_{iso}}$

of (51), (55) and (57) has the following expression

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$$(65)\quad \Delta g_T - \Delta g_{B_{iso}} = (g)_Q - \gamma_0 + 2(\gamma/r)\cdot h_n -$$

$$- \delta g_t - \delta g_c + (2/r)\cdot B_{iso} \ .$$

Comparing (62) with (65), the subsequent important relation follows, in view of (51) (57),

$$(66)\quad \Delta g_T - \Delta g_{B_{iso}} = \Delta g_I = \Delta g_{iso} + (2/r)\cdot B_{iso} \ .$$

As to the second term on the right hand side of (66), the expression $- B_{iso}$ is the change the potential at the surface point Q undergoes by the removal of the mountain masses and their mountain roots (and the mass defect of the oceans and their antiroots).

In the traditional theory of the isostatic gravity anomalies, there appears also the indirect or Bowie effect exerted on the gravity anomaly by the potential change of $- B_{iso}$, [8]. This effect refers to the geoid level, it has the shape, [8],

$$(67)\quad (2/R)\cdot B_{iso} \ .$$

The second term on the right hand side of (66) is in very close neighborhood to this Bowie effect, (67), obviously.

Finally, it seems to be useful to stress again the fact that the isostatic anomalies Δg_{iso} of (62) are much more smoothed and much more representative than the free-air anomalies Δg_T of the gravity.

In this context, we present Table 1. For certain stations in the area of the Alps, Table 1 represents the position (φ, λ), the height h, the free-air anomaly, and the isostatic anomaly of the

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Airy-Heiskanen system (of 30 km compensation depth), [8]. Obviously, the isostatic anomalies are much more smoothed and representative than the free-air anomalies. This is a well-known fact which the publication in hand makes use of with advantage.

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Table 1

Gravity anomalies in the Alps

Station	φ	λ	H, m	Anomalies: Free-air, 10^{-3} cm s^{-2}	Isostatic Airy-Heiskanen System, D=30 km, 10^{-3} cm s^{-2}
Campiglio	46°13.′9	10°51.4	1530	+ 58	+ 16
Ober-Drauburg 1	46 44.9	12 58	618	- 38	+ 33
Greifenburg 2	46 45.1	13 11	632	- 36	+ 21
Sandbüchel	46 45.3	11 01.8	2967	+116	- 44
S. Leonardo	46 48.7	11 16.4	655	-107	- 4
Lienz 1	46 50.0	12 46	674	- 51	+ 16
Möllbrücken	46 50.3	13 22	556	- 42	+ 28
Hochstradenkogl	46 50.8	15 56	607	+ 69	+ 38
Iselsberg	46 51.4	12 52	1198	+ 19	+ 22
Sterzing	46 53.9	11 26	950	- 75	- 17
Weissenbachscharte	47 01.4	10	2196	+ 71	- 30
Sonnblick	47 03.4	12 58	3099	+143	- 24
Steinach	47 05.4	11 28.4	1050	- 76	- 31
Bucheben	47 09.5	12 58	1062	- 68	- 33
Innsbruck 1	47 15.7	11 24.3	584	-127	- 44
Mixnitz	47 19.8	15 22	445	- 46	- 8
Bruck an der Mur	47 24.6	15 15	487	- 19	+ 8
Wörgl	47 29.5	12 03.9	508	-108	- 45
Semmering	47 38.0	15 50	986	+ 70	+ 26
Benediktbeuern	47 42.5	11 24.1	618	- 37	- 20
Hohenpeissenberg	47 48.1	11 00.9	996	+ 4	- 18
Wiener Neustadt 1	47 48.5	16 15	270	- 13	+ 1
Kaufbeuren	47 52.8	10 38	680	- 16	- 17

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7. The final solution

7.1. The universal final solution

The combination of (51) and (66) yields the final solution of our boundary value problem developed in terms of the isostatic anomalies Δg_{iso} , (62). This formula is universally valid, also in high mountains and in the Mittelgebirge, [1] .

$$(68)\quad \{T\} = \frac{1}{4\pi R'} \iint_w \left[\Delta g_{iso} + \frac{2}{r} \cdot B_{iso} + C_1(I) \right] \cdot S(p) \cdot dw + \{B_{iso}\} + \{\Omega_1(I)\} .$$

For the introduction in (68), B_{iso} can be computed by (38) inserting the densities of (31) (32) (33) (34). The T value on the left hand side of (68) refers to the surface u.

The potential I was described by (52).

The terms $C_1(I)$ and $\Omega_1(I)$ - produced by our here developed precise theory - construct in detail the refinements of the traditional computation of the T values in terms of the isostatic anomalies. By these refinements, the theoretical error of the resulting height anomaly T/γ gets smaller than about 1 cm.

As to the calculation of $C_1(I)$ by [5] eq. (269), $C_1(I)$ derives from the deflections of the vertical in the potential field I, substituting M by I. These deflections in the potential field I are denominated by α_1 and α_2. The potential I comes from (52) of the publication in hand.

Thus, [5] eq. (269), substituting M, μ_1, μ_2 by I, α_1, α_2,

$$(69)\quad C_1(I) = GZ \cdot \left[\frac{\partial \alpha_1}{R' \cdot \partial\varphi} + \frac{\partial \alpha_2}{R' \cos\varphi \cdot \partial\lambda} - \frac{\tan\varphi}{R'} \alpha_1 \right] .$$

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G is the global mean gravity, and $Z = H_Q - H_p$ is the height difference with regard to the test point P. α_1 and α_2 have the following equations, [5] eq. (153) (154) (156),

(70) $$\alpha_1 = - \frac{1}{R'+Z} \cdot \frac{1}{g^*} \cdot \frac{\partial I}{\partial \varphi} ,$$

(71) $$\alpha_2 = - \frac{1}{R'+Z} \cdot \frac{1}{g^*} \cdot \frac{1}{\cos \varphi} \cdot \frac{\partial I}{\partial \lambda} ;$$

with

(72) $$g^* = \left| \nabla (U+I) \right| = \left| \mathrm{grad}(U+I) \right| .$$

U is the standard potential. The values of T, U, I, α_1, α_2, $R'+Z$, g^*, $\partial I / \partial \varphi$, and $\partial I / \partial \lambda$ refer to the surface of the Earth u.

In order to express the amount of $C_1(I)$ in terms of the isostatic anomalies (62), principally, the ideas applied in [5], eq. from (274) through (292), can be used also here, (see also chapter C). Thus, we have in a self-explanatory way,

(73) $$C_1(I) = C_{1.a}(I) + C_{1.b}(I) ,$$

(74) $$C_{1.a}(I) = GZ \left[\frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_2}{\partial y} \right] ,$$

x and y are horizontal coordinates.

(75) $$C_{1.a} = - Z \left[\partial^2 I / \partial x^2 + \partial^2 I / \partial y^2 \right] .$$

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Introducing the Laplace differential equation, we have

(76) $$C_{1.a} = Z \cdot \left[\partial^2 I / \partial z^2 \right] ,$$

z is the vertical coordinate.

(77) $$C_{1.b} = C_{1.b.1} + C_{1.b.2} ;$$

the detailed developments for the two terms on the right hand side of (77) yield

(78) $$C_{1.b} = - Z \cdot \left[\frac{\partial^2 I}{\partial x \partial z} \tan \nu_x + \frac{\partial^2 I}{\partial y \partial z} \tan \nu_y \right] ,$$

(78a) $$\frac{dz}{dx} = \tan \nu_x , \frac{dz}{dy} = \tan \nu_y .$$

ν_x and ν_y is the slope of the terrain in the north-south and in the east-west direction. (62) and (66) give (79), anticipating (85) and (86), (see also [5] , eq. (274)),

(79) $$\frac{\partial I}{\partial r} = \frac{\partial I}{\partial z} = - \Delta g^*_{iso} = - \Delta g_{iso} - (2/r) \cdot T .$$

The new symbol of (79) is Δg^*_{iso} . It denotes the modified isostatic anomalies, modified according to (79), modified by the addition of (2/r) T.
The details of (79) will be derived later, below, by (85) (86).
The first term on the right hand side of (78) gives with (79), for a north-south profile,

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(80) $C_{1.b.1.} = Z \cdot \frac{(\Delta g^*_{iso})_o - (\Delta g^*_{iso})_u}{\Delta x} \cdot \frac{(H)_o - (H)_u}{\Delta x}$.

The suffix $(\ )_o$ and $(\ )_u$ refer to the end points of the considered north-south profile of the length Δx.

Here is

(80a) $\Delta x = (x)_o - (x)_u > o$,

(80b) $(x)_o > (x)_u$.

$C_{1.b.2.}$ follows in a similar way as $C_{1.b.1.}$, exchanging x for y. For the amount of $C_{1.b.1}$ and $C_{1.b.2}$ expressed by Bouguer anomalies, we found in [5] , eq. (290), for the extreme conditions in the Swiss Alps

(81) $C_{1.b.1} \cong 0.02 \cdot 10^{-3}$ cm $s^{-2} \cong 0$;

$C_{1.b.2}$ in terms of the Bouguer anomalies will have a similar amount, in the area of the Swiss Alps.

On the oceans, in the lowlands, and in the Mittelgebirge, the amounts of $C_{1.b.1}$ and $C_{1.b.2}$ in terms of the Bouguer anomalies will be much more small than (81), sure.

But now, the amounts of $C_{1.b.1}$ and $C_{1.b.2}$ expressed by the isostatic anomalies Δg^*_{iso} are in the fore, (80). These amounts are evaluated by a small test computation carried out in the profile of E. Holopainen, ([8] , page 194, Fig. 7-1). This profile crosses the Alps from Trieste to Salzburg, about. Hence, we have an extreme mountainous area. By a short computation of $C_{1.b.1}$ according to (80), in terms of isostatic gravity anomalies, for $C_{1.b.1}$ an absolute amount which is by far smaller than $0.02 \cdot 10^{-3}$ cm s^{-2}, i.e. 0.02 mgal, was found. This amount is negligible.

(81a) $\left| C_{1.b.1}(\Delta g^*_{iso}) \right| < 0.02 \cdot 10^{-3}$ cm s^{-2} .

For this Holopainen-profile, the amount of $C_{1.b.1}$ in terms of the Bouguer anomalies was computed also, by the formula of [5] , eq. (289). $C_{1.b.1}$ in terms of Bouguer anomalies proved to be a little

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greater than $C_{1.b.1}$ expressed by isostatic anomalies, (81a).

For lowland areas, for the oceans, and for the Mittelgebirge, the absolute amount of $C_{1.b.1}$ and $C_{1.b.2}$ in terms of isostatic anomalies will be much more small than $0.02 \cdot 10^{-3}$ cm s^{-2} which is the quantity found in the Alps, (81a). Thus, summarizing these test computations in the Holopainen-profile, the isostatic anomalies yield negligible amounts for $C_{1.b.1}$ and $C_{1.b.2}$.

See also chapter C of this publication.

After this excursion into the Alps, we look back to the equations (73) through (80) of the publication in hand. Now, we continue to consider the investigations about $C_1(I)$, by analogy with [5], eq. (274) through (292). In the course of the deductions connected with (73) (77) (81a), $C_1(I)$ can be expressed by the following relation approximatively valid,

(82) $C_1(I) \cong C_{1.a}(I)$,

and further (see [5] eq. (284)),

(83) $C_1(I) \cong Z \cdot \partial^2 I \,/\, \partial z^2$.

And, regarding (79),

(83a) $C_1(I) \cong - Z \dfrac{\partial}{\partial z} \Delta g^{*}_{iso}$.

Thus, finally, the formula for routine computations of $C_1(I)$ has the shape given by (84); (see Fig. 1). (see [5] , eq. (274) (284) (285) (291) (292); [6] eq. (37)).

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(84) $$C_1(I) \cong - Z \frac{1}{2\pi} \iint_v \frac{(\Delta g^*_{iso})_Y - (\Delta g^*_{iso})_Q}{e^3_{oo}} \cdot dv.$$

v is the sphere with the radius R, being the ball (in sea level).

Now, we turn towards the equation (79), especially. Belated, we supplement the verification of this equation, now. In this context, the relations (52) and (66) yield (85), considering the following differential relation for Δg_I ,

(84a) $$\Delta g_I = - \partial I/\partial r - (2/r)\cdot I.$$

(85) $$- \Delta g_{iso} = - \Delta g_I + (2/r)B_{iso} =$$

$$= \partial I/\partial r + (2/r)\cdot I + (2/r)B_{iso} =$$

$$= \partial T/\partial r - \partial B_{iso}/\partial r + (2/r)(T-B_{iso}) + (2/r)B_{iso} =$$

$$= \partial (T-B_{iso})/\partial r + (2/r)\cdot T =$$

$$= \partial I/\partial r + (2/r)\cdot T \quad .$$

(85) turns to (86), (see (79)),

(86) $$\partial I/\partial r = - \Delta g_{iso} - (2/r)\cdot T = - \Delta g^*_{iso} .$$

The last term of (86) with the star index has the meaning of an abbreviating symbol.

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After the preceding developments from (69) through (86), we can finish now the description of the details of the computation of the amending term $C_1(I)$ of our basing expression (68) for the perturbation potential T.
(68) expresses T by the isostatic anomalies Δg_{iso} in the form of a universally valid formula, valid also in the high mountains.

Now, the details of the computation of the amending term $\Omega_1(I)$ of (68) are in the fore.

The relation (224) of [5] gives, substituting M by I,

$$(87) \qquad \Omega_1(I) = \frac{3}{(4\pi R')^2} \iint_w F(I)\cdot S(p)\cdot dw \; +$$

$$+ \frac{1}{2\pi} \iint_w \Delta g_I \, \frac{Z}{R} \cdot \left[2 - \frac{1}{y+y^2} \right] \cdot \frac{1}{e'} \cdot dw \; +$$

$$+ \frac{1}{2\pi} \iint_w \frac{I}{R} \cdot \frac{Z}{R} \cdot \left[3 - \frac{2}{y+y^2} \right] \cdot \frac{1}{e'} \cdot dw \; +$$

$$+ \frac{1}{2\pi} \iint_w \frac{I}{R} \cdot \frac{v_1}{R} \cdot dw \; +$$

$$+ \frac{1}{2\pi} \iint_w \frac{\partial I}{R\,\partial p} \cdot \left[-\frac{1}{R} \cdot \frac{(\cos p/2)^2}{\sin p} \cdot b_7 - \frac{Z}{2\,(R')^2} \cdot \frac{dS(p)}{dp} \right] \cdot dw \; +$$

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$$+ \frac{1}{2\pi} \iint \Delta g_I \frac{-x^2}{y+y^2} \cdot de' \cdot dA \quad +$$

$$+ \frac{1}{2\pi} \iint \frac{I}{R} \cdot \left[\frac{-2x^2}{y+y^2} + v_3 \right] \cdot de' \cdot dA \quad +$$

$$+ \frac{1}{2\pi} \iint \frac{\partial I}{\partial e'} (v_2 - b_{11}) \cdot de' \cdot dA \quad +$$

$$+ \frac{1}{2\pi} \iint (-GZ) \cdot \Phi(x^* \alpha_1, x^* \alpha_2) \cdot de' \cdot dA \quad .$$

The radius of the sphere w is R + H_p, (see Fig. 1). In (87), we have from [5], (225) through (225h),

(88) $$F(I) = \sum_{i=1}^{8} f_i(I) \quad ;$$

(89) $$f_1(I) = \iint_w \Delta g_I \frac{Z}{R} \cdot \left[2 - \frac{1}{y+y^2} \right] \cdot \frac{1}{e'} \cdot dw \ ,$$

(90) $$f_2(I) = \iint_w \frac{I}{R} \frac{Z}{R} \cdot \left[3 - \frac{2}{y+y^2} \right] \cdot \frac{1}{e'} \cdot dw \ ,$$

(91) $$f_3(I) = \iint_w \frac{I}{R} \frac{v_1}{R} \cdot dw \ ,$$

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$$(92) \quad f_4(I) = - \iint_w \frac{\partial I}{R \cdot \partial p} \frac{1}{R} \frac{(\cos p/2)^2}{\sin p} \cdot b_7 \cdot dw \quad ,$$

$$(93) \quad f_5(I) = - \iint \Delta g_I \frac{x^2}{y+y^2} \cdot de' \cdot dA \quad ,$$

$$(94) \quad f_6(I) = \iint \frac{I}{R} \cdot \left[\frac{-2x^2}{y+y^2} + v_3 \right] \cdot de' \cdot dA \quad ,$$

$$(95) \quad f_7(I) = \iint \frac{\partial I}{\partial e'} \cdot (v_2 - b_{11}) \cdot de' \cdot dA \quad ,$$

$$(96) \quad f_8(I) = - \iint GZ \quad \cdot \Phi(x^* \cdot \alpha_1, x^* \cdot \alpha_2) \cdot de' \cdot dA \quad .$$

By the relation from (68) through (96), the precise and universal formula for the perturbation potential T in terms of isostatic gravity anomalies along the Earth´s surface is developed in good detail. The theoretical error for the height anomalies T/γ will not be greater than about 1 cm, basing on (68).

As to further details, the precise and complete expressions for b_7, b_{11}, v_1, v_2, v_3, x^*, x, y , which appear from (87) through (96), can be found in [5], eq. (75) (76) (78), (80) to (84), and also in the appendix of [5].
The formula (68) is of use especially if the height anomalies $\zeta = T/\gamma$ have to be computed up to a precision of $\pm$ 1 cm in the mountains. This case is very rare. In most applications, the relative simple lowland version of this solution will suffice. This case is discussed subsequently, it will suit the purposes

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for test points P situated on the oceans, in the lowlands, and in the Mittelgebirge, as long as the slopes of the terrain are not too great.

7.2. The lowland version of the final solution

The combination of the relation (57) with (66) yields the lowland version of the final solution in terms of isostatic anomalies,

$$\{T\} = \frac{1}{4\pi R'} \iint_w \left[\Delta g_{iso} + \frac{2}{r} B_{iso} + C_1(I) \right] \cdot S(p) \cdot dw + \{B_{iso}\} + \{\Omega^*_1(I)\} . \tag{97}$$

In (97), the potential B_{iso} on the surface of the Earth u can be computed by (38) with the densities of (31) (32) (33) (34). The T value on the left hand side of (97) refers to the surface u, too.
The term $C_1(I)$ constructs one of the amendment terms, being amendments which correct the traditional theory.
It can be computed along the lines of (69) through (86).
The term $\Omega^*_1(I)$ of (97) is the second amending term of this lowland solution. It is a simplification of (87), this simplification is induced by the lowland constraint (7). The lowland relation (230) of [5] leads to (98), if M is replaced by I.

$$\Omega^*_1(I) = \frac{3}{(4\pi R)^2} \iint_w F^*(I) \cdot S(p) \cdot dw + \frac{1}{2\pi} \iint_w \Delta g_I \frac{Z}{R} \frac{3}{2} \frac{1}{e_o} \cdot dw + \tag{98}$$

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$$+ \frac{1}{2\pi} \iint_w \frac{I}{R} \frac{Z}{R} \frac{1}{e_o} \cdot dw \quad -$$

$$- \frac{1}{8\pi R^2} \iint_w \frac{\partial I}{R\, \partial p} Z \cdot \left[\frac{\cos p/2}{(\sin p/2)^2} + 2 \frac{dS(p)}{dp} \right] \cdot dw \quad .$$

In (98), we have the subsequent formula (99) representing $F^*(I)$, obtained from the relations (227) through (228) of [5],

$$(99) \qquad F^*(I) = \sum_{i=1}^{3} f_i^*(I) \quad ,$$

with

$$(100) \qquad f_1^*(I) = \iint_w \Delta g_I \frac{Z}{R} \frac{3}{2} \frac{1}{e_o} \cdot dw \quad ,$$

$$(101) \qquad f_2^*(I) = \iint_w \frac{I}{R} \frac{Z}{R} \frac{1}{e_o} \cdot dw \quad ,$$

$$(102) \qquad f_3^*(I) = - \iint_w \frac{\partial I}{R\, \partial p} \cdot \frac{Z}{4R^2} \cdot \frac{\cos p/2}{(\sin p/2)^2} \cdot dw \quad .$$

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(103) $e_0 = 2 R \sin p/2$.

As to the computation of B_{iso}, (38), see also [2] [7] [8] [9].

As to the impact $C_1(I)$ exerts on T by the formula (97), we recommend warmly to read the section 12.2. of [5] , especially its equations from (293) through (305); further, the section 5 of [6] is recommended likewise warmly, as so as the chapter C of the publication in hand.

8. Conclusions

The isostatic anomalies of the gravity are defined in a new way for points at the surface of the Earth considering the transition from the orthometric heights to the normal heights, effecting small changes.

In terms of these anomalies, it is shown that a precise formula for routine calculations of the height anomalies can be developed, having a theoretical error of not more than about ± 1 cm, (97) (68). This method profits from the fact that the isostatic anomalies have smoothed values.
The routine application of the final formulas for the height anomalies expressed by the isostatic gravity anomalies is facilitated enormously by the modern technical progress. For instance, the numerical application of the obtained formulas can profit from the use of electronic computers in the computation of the isostatic anomalies.
Recent progresses bring the required data in a new light, now: Now, we have more complete terrestrial gravity material, and, last not least, we have global sets of 1° x 1° mean heights, supplemented by dense grids of digitized heights of regional extension, [9] .

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8. Density distribution in the Earth's mantle by gravimetrical and seismological data

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Summary

For the area of the mantle of the Earth, it is investigated how far the real density values deviate from the standard values of an Earth being in hydrostatic equilibrium. The order of the r, m,s, value of these deviations is estimated. This r.m.s. value is found to cover the range of ± 6 to ± 14 kg m^{-3} for a distance of 3470 km to 5970 km from the gravity center, i.e. a depth range of 400 km to 2900 km (the core). The global density anomalies are modelled in terms of low-degree spherical harmonics. They comprise both the effect of the chemical composition variation and an eventual effect of elastic compression or extension. These density anomalies are determined from an observational material that consists of both the global variation of the gravity potential and the lateral variation of the seismic velocity in the upper layers of the mantle. The here treated model Earth is made up by a superposition of 4 phenomenons: 1. The Earth in hydrostatic equilibrium; 2. The Airy-Heiskanen isostatic system of the mountains, the oceans, their roots and their antiroots; 3. The density anomalies in the upper layers down to a depth of 400 km. 4. The density anomalies deeper than 400 km, down to the core-mantle boundary.

It are the latter density anomalies which are to be determined here. The working hypothesis is the demand to find the minimum of the r.m.s. value of these anomalies situated in the depth range of 400 km to 2900 km depth. Finally, for the area of the deep mantle, a comparison of the density anomalies here computed and of the anomalies of the seismic velocities obtained by other authors is carried out.

Zusammenfassung

Für den Bereich des Erdmantels wird untersucht wie sehr die wirklichen Dichtewerte von ihren Standardwerten abweichen, wobei die letzteren Werte sich aus einer Erde im hydrostatischen Gleichgewicht ableiten. Die Größenordnung des mittleren quadratischen

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Wertes m^2 dieser Abweichungen wird ermittelt. Der Wert von m liegt zwischen $\pm$ 6 und $\pm$ 14 kg m^{-3} für eine Entfernung von 3470 km bis 5970 km vom Mittelpunkt der Erde, d.i. eine Tiefe von 400 km bis 2900 km (Kern-Mantel-Grenze). Die globalen Dichteanomalien und die anderen Daten werden dargestellt durch eine Kugelfunktionsentwicklung, die nur die Glieder geringeren Grades umfaßt. Diese Dichteanomalien reflektieren nicht nur den Effekt der Änderung der chemischen Zusammensetzung, sondern auch einen eventuellen Effekt der elastischen Deformation. Diese Dichteanomalien leiten sich ab aus einem Beobachtungsmaterial, das das globale Schwerepotential umfaßt und darüberhinaus auch die horizontale Veränderung der seismischen Geschwindigkeiten in den oberen Schichten des Mantels.

Die hier eingeführte Modellerde besteht aus der Superposition von 4 Teilen: 1. Die Erde in hydrostatischem Gleichgewicht; 2. Die Gebirge, die Ozeane, die Gebirgswurzeln und die ozeanischen Gegenwurzeln im Sinne des isostatischen Systems von Airy-Heiskanen; 3. Die Dichteanomalien in den oberen Schichten bis zu einer Tiefe von 400 km; 4. Die Dichteanomalien zwischen einer Tiefe von 400 km und der Kern-Mantel-Grenze. Die zuletzt genannten Dichteanomalien sind die Werte, die hier zu bestimmen sind. Das Minimum des mittleren quadratischen Wertes dieser Anomalien im Bereich zwischen 400 km und 2900 km Tiefe zu finden, das ist die hier eingeführte Arbeitshypothese.

Schließlich werden die so bestimmten Dichteanomalien mit den von anderen Autoren für den Bereich des tiefen Erdmantels empirisch gefundenen Anomalien der seismischen Geschwindigkeiten verglichen.

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Аннотация

Для зоны мантии Земли исследуются отклонения действительных значений плотности от их стандартных значений, причем последние значения выводятся для Земли в гидростатическом равновесии. Определяется порядок величин среднего квадратического значения m^2 этих отклонений. Значение m находится в пределах ± 6 и 14 кг $м^{-3}$ при удаленности от 3470 до 5970 км от центра Земли, что является глубиной от 400 до 2900 км (граница между мантией и ядром). Глобальные аномалии плотности и другие данные предоставляются на основе разложения шаровой функции, которое включает в себя только члены меньшей степени. Эти аномалии плотности отражают не только эффект изменения химического состава, но также возможный эффект упругой деформации. Эти аномалии плотности выводятся из материала наблюдений, который включает в себя глобальный гравитационный потенциал и, кроме того, горизонтальное изменение сейсмических скоростей в верхних слоях мантии. Приведенная здесь модель Земли состоит из суперпозиций 4 частей :

1. Земля, в гидростатическом равновесии ;
2. Горы, океаны, корни гор и противокорни океанов в смысле изостатической системы Airy - Heiskanen ;
3. Аномалии плотности в верхних слоях до глубины 400 км ;
4. Аномалии плотности между глубиной 400 км и границей между мантией и ядром.

Названные последними аномалии плотности являются значениями, которые намечается определить. Определить минимум среднего квадратического значения этих аномалий в зоне между 400 и 2900 км глубины является приведенной здесь рабочей гипотезой.

В заключении эти опеределенные таким образом аномалии плотности сравниваются с аномалиями сейсмических скоростей, эмперически найденными другими авторами для зоны глубокой мантии Земли.

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1. Introduction

In this article, we model density anomalies in the interior of the Earth down to the depth of the core. These density anomalies depend on the latitude, the longitude and the radius. The observation material comes from seismology and from gravimetry. Thus, geophysical and geodetical ideas meet in this elaboration. Density variations in the Earth are deviations of the real density (or better: The model of the real density obtained within the potentialities of the here applied methods) from the standard density of an Earth model of a density law with pure radial variations of the density.

The velocity of the seismological waves depend on the density of the masses crossed. The gravity along the surface of the Earth depends on the density values in whole the body of the Earth. Thus, the inversion of these relations leads to a non-unique estimation of the density anomalies in the interior of the Earth, using seismological and gravimetrical data which play here the role of the underlying observation material.

Here, all the values are given in terms of low-degree spherical harmonics. The density anomalies are determined in relation to the global variation of the gravity potential and to the lateral variation of the seismic velocity in the upper layers of the mantle.

We first discuss the basic observational evidence that bears upon density distribution in the Earth. We next present mathematical models for computing density from measurements of gravity potential and seismic wave velocity. Finally, we discuss the results of a model computation for the distribution of density anomalies in the mantle, [6][7]. In comparison with [6][7], the publication in hand is a more detailed description, which I was asked for.

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2. A surface Bouguer layer

The gravity anomalies on the surface of the Earth are caused by density anomalies within whole the body of the Earth. For a moment, in a very simple version, the density anomalies can be taken to be distributed in a small depth, only. Thus, they can be represented by a Bouguer plate. For a plate of the density $\rho_0 = 2650\ kg\ m^{-3}$, we have the well-known formula

(1) $(\Delta g)_{mgal} = 0.1 \cdot (T)_{meter}$,

if T is the widst of the plate. Considering the relation (1), its coefficient 0.1 is proportional to the density,

(2) $0.1 = k_1 \cdot \rho_o$,

k_1 is a constant value. Thus, for a homogeneous Bouguer plate of the arbitrary density anomaly $\delta\rho$, we have the gravity of (3),

(3) $(\Delta g)_{mgal} = 0.1(\delta\rho/\rho_o)(T)_{meter}$.

T is the width of the plate, Fig. 1. For a Bouguer anomaly of $\Delta g = 20$ mgal, ($0.02\ cm\ s^{-2}$), and for T = 400 km, (upper mantle), we have by (3)

(4) $\delta\rho \cong 1\ kg\ m^{-3}$, (i.e. $1/1000\ g\ cm^{-3}$).

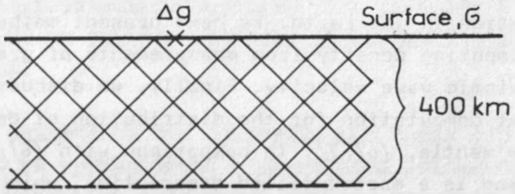

Fig. 1. A modelling of the gravity anomalies by a Bouguer plate of 400 km width.

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3. The potential of a shell

In the deep interior of the Earth, for great values of the depth t, we can introduce a gravitating spherical shell being the source of the gravity anomalies. The widst of this shell may be equal to T, whereat T is much more small than the radius R, $T \ll R$ (R: radius of the Earth). Within this shell or within this layer, we have a density distribution of lateral variation only. This layer of density anomalies $\delta\varrho$ in the mantle can be replaced by a surface distribution ($\Theta = \Theta(\varphi,\lambda)$) in the mean depth of this layer. (φ is the geocentric latitude and λ the longitude).

A spherical shell of the density $\delta\varrho$, of the width T, and of the mean depth t may play the role of the underlying gravitating body. This shell causes the potential Y. Thus, we have the following potential Y for test points P situated at the surface of the Earth,

$$(5)\quad Y = Y(P) = G\iiint\limits_{V} \frac{1}{e(P,Q)}\cdot\delta\varrho(Q)\cdot dV_Q ,$$

G is the gravitational constant, V is the volume of the gravitating shell, the meaning of e(P,Q) comes from Fig. 2. In (5), within the shell of the width T (see Fig. 2), $\delta\varrho$ or $\delta\varrho(Q)$ does not depend on the radius. For $T \ll R$, the relation (5) can be approximated by the potential of a surface distribution in the mean depth t of the shell

$$(6)\quad Y = Y(P) \cong G\iint\limits_{\varkappa} \frac{1}{e(P,Q)}\cdot\delta\varrho(Q)\cdot T\cdot d\varkappa_Q .$$

The radius of $\varkappa$ is (R-t). Thus

$$(7)\quad Y = Y(P) \cong G\iint\limits_{\varkappa} \frac{1}{e(P,Q)}\cdot\Theta\cdot d\varkappa_Q ;$$

with

$$(8)\quad \Theta = \Theta(\varphi,\lambda) = \delta\varrho\cdot T = \delta\varrho(\varphi,\lambda)\cdot T .$$

Θ is the gravitating surface distribution, see Fig. 2.

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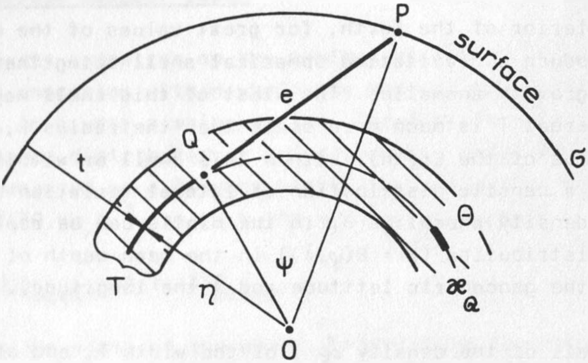

Fig. 2. A modelling of the gravity anomalies by a spherical shell.

We have the harmonics development for 1/e in terms of the Legendre functions P_n , [11],

$$(9) \quad \frac{1}{e} = \sum_{n=0}^{\infty} \frac{(R-t)^n}{R^{n+1}} \cdot P_n(\cos\psi) \ ,$$

$$\eta = (R-t) < R \ .$$

Now, the surface spherical harmonics $S_n(\varphi,\lambda)$ are introduced, (9a); integrating over the unit sphere, we have

$$(9a) \quad \iint \left[S_n(\varphi,\lambda) \right]^2 \cdot \cos\varphi \cdot d\varphi \cdot d\lambda = 4\pi \ .$$

This are fully normalized harmonics. The symbol $S_n(\varphi,\lambda)$ represents all the surface spherical harmonic functions of degree n, whatever the order of them may be. Or, with other words, only the zonal harmonics of degree n are written down, since the tesseral and sectorial harmonics of the degree n have similar relations as the zonal harmonics. This is an often used abbreviating style. By the decomposition formula of the harmonics, P_n can be expressed by the surface spherical harmonics of degree n (being

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$S_n(\varphi,\lambda)$). Thus, the relation (9) turns to

$$(10)\quad \frac{1}{e} = \sum_{n=0}^{\infty} \frac{(R-t)^n}{R^{n+1}} \cdot \frac{1}{2n+1} \cdot S_n(\varphi,\lambda) \cdot S_n(\varphi',\lambda') ,$$

for

$$(10a)\quad \eta = (R-t) < R .$$

φ and λ are the co-ordinates of P; φ' and λ' are those of Q. Thus, we find the following form for Y(P) which is the potential of a shell, (5)(6)(7),

$$(11)\quad Y(P) = \sum_{n=0}^{\infty} y_n \cdot S_n(\varphi,\lambda) ,$$

with the subsequent expression (12) for the Stokes constants y_n, and with $k_2 = G$, (8)(9a), (R-t : Radius of the sphere $æ_Q$, Fig.2),

$$(12)\quad y_n = k_2 \iint_{æ} T \cdot \frac{(R-t)^n}{R^{n+1}} \cdot \delta\varrho \cdot S_n(\varphi',\lambda') \cdot \frac{1}{2n+1} \cdot d æ_Q .$$

Or, inserting

$$(12a)\quad \delta\varrho = \sum_{n=0}^{\infty} (\delta\varrho)_n \cdot S_n(\varphi',\lambda') ,$$

$$(13)\quad y_n = 4\pi k_2 \cdot T \cdot \frac{(R-t)^n}{R^{n+1}} \cdot \frac{(R-t)^2}{2n+1} \cdot (\delta\varrho)_n .$$

The surface distribution Θ along the sphere æ has the harmonics development, (8)(12a),

$$(14)\quad \Theta = \sum_{n=0}^{\infty} \vartheta_n \cdot S_n(\varphi,\lambda) = T \sum_{n=0}^{\infty} (\delta\varrho)_n \cdot S_n(\varphi,\lambda) .$$

Hence,

$$(15)\quad y_n = 4\pi k_2 \frac{(R-t)^n}{R^{n+1}} \frac{(R-t)^2}{2n+1} \vartheta_n .$$

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The potential $Y(P)$, described by (11)(13)(15), is valid for test points P on the Earth's surface of radius R. The value

$$(16) \quad \frac{(R-t)^n}{R^{n+1}}$$

appearing in (13) and (15) is the smaller the greater the parameter n, because of (10a). For

$$(17) \quad R-t = \frac{1}{2} R ,$$

and for

$$(18) \quad n = 20 ,$$

we find

$$(19) \quad \left(\frac{R-t}{R}\right)^n = \left(\frac{1}{2}\right)^{20} \cong 10^{-6} .$$

But, the smaller $[(R-t)/R]^n$, the greater ϑ_n , if y_n is understood that it is fixed, (15).

Since, in (13) and (15), always the product of (16) with the Stokes constants ϑ_n resp. $(\delta\varrho)_n$ appear, the effect of a change of the t value can be compensated by the effect of a corresponding change of the ϑ_n or $(\delta\varrho)_n$ value.

Thus, basing on $Y(P)$ as a given function, it is not possible to compute the precise value of the depth of the density anomalies in terms of the surface potential values, or, what is equivalent, in terms of the surface gravity anomalies.

What is possible by these methods without the introduction of any hypothesis, that is the computation of the whole mass δM of the density anomalies. The concerned formula (20) follows from the Gauss theorem, [11]. We have, integrating over the surface σ of the Earth,

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$$(20)\quad \delta M = k_3 \iint_{\sigma} \delta g \cdot d\sigma \; .$$

k_3 is a constant quantity, δg is the gravity perturbation along σ (being the radial derivative of the perturbation potential). The derivation of (20) can be found in the text books on potential theory.

4. On the potential of the surface distribution

The density anomalies, (8),

$$(21)\quad \delta\varrho = \delta\varrho\,(\varphi,\lambda) = \frac{1}{T}\cdot\theta(\varphi,\lambda)$$

within a certain layer of the width T have the surface spherical harmonics development, (12a)(14),

$$(22)\quad \delta\varrho\,(\varphi,\lambda) = \sum_{n=0}^{\infty} (\delta\varrho)_n \cdot S_n(\varphi,\lambda)\; .$$

In order to be clear, the right hand side of (22) is the abbreviated shape of the detailed form (23),

$$(23)\quad \delta\varrho = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \bar{P}_{n.m}(\varphi)\left[(\delta\varrho)_{1.n.m}\cdot\cos m\lambda + (\delta\varrho)_{2.n.m}\cdot\sin m\lambda\right].$$

$\bar{P}_{n.m}(\varphi)\cdot\cos m\lambda$

and

$\bar{P}_{n.m}(\varphi)\cdot\sin m\lambda$

are the fully normalized surface spherical harmonics, (9a).

$(\delta\varrho)_{1.n.m}$

and

$(\delta\varrho)_{2.n.m}$

are the Stokes constants.

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Sure,(22) can be understood in such a manner that only the zonal harmonics of the degree n are written down, the sectorial and tesseral harmonics of the degree n will transform in the same way, in the course of the subsequent deliberations.

The r.m.s. value of $\delta\varrho$ within the volume V of the considered layer, (5), is $(\delta\varrho)_a$. For this r.m.s. value, we have

$$(24)\quad (\delta\varrho)_a^2 = \frac{1}{V}\cdot\iiint\limits_V (\delta\varrho)^2\cdot dV_Q \,.$$

As to (24), within the shell of the volume V, the density anomaly $\delta\varrho$ does not depend on the radius, (12a). Thus, the volume integral (24) can be substituted by the subsequent surface integral covering the sphere $\varkappa_Q$, (see Fig. 2), (22),

$$(25)\quad (\delta\varrho)_a^2 = T\cdot\frac{1}{V}\iint\limits_{\varkappa_Q} (\delta\varrho)^2\cdot d\varkappa_Q \,.$$

Now, in the integrand of (25), $\delta\varrho$ is replaced by the expression (22). Considering the relation (26)

$$(26)\quad d\varkappa_Q = \eta^2\cdot\cos\varphi\cdot d\varphi\cdot d\lambda$$

with

$$(27)\quad \eta = R - t\,,$$

and accounting for (9a), the relation (25) turns to

$$(27a)\quad (\delta\varrho)_a^2 = 4\pi\cdot T\cdot\eta^2\cdot\frac{1}{V}\cdot\sum_{n=0}^{\infty}(\delta\varrho)_n^2 \,.$$

The volume of the shell can be approximated by

$$(27b)\quad V \cong 4\pi\cdot T\cdot\eta^2 \,.$$

Hence, (27a)(27b),

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(27c) $(\delta\rho)^2_a \simeq \sum_{n=0}^{\infty} (\delta\rho)^2_n .$

5. The gravity potential

The gravity potential W of the Earth can be approximated by the following spatial spherical harmonics series expression valid in the exterior of the body of the Earth, [3][4][5]. This series is of common use,

(28) $$W = \frac{GM}{r}\left[1 + \sum_{n=2}^{N} \left(\frac{a_e}{r}\right)^n \cdot w_n \cdot S_n(\varphi,\lambda)\right] + Z ,$$

(28a) $$Z = \left(\frac{1}{2}\right)\cdot \omega^2 \cdot r^2 \cdot \cos^2\varphi \quad .$$

M is the mass of the Earth, w_n are the concerned Stokes constants, and Z is the potential of the centrifugal force. ω is the angular velocity of the Earth's rotation. a_e is the equatorial radius of the mean Earth ellipsoid.

We took the w_n values of GEM-10. Meanwhile, refined values are available. Table 1 gives the w_n values, (see the appendix), [7] [11].

6. The convergence of the spatial spherical harmonics series development of the potential down to the surface of the Earth

Here, in our investigations, the spatial harmonic potentials are represented by spatial spherical harmonics series developments. This series is uniform convergent in whole the mass-free exterior [3][4][5].

During the last years, some authors published "retorts". These "counter-proofs" have no foundation: A counter-proof is possible only in case the problem is unique. But, even this case is the crux: The harmonic downwards continuation of a potential function

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is unique only if the potential function is continuous, [3][4] [5][11]. If the downwards continuation is divergent, it is simultaneously discontinuous, too. Thus, there is no uniqueness. Consequently, the counter-proof examples are paralysed because they forget the continuity contraint.

In case the harmonic potential in the exterior of the Brillouin sphere undergoes a harmonic downwards continuation, we have two branches.

The first branch leads to discontinuous harmonic functions, divergent series, and it leads to a field being of no use for natural science; it cannot lead to the potential of a gravitating body.

The second branch leads to continuous harmonic functions, convergent series, and it is, thus, the branch which cultivates natural science. It is the branch of our choice. In the downwards continuations, the constraint of continuity is indispensable, [3][4][5].

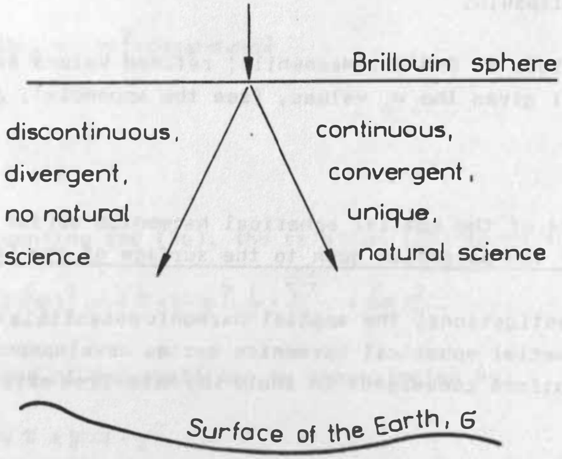

Fig. 3. The continuous and the discontinuous branch in the downwards continuation of a harmonic function.

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7. The potential of the isostatic masses

The potential of the isostatic masses consists of the potential of the mountain masses above sea level, of the potential of the compensating mountain roots, of the potential of the oceanic mass defects, and of the potential of the oceanic antiroots, (see chapter A of the publication in hand, especially the equations (37) and (31)(32)(33)(34) of chapter A). The isostatic system according to Airy-Heiskanen having a compensating depth of $T^* = 30$ km is well-proved even by recent computations, [14] [17], Fig. 9.

The isostatic potential W_I comes from the isostatic masses m_I by

$$(29) \quad W_I = G \iiint_{V_I} \frac{1}{e}\, dm_I \, .$$

We have a development for W_I in terms of the height of the mountains H and in terms of the depth b of the mountain roots. It is represented by the form (30).

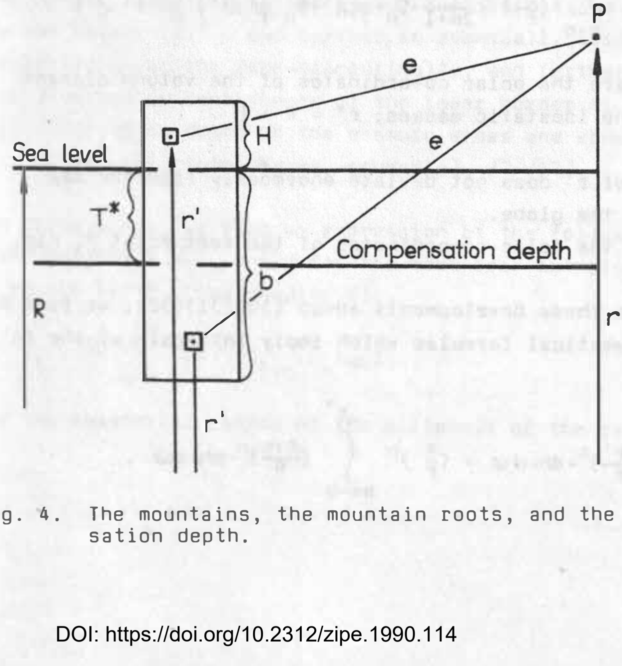

Fig. 4. The mountains, the mountain roots, and the compensation depth.

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$$(30)\quad W_I = G\left[\Delta\varrho \int_{h=-b}^{-T^*} + \varrho^0 \int_{h=o}^{H}\right]\cdot\iint_{\varphi\lambda}\frac{1}{e}\cdot dh\cdot d\omega ,$$

$$(31)\quad d\omega = (r')^2\cdot\cos\varphi\cdot d\varphi\cdot d\lambda \quad ,$$

$$(31a)\quad \Delta\varrho = -\ 600\ \mathrm{kg\ m^{-3}}\ ,$$

$$(31b)\quad \varrho^o = 2650\ \mathrm{kg\ m^{-3}}\ ;$$

(see eq. (31)(33) of chapter A).

Respecting the oceans and their antiroots also, (30), an equivalent rock topography was introduced, [8]. Thus, in (30), the H values for both the mountains and the oceans are represented by one single globally valid mathematical development, (38), being convenient for the isostatic computations.

The inverse value of the distance e is developed by the relation (32); (see also (10)(10a), [11], Fig. 4).

$$(32)\quad \frac{1}{e} = \frac{1}{r}\sum_{n=o}^{\infty}\left(\frac{r'}{r}\right)^n\cdot\frac{1}{2n+1}\cdot S_n(\varphi,\lambda)\cdot S_n(\varphi',\lambda')\ ,\quad r' < r\ .$$

r', φ', λ' are the polar co-ordinates of the volume element $dh\cdot d\omega$ of the isostatic masses; $r' = R + h$.

The amount of r' does not deviate enormously from the mean radius R of the globe.
r, φ, λ are the polar co-ordinates of the test point P, Fig. 4.

Following up these developments about (30)(31)(32), we find definite mathematical formulas which imply integrals of the following shape

$$(33)\quad \int_{h=-b}^{-T^*}\left(\frac{r'}{r}\right)^n\cdot dh\cdot d\omega = \left(\frac{R}{r}\right)^n\int_{h=-b}^{-T^*}\left(\frac{R+h}{R}\right)^n\cdot dh\cdot d\omega\ ,$$

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and

$$(34)\quad \int_{h=o}^{H} \left(\frac{r'}{r}\right)^n \cdot dh \cdot d\omega = \left(\frac{R}{r}\right)^n \int_{h=o}^{H} \left(\frac{R+h}{R}\right)^n \cdot dh \cdot d\omega \,.$$

With

$$(35)\quad \left(\frac{R+h}{R}\right)^n \cong \left(1+\frac{h}{R}\right)^n ,$$

the integrands of (33) and (34) can be developed in terms of powers of $\frac{H}{R}$,

$$(36)\quad \frac{H}{R},\ \left(\frac{H}{R}\right)^2, \ldots ,$$

they are convergent since

$$(36a)\quad h \ll R \,.$$

A thorough investigation about these questions is found in [7] [17]. In connection with our investigations, it suffices to take into account the linear term $\frac{H}{R}$, only. For the H values, spherical harmonics developments are given. Sophistications should involve the powers $\left(\frac{H}{R}\right)^2$, and further, an eventually existing overcompensation along the Moho-discontinuity, and further on, the possibly variating depth-range of the lower border of the lithosphere(about 70 km depth in the oceanic areas and about 140 km depth in the continental areas, probably), [7][17].

Along these lines, we find an expression of the following shape for the potential W_I , valid for the mass-free exterior of the body of the Earth, (see chapter 6),

$$(37)\quad W_I = \frac{GM}{r} \sum_{n=o}^{\infty} \left(\frac{a_e}{r}\right)^n \cdot w_{I.n} \cdot S_n(\varphi,\lambda) \ ;$$

a_e is the equatorial radius of the ellipsoid of the Earth.

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Table 2 shows the Stokes constants of the spatial spherical harmonics development for the isostatic potential $w_{I.n}$ according to Lachapelle [14], and further on, for the heights H according to [8], (equivalent rock topography).

The heights have the development

$$(38)\quad H = \sum_{n=o}^{\infty} H_n \cdot S_n(\varphi,\lambda) .$$

8. The reference potential U of the hydrostatic equilibrium figure

The level ellipsoid is not a convenient reference figure in our context. It cannot be generated by an equilibrium figure, or by a stratification which is physically plausible. It is a pure mathematical fiction. Here, a reference potential is introduced, the underlying masses of which have the stratification of hydrostatic equilibrium. The density anomalies treated later on describe deviations from this state of equilibrium.

The hydrostatic Earth of G. Darwin is recommended here [6][7][13] [15]. The parameters of this reference potential (40) derive as follows:

The coefficient J of (40) is obtained from J_2 by $J = -\frac{3}{2}\cdot J_2$.
Here, the coefficient J_2 of the zonal spherical harmonic of 2. degree comes empirically from satellite observations, [16].
As to J and J_2 , it may be stressed that J is here not computed from the dynamic flattening H^*, [13],

$$(39)\quad H^* = \frac{C-A}{C} = J\cdot\frac{1}{q} = +\frac{3}{2}\cdot\frac{C-A}{Ma^2}\cdot\frac{1}{q} .$$

C and A are in (39) the main moments of inertia. The meaning of the q value is found in [13], page 12.

The K value of (40) comes from the theory of the hydrostatic equilibrium in the interior of the Earth, it is the value computed by Bullard [13].

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We have for a rotating model, (28a), [7][13],

$$(40)\quad U = \frac{GM}{r}\left[1-\left(\frac{a}{r}\right)^2\cdot\frac{2}{3}\cdot J\cdot P_2(\sin\varphi)+\frac{4}{15}\cdot\left(\frac{a}{r}\right)^4\cdot K\cdot P_4(\sin\varphi)\right]+Z\,.$$

In (40), the term a is the equatorial radius of the surface of the hydrostatically stratified masses; this surface is simultaneously a level surface. Z is the zentrifugal potential (28a), and $P_i(\sin\varphi)$ are Legendre functions. From the literature, [7] [13], we take

$$(41)\quad J = 162\ 395\cdot 10^{-8}\,,$$

$$(42)\quad K = 1.127\cdot 10^{-5}\,.$$

As to details about the theory of equilibrium figures, please, consult the chapter contributed by H. Moritz to the Hungarian Winter School 1989 in Sopron.

The relation (40) was extended up to the harmonic $P_6(\sin\varphi)$ by Lanzano, recently, [15].

9. The law of Birch

The gravitation law of Newton expresses the gravitational force in terms of the density of the gravitating masses. The law of Birch relates the density of the masses in the upper 400 km of the Earth with the velocity of the seismic P-waves. This velocity V_P of the P-waves in the upper 400 km of the Earth depends on the density ϱ of these upper layers by a linear expression, in good approximation, [9]. We have,

$$(43)\quad V_P = -\,0.665 + 0.002\ 64\cdot\varrho\,,\quad 0 \leq t \leq 400\ \text{km},$$

or, abbreviating,

$$(44)\quad V_P = a + b\cdot\varrho\,.$$

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V_P in km/s, ϱ in kg/m^3. t is the depth.

Now, we do the following consideration: The layers in the upper 400 km are crossed in vertical direction by a seismic P-wave; and in the area of these layers, the density ϱ deviates from the standard density by $\delta\varrho$, $\delta\varrho$ being constant along this part of the way of the P-wave, being the way through the upper 400 km. Such a change of the ϱ value by $\delta\varrho$ leads to a change of the V_P value in the depth-range $0 \leq t \leq 400$ km, as it is evidenced by (43),

(45) $\delta V_P = 0.002\,64 \cdot \delta\varrho$.

Further, such a change of the V_P-value over a distance of about 400 km range leads to a time delay u of the travel time of the P waves crossing the layers of the upper 400 km.

If, the seismic P-waves run over a distance s within the time l, the V_P value is defined by

(46) $V_P = \frac{s}{l}$.

For the variation of the velocity V_P in terms of the travel time variation(δV_P, δl), we find in a self-explanatory way,

(47) $\delta V_P = -\frac{s}{l^2} \cdot \delta l = -(V_P)^2 \cdot \frac{\delta l}{s}$.

From (44), we find (48),

(48) $\delta V_P = b \cdot \delta\varrho$.

With (47) and (48), the relation (49) yields,

(49) $b \cdot \delta\varrho = -(V_P)^2 \cdot \frac{\delta l}{s}$.

Thus,

(50) $\delta l = -\frac{b}{(V_P)^2} \cdot s \cdot \delta\varrho$.

The quantity of s can be identified with T = 400 km.

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δl can be identified with the above introduced travel time delay u. Consequently,

(51) $\delta\varrho = -(V_P)^2 \cdot \frac{1}{b} \cdot \frac{1}{T} \cdot u$; (T = 400 km).

Inserting the values of V_P , b , and T , (51) yields

(52) $\delta\varrho = -\frac{1}{15} \cdot 10^3 \cdot u$.

$\delta\varrho$ is measured in kg/m^3 and u in time seconds.

A value of u = + 0.5 s leads to about $\delta\varrho = -33\ kg/m^3$.

u is the deviation of the observed travel time from its standard value found by the Travel Time Tables. $\delta\varrho$ is the deviation from the density of a standard Earth which is described later in the section 11 about the mathematical model.

10. The seismological data

The seismologically obtained data to be introduced in our computations should be described more thoroughly, now; [1][2][10][12][18][19][20].

At one selected place on the surface of the Earth, we have a seismological station which records the arrival times of the seismic waves radiated from the different earthquakes which happen at the different foci all over the world (taking over epicentral distances of the range 20^0 to $1o5^0$). The geographical positions of these different earthquake foci can be considered to be known, as so as the time at which the earthquakes did happen. The time the seismic wave needs to reach our seismological station, this is the travel time of the wave considered. If the recording seismological station is labelled by P_i, if the considered earthquake focus has the notation Q_k, so, the travel time observed (which the seismic wave needs to travel from Q_k to P_i) is denoted by

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(53) $(l_{i.k})_{obs.}$.

On the other hand, for a standard Earth, having a density which depends on the radius only, ($\rho = \rho(r)$), the standard value of the travel time can be interpolated in the seismological Travel Time Tables. Along these lines, the standard value

(54) $(l_{i.k})_{comp.}$

is obtained. This computed travel time of (54) is compared with the really observed travel time of (53). The difference between these two kinds of travel times is the travel time residual, which is denoted by

(55) $\tau_{i.k}$.

Thus,

(56) $\tau_{i.k} = (l_{i.k})_{obs.} - (l_{i.k})_{comp.}$.

$(l_{i.k})_{comp.}$ implies corrections for the flattening of the Earth.

From the foci of the different earthquakes distributed all over the globe, all the seismic waves arrive at our recording station, P_i. The average value of $\tau_{i.k}$ covering all the earthquakes recorded at our one single P_i station is obtained by

(57) $\tau_i = \frac{1}{F_i} \cdot \sum_{k=1}^{F_i} \tau_{i.k}$.

F_i is the number of the earthquakes recorded at the P_i station.

Fig. 5 shows clearly that the P-waves recorded at a certain station have paths which diverge in a fan-shaped form, according to the geographical positions of the different foci.

Only the P-waves are considered in this context. Within the layers of the depth $0 \leq t \leq 400$ km below the seismological station P_i at the Earth's surface, there is a kind of a narrow pass for all

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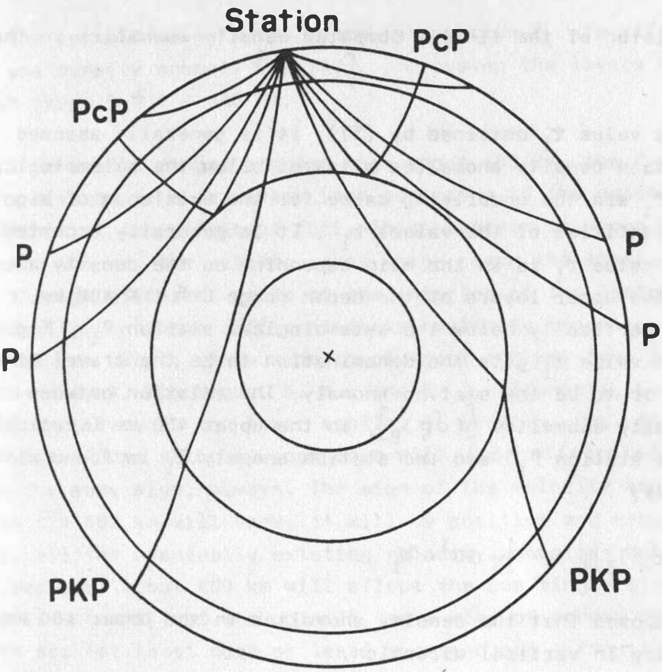

Fig. 5. The fan-shape form of the paths of the seismic waves reaching one seismological station.

the P-waves which are running to this one single seismological station P_i . This speciality is clearly recognized looking on Fig. 5.

The τ_i values of (57) here introduced are determined for many continental stations (some hundreds). But, the τ_i values are rare on the oceans; only at some island stations in the midst of the oceans, the τ_i values are recorded, see Fig. 6, 7. Seismological recording instrumentations at the ocean botton will be a help.

Especially, the addition of some more seismological stations situated on the islands in the midst of the oceans will improve

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the precision of the finally computed density anomalies in the mantle.

As to the value τ_i obtained by (57), it is generally assumed that certain density anomalies situated below the seismological station P_i are the underlying cause for the existence of significant quantities of the values τ_i . It is generally accepted that the value τ_i is in the main depending on the density anomalies in the upper layers of the depth range $0 \leq t \leq 400$ km, situated vertically below the seismological station P_i , Fig. 5. Thus, the value τ_i gets the denomination to be the travel time residual or to be the station anomaly. The relation between the mean density anomalies $[(\delta\varrho)_B]_i$ in the upper 400 km (vertically below the station P_i) and the station anomaly τ_i is found with (52). Thus,

$$(58) \quad [(\delta\varrho)_B]_i = - \frac{1}{15} 10^3 \cdot \tau_i \quad .$$

It is supposed that the density anomalies in the upper 400 km do not vary in vertical direction.

The label $[(\delta\varrho)_B]_i$ of (58) signifies that we have here a discrete value of the global function $(\delta\varrho)_B$, this discrete value refers to the seismological station P_i . This global function $(\delta\varrho)_B$ depends on φ and λ by (59),

$$(59) \quad (\delta\varrho)_B = \alpha(\varphi,\lambda) \; .$$

Along the surface of the Earth, also the station anomalies τ do vary only in dependence on φ and λ , obviously. Hence,

$$(60) \quad \tau = \tau(\varphi,\lambda) \quad .$$

The station anomalies τ (labelled also by τ_i , being the τ value at the point P_i) depend in the main on the $(\delta\varrho)_B$ value in the upper 400 km, situated below the station P_i . This dependence is arranged by the law of Birch, (43)(52). As long as the depth below the point P_i does not surpass the value of $t = 400$ km,

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all the seismic waves which reach the station P_i are affected by the one density anomaly $[(\delta\varrho)_B]_i$, crossing the layers of the depth range $0 \leq t \leq 400$ km.

But, for depths ranges greater than about 400 km, the P-wave paths diverge in a fan-shape form according to the geographical positions of the different foci. This pattern is shown by Fig. 5. For $t > 400$ km, the seismic waves which reach the station P_i will run through different parts of the interior of the Earth. Eventually, in these different parts, velocity anomalies of the P-waves can exist, below a depth of about 400 km, (see for example: Dziewonski, A. M.; Hager, B. H., and R. J. O'Connell, Large-scale heterogeneities in the lower mantle. J. geophys. Res. 82 (1977), 239-255). These velocity anomalies will (as anticipated) not have the same sign, always. The sign of the velocity anomalies below t = 400 km will vary, it will be positive and negative. Thus, all the eventually existing velocity anomalies below of the depth of about 400 km will affect the one single station anomaly τ (or τ_i) of the point P_i as a kind of random variances which are (at least more or less) averaged out - this fact is essential - in the mean value obtained by (57).

The average value which the station anomalies τ_i have on the surface of the Earth within a $5^0 \times 5^0$ grid cell, this value can be computed. Fig. 6 shows the global pattern of such mean grid cell values of τ. Fig. 6 comes from Toksöz, Arkani-Hamed, and Knight, [19].

Fig. 7 was taken from Toksöz, Arkani-Hamed, [18]. Fig. 7 shows the geographic distribution of data of seismic station anomalies τ (travel-time residuals) which are obtained by an averaging within the cells of a $5^0 \times 5^0$ grid. Solid circles indicate positive residuals, open circles negative residuals, Fig. 7.

Fig. 8 was published by Arkani-Hamed and Toksöz, [2]. It shows the contours of the seismic travel-time residuals τ (in seconds) based on spherical harmonics up to the 3rd degree. The coefficients of this development are tabulated in Table 3.

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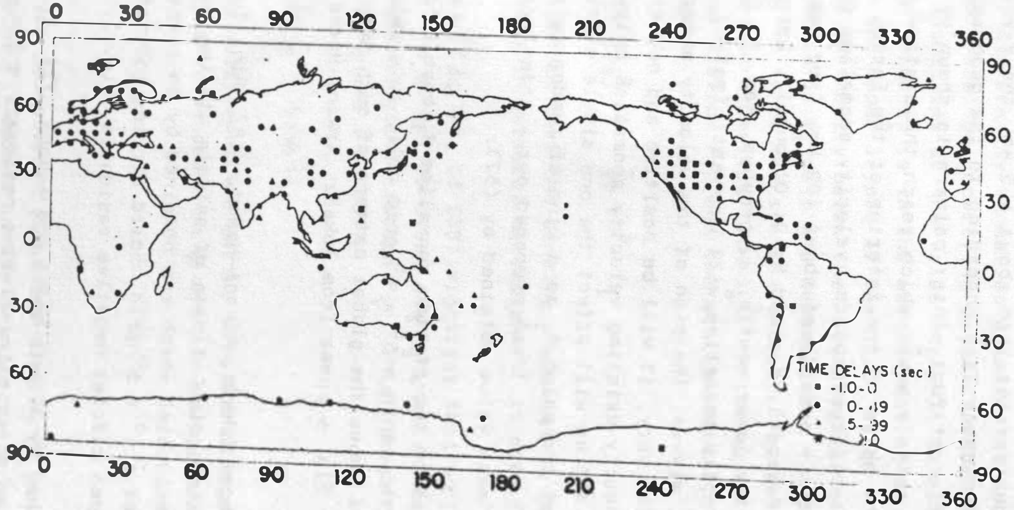

Fig. 6. Distribution of averaged (5^0 x 5^0 grid) travel-time residuals used for spherical harmonic expansion.

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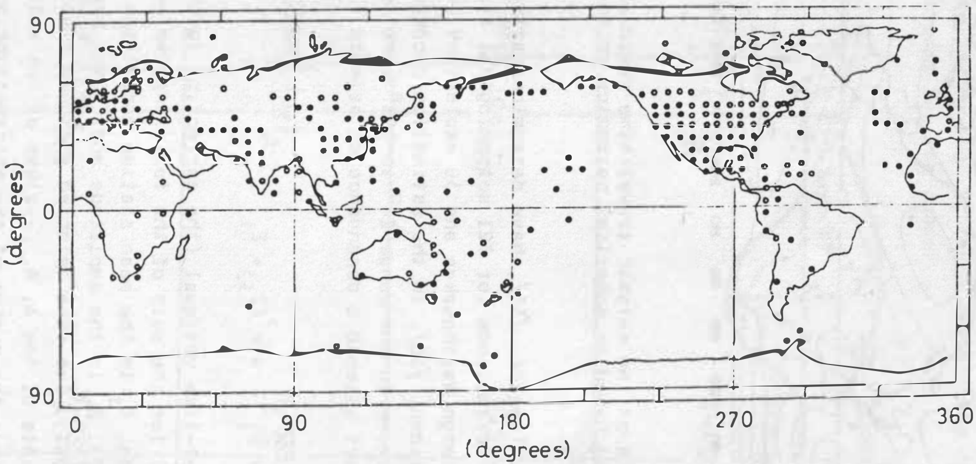

Fig. 7. Geographic distribution of data on seismic travel-time residuals after averaging over a 5^{o} by 5^{o} grid. Solid circles indicate positive residuals; open circles, negative.

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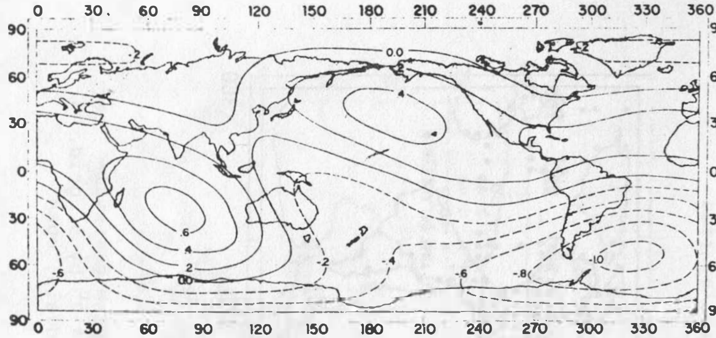

Fig. 8. Contours of the seismic travel-time residuals (in seconds) based on spherical harmonics up to the 3rd degree.

E. Herrin and J. Taggert, [10], have determined azimuthally dependent station corrections for 321 seismological stations. The records of 400 large earthquakes and 30 explosions were considered in these evaluations, [10]. In the estimation procedure, data for epicentral distances in the range 20^{o} to 105^{o} were used, only. Herrin and Taggert assumed a dependence on azimuth (Z_{ij}) of the form

$$(61) \quad C_{ij} = A_i + B_i \cdot \sin (Z_{ij} + E_i) .$$

C_{ij} is the travel-time residual (the correction to be added to the tabled time) for the pair of the following two points: Q_j, P_i - focus and station. A_i is the mean station correction, equivalent to our τ_i , (57). B_i is the amplitude and E_i the phase of the second term of (61). For some selected european stations, Table 4 shows the amounts of the A, B, E-values of the relation (61). In Table 4, N gives the number of the observations, and δ^2 is a measure for the variance of the random errors of the travel-time residuals.

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11. The mathematical model

A certain model for the density distribution in the interior of the Earth is now introduced. For this purpose, the interior of the Earth from the surface down to the core is divided into 4 spherical shells. By Fig. 9, these 4 shells in the earth's interior are pictured for the reader.

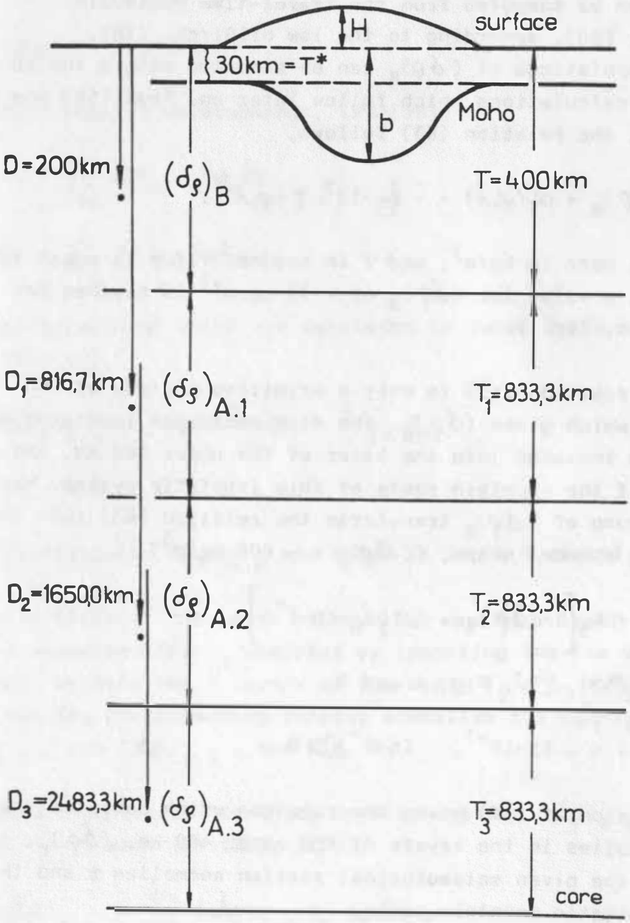

Fig. 9. The 4 shells in the Earth's interior.

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The density anomalies $(\delta\varrho)_B$ in the crust and upper mantle (in the depth range $0 \leq t \leq 400$ km) are considered to be not dependent on the radius r, but on φ and λ only. They can be expressed by spherical harmonics in the following form, (59),

$$(62) \quad (\delta\varrho)_B = \alpha(\varphi,\lambda) = \sum_{n=0}^{3} (\delta\varrho)_{B.n} \cdot S_n(\varphi,\lambda) .$$

$(\delta\varrho)_B$ can be computed from the travel-time residuals $\tau(\varphi,\lambda)$, (60), according to the law of Birch, (58). These computations of $(\delta\varrho)_B$ can be executed before the adjustment calculations which follow later on. From (58) and (59) (60)(62), the relation (63) follows,

$$(63) \quad (\delta\varrho)_B = \alpha(\varphi,\lambda) = -\frac{1}{15} \cdot 10^3 \cdot \tau(\varphi,\lambda) .$$

$(\delta\varrho)_B$ is here in kg/m^3, and τ in seconds. If τ is equal to + 0.5 s , a value for $(\delta\varrho)_B$ of - 33 kg m^{-3} is reached for T = 400 km, Fig. 9.

But, the relation (63) is only a primitive picture of the function which gives $(\delta\varrho)_B$. The Airy-Heiskanen isostatic system has to be included into the layer of the upper 400 km. The inclusion of the mountain roots of this isostatic system, having a density jump of $(\Delta\varrho)_B$ transforms the relation (63) into the following amended shape, ($(\Delta\varrho)_B = -600\ kg/m^3$),

$$(64) \quad \tau = k_5 \left[T \cdot (\delta\varrho)_B + (\Delta\varrho)_B \cdot (b - T^*) \right] .$$

Here is, (52), [7], Fig. 4 and 9,

$$(65) \quad k_5 \cdot T = -15 \cdot 10^{-3}, \quad (b - T^*) > 0 .$$

The inversion of (64) gives the relation which computes the desity anomalies in the layers of the upper 400 km, $(\delta\varrho)_B$, in terms of the given seismological station anomalies τ and the given isostatic mountain roots,

$$(66) \quad (\delta\varrho)_B = c_1 \cdot \tau + c_2 \cdot \varrho_0 \cdot H ,$$

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or, [7],

(67) $(\delta\varrho)_B = -\frac{1}{15}\,10^3\tau + \frac{1}{T}\cdot\varrho_o\cdot H$.

The Airy- Heiskanen system is governed by the relations, (see (31a)(31b)),

(68) $(\Delta\varrho)_B\cdot(b-T^*) + \varrho_o\cdot H = 0$,

(69) $(\Delta\varrho)_B = -\ 600\ \mathrm{kg/m^3}$.

With the harmonics developments, (60)(38),

(70) $\tau = \sum_{n=o}^{3} (\tau)_n\cdot S_n(\varphi,\lambda)$

and

(71) $H = \sum_{n=o}^{3} H_n\cdot S_n\ (\varphi,\lambda)$,

the coefficients of which are tabulated in Table 2 and 3, we have, (62)(67),

(72) $(\delta\varrho)_{B.n} = -\frac{1}{15}\,10^3\cdot(\tau)_n + \frac{1}{T}\cdot\varrho_o\cdot H_n$;

(72a) $n = 0,1,2,...$.

These coefficients $(\delta\varrho)_{B.n}$ are shown in Table 5, third column.

In the interior of the Earth below of the uppermost layer with the density anomalies $(\delta\varrho)_B$ computed by inserting the law of Birch, (67)(72), we have the 3 layers of the width $T_1 = T_2 = T_3 =$ 833,3 km. The corresponding density anomalies are $(\delta\varrho)_{A.1}$, $(\delta\varrho)_{A.2}$, and $(\delta\varrho)_{A.3}$, see Fig. 9.

The surface spherical harmonics development for $(\delta\varrho)_{A.i}$, (i=1,2,3), is with (21)(22), [6][7],

(73) $(\delta\varrho)_{A.i} = \sum_n (\delta\varrho)_{i.n}\cdot S_n\ (\varphi,\lambda)$,

(74) $(i = 1,2,3)$.

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It can be taken from (73), the density anomalies in the 3 individual layers of the width 833,3 km do not vary in radial direction.

The coefficients of the 3 harmonics developments of (73), that are the unknowns of our problem which we have to determine. They represent the beforehand unknown density anomalies in the deep mantle,i.e.the depth range between $t = 400$ km and the core-mantle boundary.

Now, we come to the detailed definition of our mathematical model.

The model of the gravity potential W has the following expression in terms of the different gravitating scources,

$$(75) \quad W = U + W_I + W_B + W_A .$$

This equation is fundamental for our investigations.

The gravity potential W is explained by (28), the reference potential U has the representation (40), both of these expressions are given in spatial spherical harmonics. The isostatic potential W_I has the mass integral (29) and the spatial harmonics development (37). The potential W_B is the potential of the beforehand known density anomalies in the crust and upper mantle, $(\delta\rho)_B$, $(0 \leq t \leq 400$ km); see (62)(72). W_B has the following shape of a mass integral

$$(76) \quad W_B = G \iiint_{V_B} \frac{1}{e} \cdot dm_B .$$

Here,V_B is the volume of the shell situated between the depths $0 \leq t \leq 400$ km. With the volume element dV, we have for the mass element of (76),

$$(77) \quad dm_B = dV \cdot (\delta\rho)_B .$$

The potential W_B of (76) can be brought into the shape of the

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potential of a surface distribution in the depth of D = 200 km, Fig. 9. Considering (5)(6)(11)(14)(15)(72), the mass integral (76) turns to (78) for test points P on the surface of the Earth, [6][7],

$$(78)\quad W_B = W_B(P) = 4\pi \sum_n k_2 T \frac{(R-D)^n}{R^{n+1}} \frac{(R-D)^2}{2n+1} (\delta\rho)_{B.n} \cdot S_n(\varphi,\lambda) \; ;$$

$$(78a)\quad k_2 = G \; .$$

In (75), the potential W_A comes from the a priori unknown density anomalies; thus, it is given in terms of $(\delta\rho)_{A.1}$, $(\delta\rho)_{A.2}$, and $(\delta\rho)_{A.3}$ which are the density anomalies in the 3 different layers of the lower mantle. All these three layers have the same width of 833,3 km, (see Fig. 9). If V_A is the volume between the depth of t = 400 km and the core-mantle boundary, W_A has the mass integral (integrating over these 3 shells)

$$(79)\quad W_A = G \iiint_{V_A} \frac{1}{e}\, dm_A \; ,$$

with - for the 1st shell -

$$(80)\quad dm_A = dV \cdot (\delta\rho)_{A.1} \; ,$$

with - for the 2nd shell -

$$(81)\quad dm_A = dV \cdot (\delta\rho)_{A.2} \; ,$$

and with - for the 3rd shell -

$$(82)\quad dm_A = dV \cdot (\delta\rho)_{A.3} \; .$$

Hence, (80)(81)(82) show how to divide the integral (79), accounting for the densities of the different 3 layers filling the volume V_A described above.

If D_1, D_2, D_3 are the mean depths of these 3 layers, Fig. 9, and if T_1, T_2, T_3 are the width values of these 3 layers, we find according to (73)(78), [6][7],

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$$(83)\quad W_A = W_A(P) = 4\pi \sum_n k_2 T_1 \frac{(R-D_1)^n}{R^{n+1}} \frac{(R-D_1)^2}{2n+1} (\delta\rho)_{1.n} \cdot S_n(\varphi,\lambda) +$$

$$+ 4\pi \sum_n k_2 T_2 \frac{(R-D_2)^n}{R^{n+1}} \frac{(R-D_2)^2}{2n+1} (\delta\rho)_{2.n} \cdot S_n(\varphi,\lambda) +$$

$$+ 4\pi \sum_n k_2 T_3 \frac{(R-D_3)^n}{R^{n+1}} \frac{(R-D_3)^2}{2n+1} (\delta\rho)_{3.n} \cdot S_n(\varphi,\lambda) \; .$$

Herewith , considering (28)(40)(37)(78)(83), the expressions on the right and left hand side of (75) are explained; they can be represented by harmonics developments, [6][7].

In (75), the potentials W, U, W_I , and W_B have beforehand known functions. Thus, the unknown function W_A has the following constraint which is also a constraint for the a priori unknown coefficients of it, $(\delta\rho)_{i.n}$, (i = 1, 2, 3),

$$(84)\quad W_A = G \iiint_{V_A} \frac{1}{e}\, dm_A = W - U - W_I - W_B \; .$$

For the above constraint (84), we can introduce the symbol θ.

$$(85)\quad \theta = W - W_A - W_I - W_B - U = 0 .$$

θ can be decomposed into surface spherical harmonics,

$$(86)\quad \theta = \sum_n \theta_n \cdot S_n(\varphi,\lambda) \; .$$

Consequently, θ_n is the symbol for the following constraints,

$$(87)\quad \theta_n = (W-Z)_n - (W_A)_n - (W_I)_n - (W_B)_n - (U-Z)_n = 0,$$

$$(88)\quad n = 0, 1, 2, \ldots \; .$$

On the right hand side of (87), the shares of the harmonics of degree n of the individual 5 potentials can be found.
For instance for n = 2 , we find for test points in the exterior space,

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$$(89)\quad \sigma = \theta_2 = \frac{GM}{r}\left(\frac{a_e}{r}\right)^2 w_2 + \frac{GM}{r}\left(\frac{a}{r}\right)^2 \frac{2}{3} J\cdot\zeta - \frac{GM}{r}\left(\frac{a_e}{r}\right)^2 w_{I.2} -$$

$$-4\pi\, k_2 T \frac{(R-D)^2}{r^3} \frac{1}{5}(R-D)^2 (\delta\rho)_{B.2} -$$

$$-4\pi\, k_2 T_1 \frac{(R-D_1)^2}{r^3} \cdot \frac{1}{5}(R-D_1)^2 \cdot (\delta\rho)_{1.2} -$$

$$-4\pi\, k_2 T_2 \frac{(R-D_2)^2}{r^3} \frac{1}{5}(R-D_2)^2 \cdot (\delta\rho)_{2.2} -$$

$$-4\pi\, k_2 T_3 \frac{(R-D_3)^2}{r^3} \frac{1}{5}(R-D_3)^2 \cdot (\delta\rho)_{3.2} .$$

ζ stands for the transition from the Legendre function of 2. degree (P_2) to the harmonic $S_2(\varphi,\lambda)$ in the course of the full normalization, $P_2 = \zeta\cdot S_2$. (89) can be written in the following abbreviating form,

$$(90)\quad \sigma = \theta_2 = \left[W-U-W_I-W_B\right]_2 + k_{6.2}(\delta\rho)_{1.2} + k_{7.2}(\delta\rho)_{2.2} + k_{8.2}(\delta\rho)_{3.2} .$$

Obviously, (90) allows symbolically the following generalization for all degrees n,

$$(91)\quad \sigma = \theta_n = \left[W-U-W_I-W_B\right]_n + \sum_{j=1,2,3} \bar{k}_{j.n}(\delta\rho)_{j.n} ;$$

$$(92)\quad n = 0,\ 1,\ 2,\ \ldots .$$

$\bar{k}_{j.n}$ are given constants, the formulas of these constants can be obtained by a comparison with (87)(89). The comprehension and the clear understanding of the essentials of our coming deliberations will not be impaired considerably by the fact that not the detailed formulas for all the coefficients $\bar{k}_{j.n}$ can be given here, (see [7]).

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12. The determination of the density anomalies in the deep mantle

The relations (91) have the character of condition equations for the $(\delta\varrho)_{j.n}$ values (j = 1, 2, 3; n = 0, 1, 2, ...), which can be found in the last terms on the right hand side of (91). Each individual equation of the type (91) assigned to the index n has the character of one relation for the 3 unknown values $(\delta\varrho)_{1.n}$, $(\delta\varrho)_{2.n}$, $(\delta\varrho)_{3.n}$. Thus, the relations (91) do not suffice to determine the coefficients

(93) $(\delta\varrho)_{j.n}$;

(94) j = 1, 2, 3;

(95) n = 0, 1, 2, ...

in a unique way. The reason lies in the fact that we have only n equations for 3n unknown values.

Furthermore, in connection with the equations (16)(17)(18)(19), it was already discussed that the surface values of a potential do not allow to find precise and unique values for the amount and the spatial place (depth) of the gravitating masses in the interior of the Earth: The integrals of the type (5), giving the potential in terms of the gravitating scources $\delta\varrho$, have not a unique inversion. This is a fact well-known from exploration gravimetry.

Further on, it is not necessary to decompose our density anomalies distributed in the interior of the Earth, (i.e. $(\delta\varrho)_B$, $(\delta\varrho)_{A.1}$, $(\delta\varrho)_{A.2}$, $(\delta\varrho)_{A.3}$), into the part of them which is caused by elastic compression and, on the other hand, into the part of them which is caused by a spatial variation of the chemical composition. The reason is, that our fundamental relations, as the law of Birch (43) and the integral relations of the type (5), relate the density with the velocity of the P-waves and, further, the density with the gravity potential values in the exterior of the body of the Earth, irrespective to the deeper

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reasons which cause the density anomalies, may they be generated by elastic compression or may they be generated by distinctions in the chemical composition. This fact is a relief for our computations.

In order to find a reliable and plausible solution for the unknowns marked by (93), a reasonable working hypothesis is introduced. It makes the integrals over the squares of the unknown density anomalies $(\delta\rho)_{A.1}$, $(\delta\rho)_{A.2}$, $(\delta\rho)_{A.3}$ (given by (73) (74)) to a minimum value accounting simultaneously for the constraints of (91).

We define the following fundamental Γ operator as given by (96), (79)(80)(81)(82)(89)(91), [7],

$$(96) \quad \Gamma = \sum_{i=1,2,3} \iiint_{V_{A.i}} (\delta\rho)^2_{A.i} \, dV + \sum_{n=0}^{3} X_n \cdot \theta_n .$$

$V_{A.i}$ is the volume of the one single shell of the number i situated in the deeper mantle, Fig. 9; it has the mean depth D_i, the density anomaly $(\delta\rho)_{A.i}$, and the width T_i, (i=1,2,3). The symbols X_n mean Lagrange multipliers, they are a priori unknown.

Our working hypothesis is, [6][7],

$$(97) \quad \Gamma \longrightarrow \text{Minimum},$$

or, more detailed, in terms of the Stokes constants $(\delta\rho)_{i.n}$, (73),

$$(98) \quad \Gamma\left\{(\delta\rho)_{i.n}\right\} \longrightarrow \text{Minimum};$$

(98a) i = 1, 2, 3;   n = 0, 1, 2, 3, ...

For the subsequent mathematical derivations, considering the relation (96), it is recommended to change over from the $(\delta\rho)_{A.i}$ values to the unknown coefficients of the series developments of them, (73)(93). With regard to (24)(25)(26)(27)(27a), the relations

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relations (96)(98) turn over to

(99) $\Gamma = 4\pi \sum_{i=1}^{3} \eta_i^2 \cdot T_i \cdot \sum_{n=0}^{3} (\delta\rho)^2_{i.n} + \sum_{n=0}^{3} X_n \cdot \theta_n$;

with, (27b),

(99a) $v_{A.i} \cong 4\pi T_i \cdot \eta_i^2$.

In the depth range between t = 400 km and the core-mantle boundary, η_i is the mean radius of the spherical shell of the number i and the mean depth D_i , (i = 1,2,3), Fig. 2, 9.

The minimum principle (98) demands the fulfillment of the following relations, observing (91):

(100) I.) $\frac{\partial \Gamma}{\partial (\delta\rho)_{1.n}} = 0$,

(101) II.) $\frac{\partial \Gamma}{\partial (\delta\rho)_{2.n}} = 0$,

(102) III.) $\frac{\partial \Gamma}{\partial (\delta\rho)_{3.n}} = 0$;

(103) IV.) $\theta_n = 0$;

(104) n = 0, 1, 2, ...

The relation (103) comes from (91), it has to be observed simultaneously with the derivatives (100)(lo1)(lo2). The equations from (100) to (103) construct the system which allows the determination of the unknown values of (93) and the unknown X_n values.

After the Stokes constants of (93) will be found by the inversion of the determining system I, II, III, and IV, we will find the series development (73). The reader will be well-acquainted with inversion calculations of this kind. Therefore, the author can dispense himself from the task to give a comprehensive descrip-

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tion of these calculations at great length; (Gaussian algorithm).

The last 3 columns of Table 5 give the detailed amounts of the Stokes constants of (93) obtained by (100) to (104), specified for the three shells in the individual depth ranges of 400 km $\leq t \leq$ 1233 km, 1233 km $\leq t \leq$ 2067 km, and 2067 km $\leq t \leq$ 2900 km; this are the three layers in the deep mantle of the Earth, Fig. 9.

Table 5 shows the results of all our investigations. As can be taken from Table 5, the masses of the density $(\delta\varrho)_B$ are rather well compensated by the masses of the density $[(\delta\varrho)_{A.1} + (\delta\varrho)_{A.2}]$. Consequently, the density anomalies in the depth range $0 \leq t \leq$ 400 km are rather well compensated by the density anomalies in the depth range 400 km $\leq t \leq$ 2067 km. We have, (62)(72)(73),

(104a) $(\delta\varrho)_{B.n} \cong -(\delta\varrho)_{1.n} - (\delta\varrho)_{2.n}$,

or, summing-up over the harmonics of all the degrees and orders here considered,

(105) $(\delta\varrho)_B \cong -(\delta\varrho)_{A.1} - (\delta\varrho)_{A.2}$.

Finally, it is useful to execute the step from the Stokes constants of (93) to the full density anomalies covering all the harmonics here involved,

(106) $(\delta\varrho)_B$, $(\delta\varrho)_{A.1}$, $(\delta\varrho)_{A.2}$, and $(\delta\varrho)_{A.3}$.

Along the deliberations connected with (24)(25)(26)(27)(27c), the r.m.s. values for the 4 functions of (106) are computed. With (24)(27c), these r.m.s. values are denominated by the terms of (107), (see [6][7]),

(107) $((\delta\varrho)_B)_a$, $((\delta\varrho)_{A.1})_a$, $((\delta\varrho)_{A.2})_a$, $((\delta\varrho)_{A.3})_a$.

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The individual amounts of the 4 r.m.s. values of (107) are figured in the graph of Fig. 10, in dependence on the distance from the center of the Earth. In the crust and upper mantle, the r.m.s. value of the density anomalies is about 25 kg/m^3, for $0 \leq t \leq$ 400 km. In the 3 shells in the deep mantle, we have the r.m.s. values of 14, 10, and 6 kg/m^3, respectively.

The pure gravimetric evaluation type without seismological data gave the amount of 1 kg/m^3 only, (4). This value - being free of seismology - is by far too small, consequently. (See also the discussion about the work of Kaula and that of Tscherning/Sünkel presented in the final remarks,chapter 13. These authors found quantities one order too small.)

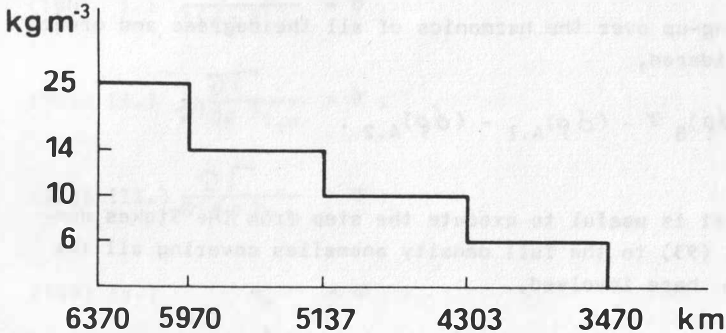

Fig. 10.

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13. Final remarks

Further amendments of the investigations above can possibly happen along the following lines:

1. Constraints for the inertial moments of the Earth can be introduced.

2. A constraint for the dynamical flattening H^*, (39), can be of help.

3. Further condition equations, already applied in [7], can come from a consideration of the gravity potential field in the interior of the Earth's core. In this context, it is of interest that the mass in the exterior core is commonly regarded as a fluid. Therefore, in this area, the gravity potential has to be represented by the zonal harmonics of the 0th and 2nd degree of the potential U, only, (40). Following up this concept, along the core-mantle boundary, a condition for the isostatic potential and for the potential caused by the density anomalies (situated between the surface of the Earth and the core) follows. This condition prohibits in the exterior core that tesseral and sectorial harmonics of 2nd degree come into existence, as so as all the harmonics of degree 3, 4, ... In [7], this speciality was considered by computing a special version of our mathematical model.

4. The fact can be put into the fore that Europe and North America have a relative dense coverage by the τ values of the travel time residuals, Fig. 6, 7. Thus, for these areas, it will be of interest to find out what will come out if the density anomalies are represented by finite elements, instead of the usually used spherical harmonic development. These finite elements have to have the shape of bodies of three-dimensional extension. Along these lines, it is possible to check how far our finally obtained density anomalies are biassed by the fact that spherical harmonics were favoured in this publication in hand, in the mathematical representation of the data material.

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5. Further, the isostatic potential W_I can be extended and refined by the inclusion of the terms quadratic in the heights, $(H/R)^2$, [17], (see also: Arnold, K., The isostatic potential including the 2nd - order terms. Gerlands Beiträge z. Geophysik 89(1980), 287-293).

Finally, in this context, it should be mentioned that significant values for spherical harmonics developments of the velocity anomalies of the seismic waves have been determined by Dziewonski et al., see [7]. In [7], we computed the r.m.s. values of these velocity anomalies, and we compared them with the r.m.s. values of the density anomalies in the deep mantle, (107), Fig. 10, Table 5; $((\delta\rho)_{A.i})_a$, (i = 1,2,3). The quotient of these two r.m.s. values (thus, this quotient is determined by the definition: The r.m.s. value of the seismic velocity anomaly has to be divided through the r.m.s. value of the density anomaly) was in the mean about $x = 0.0022$. For the upper 400 km, the corresponding coefficient x obtained by the law of Birch was $x = 0.00264$, (43). Both these values are in good neighbourhood.

At an earlier time, Kaula evaluated the density anomalies in the deep mantle, (Elastic models of the mantle corresponding to variations in the external gravity field. J. Geophys. Res. 68 (1963), 4967-4978). Data from seismology were not introduced. Kaula found a r.m.s. value for the density anomalies in the deep mantle of about $\pm$ 1 kg m^{-3}. This value is too small by one order (factor 0.1). Thus, this value is not a realistic one. The real value will be about 10 time greater, because otherwise the constraints from the seismological data cannot be fulfilled. The same statement is valid for a more recent paper by Tscherning and Sünkel, (A method for the construction of spheroidal mass distributions .... Veröff. Zentralinst. Physik d. Erde, Potsdam 63 (1981)II, 481-500).

The here considered mathematical model is relative simple, since the here unnecessary elastic deformation considerations are not involved. Indeed, the density anomalies here obtained will cause gravitational forces which are relative small and long-time

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effective. Thus, regarding the rigidity of the material in the interior, and regarding these above discussed small gravitational forces, it will be questionable whether we are over the concerned threshold value which opens the door to enter the area where the common elasticity theory is valid. This Earth model here discussed is in good harmony with both the geophysical and geodetic conceptions.

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15. Tables

Table 1

Coefficients of the gravity potential GEM 10, w_n values.

n	m	$w_n \cdot 10^6$
1	0	0
1	1 c	0
1	1 s	0
2	0	-484.165
2	1 c	0
2	1 s	0
2	2 c	2.43
2	2 s	-1.40
3	0	0.96
3	1 c	2.03
3	1 s	0.25
3	2 c	0.89
3	2 s	-0.62
3	3 c	0.70
3	3 s	1.41

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Table 2

The Stokes constants of the isostatic potential and of the heights, $w_{I.n}$ and H_n .

n	m	$w_{I.n} \cdot 10^6$	H_n /meter/
1	0	0.109	447
1	1 c	0.106	385
1	1 s	0.086	273
2	0	0.134	288
2	1 c	0.054	200
2	1 s	0.081	227
2	2 c	-0.090	-274
2	2 s	-0.005	- 33
3	0	-0.095	- 99
3	1 c	-0.039	- 99
3	1 s	0.048	78
3	2 c	-0.124	-313
3	2 s	0.108	299
3	3 c	0.021	71
3	3 s	0.111	344

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Table 3.

Spherical Harmonics Development for the Station Anomalies τ.

n	m	τ, s
1	0	0.159
1	1c	- 0.014
1	1s	0.086
2	0	- 0.149
2	1c	0.002
2	1s	- 0.159
2	2c	- 0.062
2	2s	0.100
3	0	- 0.040
3	1c	- 0.089
3	1s	0.080
3	2c	0.113
3	2s	- 0.053
3	3c	- 0.015
3	3s	- 0.013

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Table 4

Stations Corrections

Code	Station	N	A	B	E	δ^2
ABE	Aberdeen, Scotland	12	1.86	1.75	156	4.04
ATH	Athens, Greece	59	.05	1.00	222	1.35
BOB	Bagnerres de Bigorre, France	39	- .45	.51	138	2.21
BNS	Bensberg, Germany	45	.14	.32	322	.43
BEO	Beograd (Belgrade), Yugoslavia	59	.88	.47	156	1.40
BES	Besancon, France	46	- .40	.53	113	.82
BRA	Bratislava,Czechoslovakia	57	- .01	.56	121	.80
BUC	Bucharest, Romania	20	2.49	2.91	267	3.57
BUD	Budapest, Hungary	35	- .44	1.75	85	1.79
CRT	Cartuja (Granada), Spain	39	.85	.76	175	3.49
CHE	Cheb, Czechoslovakia	25	.25	1.06	126	3.27
CFF	Clermont Ferrand, France	43	.45	.87	112	.94
CLL	Collmberg, Germany	108	.00	.25	191	.51
COP	Copenhagen, Denmark	98	.85	.51	177	.76
DBN	Debilt, Holland	28	1.90	.71	289	2.01
DUR	Durham, England	39	.86	.23	67	1.50
FIR	Firence, Italy	25	2.14	3.91	188	7.57
FLN	Foliniere, France	63	- .18	.37	131	1.11
GOT	Goteborg, Sweden	60	- .05	.84	185	.94
HEL	Helsinki, Finland	47	- .03	.62	90	.73
JEN	Jena, Germany	110	- .28	.28	115	.88
KRL	Karlsruhe, Germany	18	- .07	1.83	66	3.72
KHC	Kasperske Hory, Czechoslovakia	62	- .60	.52	75	.80
KRA	Krakow, Poland	99	.02	.36	72	.76
LIS	Lisbon, Portugal	31	.68	.47	151	1.68
LJU	Ljubljana, Yugoslavia	54	.12	.49	157	.87
MOS	Moskow, USSR	156	.04	.17	181	.86
MWG	Münster-Westfalen, Germany	11	.47	.56	119	.67
PAR	Paris, France	27	.06	1.16	80	.98
PRA	Prague, Czechoslovakia	38	.56	.76	161	1.47
PUL	Poulkovo, USSR	140	- .14	.29	36	.87
REY	Reykjavik, Iceland	26	2.13	.59	359	1.19
STR	Strasbourg, France	96	.13	.47	108	.75
STU	Stuttgart, Germany	143	- .30	.62	74	.75

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Table 5

Final Spherical Harmonics Developments for the Density Anomalies in the Earth's Mantle.

n	m	$(\delta\rho)_B$ Depth 0 - 400 km, kg/m^3	$(\delta\rho)_{A.1}$ Depth 400-1233 km, kg/m^3	$(\delta\rho)_{A.2}$ Depth 1233-2067 km, kg/m^3	$(\delta\rho)_{A.3}$ Depth 2067-2900 km, kg/m^3
1	0	- 7.7	3.0	2.6	2.1
1	1c	3.5	- 1.4	- 1.2	- 1.0
1	1s	-3.9	1.6	1.3	1.1
2	0	11.9	- 6.2	- 4.5	- 3.0
2	1c	1.2	- 0.6	- 0.5	- 0.3
2	1s	12.1	- 6.3	- 4.6	- 3.1
2	2c	2.3	- 1.0	- 0.7	- 0.5
2	2s	- 6.9	3.5	2.5	1.7
3	0	2.0	- 1.1	- 0.7	- 0.4
3	1c	5.3	- 3.1	- 1.9	- 1.1
3	1s	- 4.8	3.1	1.9	1.1
3	2c	- 9.6	6.2	3.8	2.1
3	2s	5.5	- 3.6	- 2.2	- 1.2
3	3c	1.5	- 0.8	- 0.5	- 0.3
3	3s	3.2	- 1.8	- 1.1	- 0.6

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C. Considerations about the term $C_1(M)$

Contents

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Summary

In the solution of the geodetic boundary value problem, the term $C_1(M)$ appears in the integrand of the Stokes integral; [2], equation (3) on page 10. This term can be represented by the smoothed Bouguer anomalies for numerical routine computations; [2], equation (4) on page 10. $C_1(M)$ has positive and negative amounts which surmount 1 mgal in seldom cases, only. This mathematical expression of $C_1(M)$ in terms of the Bouguer anomalies is in the fore. It is proved that the expression (4) on page 10 of [2] is sufficient precise for our applications, the residua can be neglected.

Zusammenfassung

Die Lösung des geodätischen Randwertproblems enthält im Integranden des Stokes-schen Integrals den Ausdruck $C_1(M)$; [2], Gleichung (3), Seite 10. Dieser Ausdruck kann durch Bougueranomalien ausgedrückt werden; man erhält so eine Formel, die für numerische Routineberechnungen besonders geeignet ist, weil die Bougueranomalien einen glatten Verlauf haben; [2], Gleichung (4), Seite 10. $C_1(M)$ hat positive und negative Werte, die selten den Betrag von 1 mgal übersteigen. Dieser mathematische Ausdruck für $C_1(M)$ steht hier im Vordergrund. Es wird gezeigt, daß der Ausdruck (4) auf Seite 10 von [2] für unsere Anwendungen genügend genau ist; die dabei vernachlässigten Terme sind bedeutungslos.

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1. On the definition of the term $C_1(M)$

The term $C_1(M)$ here to be considered is defined by the equations (221), (219), and (217a) on the pages 60 and 61 of [2],

(1) $$C_1(M) = GZ \cdot \Phi(\mu_1, \mu_2)$$

with

(2) $$\Phi(\mu_1, \mu_2) = \Phi(\mu_{1.u}, \mu_{2.u}) \quad .$$

(3) $$\Phi(\mu_1,\mu_2) = \frac{1}{R'} \cdot \frac{\partial \mu_{1.u}}{\partial \varphi} + \frac{1}{R'\cos\varphi} \frac{\partial \mu_{2.u}}{\partial \lambda} - \frac{1}{R'} \tan\varphi \cdot \mu_{1.u} \quad .$$

The model potential M is

(4) $$M = T - B \quad ,$$

where T is the usual perturbation potential, and where B is the gravitational potential of the mountain masses situated above ocean level (having the standard density $\varrho_o = 2.67 \text{ g cm}^{-3}$); [2], pg. 46 and 47. G is the global mean gravity, Z is the difference between the height H_Q of the running point Q and the height H_P of the fixed test point P,

(5) $$Z = H_Q - H_P \quad .$$

(6) $$\mu_1 = \mu_{1.u}$$

and

(7) $$\mu_2 = \mu_{2.u}$$

are the north-south and the east-west components of the plumb-line deflection on the surface of the Earth u, they are computed for the potential M. R' is the radius of the test point P,

(8) $$R' = R + H_P \quad .$$

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φ and λ are the geocentric latitude and longitude. The deflection components at the surface of the Earth u are obtained from M by, ([2], pg. 48, eq. (153))

$$(9) \qquad \mu_1 = \mu_{1.u} = -\left[\frac{1}{g'''} \cdot \frac{\partial M}{R' \partial \varphi}\right]_u ,$$

and

$$(10) \qquad \mu_2 = \mu_{2.u} = -\left[\frac{1}{g'''} \cdot \frac{1}{R' \cos\varphi} \cdot \frac{\partial M}{\partial \lambda}\right]_u ;$$

with

$$(11) \qquad g''' = \left| \nabla (U + M) \right| .$$

U is the standard potential.
In (9) and (10), it is allowed to introduce some approximations. g''' can be replaced by the global mean of the gravity G, and R' can be substituted by R; these approximations involve relative errors of not more than about 1/300. μ_1 and μ_2 are two-parametric functions along the surface of the Earth, as evidenced by (9) and (10). Thus,

$$(12) \qquad \mu_1 = \alpha(\varphi, \lambda) = -\frac{1}{GR}(\vartheta_\varphi)_u ,$$

$$(13) \qquad \mu_2 = \beta(\varphi, \lambda) = -\frac{1}{GR \cdot \cos\varphi}(\vartheta_\lambda)_u .$$

Here, ϑ is the spatial function for the spatial potential M. In spatial polar coordinates r, φ, λ, we have

$$(14) \qquad M = \vartheta(r, \varphi, \lambda) .$$

The potential M, ϑ, T, and B are harmonic functions,

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(15) $\Delta M = \Delta \vartheta = \Delta T = \Delta B = 0.$

According to (15), the Laplace-operator for ϑ is, [3] [5] ,

(16) $$0 = \Delta \vartheta = \vartheta_{rr} + \frac{2}{r} \vartheta_r + \frac{1}{r^2} \vartheta_{\varphi\varphi} + \frac{1}{r^2 \cos^2 \varphi} \vartheta_{\lambda\lambda} - \frac{1}{r^2} \tan \varphi \cdot \vartheta_\varphi$$

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2. The development for the term $\Phi(\mu_1, \mu_2)$

Considering the third term on the right hand side of (16), we have the term

$$\vartheta_{\varphi\varphi} = \frac{\partial^2 \vartheta}{\partial \varphi^2} \quad . \tag{17}$$

It does contain the derivatives of ϑ along the line where only the φ values vary, but where the values r and λ are constant. The line where only the φ values vary is horizontal, and it has north-south direction. A similar property is valid for the expression $\vartheta_{\lambda\lambda}$ of (16).
But, in (12) and (13), the functions α and β describe quantities distributed along the surface of the Earth u. Thus, φ and λ are Gauss curvilinear coordinates on the surface u, in case of the functions α and β .
In this context, we are confronted with the problem to express the derivative

$$\frac{\partial \alpha}{\partial \varphi} \tag{18}$$

in terms of the second derivatives of the function ϑ.
The derivation (18) happens along the surface path from Q_a to Q_b, Fig. 1. Fig. 1 is a cross-section through the surface of the Earth u for the case that λ = const. But, if the derivations of ϑ are in the fore, Q_b can be reached from Q_a along another way by a first step from Q_a to A, and by the ensuing second step from A to Q_b, Fig. 1. During the first step, r and λ are constant. During the second step, φ and λ are constant.

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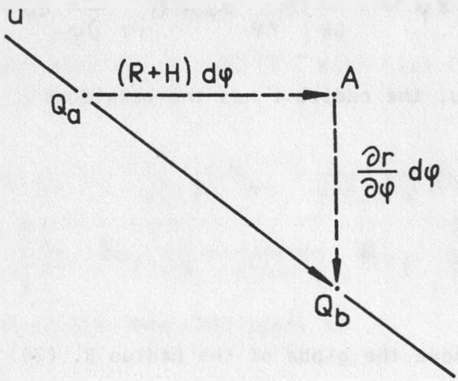

Fig. 1: The replacement of the oblique derivation in the direction of φ by a horizontal and a vertical derivation.

From Fig. 1, the following self-explanatory lines can be taken, (12) (13),

(19) $$\frac{\partial \mu_1}{(R+H)\partial\varphi}\cdot(R+H)\,d\varphi = \frac{\partial\mu_1(\varphi,\lambda)}{\partial\varphi}\,d\varphi =$$

$$= \frac{\partial\alpha(\varphi,\lambda)}{\partial\varphi}\,d\varphi = (\mu_1)_{Q_b} - (\mu_1)_{Q_a} =$$

$$= (\alpha)_{Q_b} - (\alpha)_{Q_a} = -\frac{1}{GR}\left[(\vartheta_\varphi)_{Q_b} - (\vartheta_\varphi)_{Q_a}\right] =$$

$$= -\frac{1}{GR}\left[(\vartheta_\varphi)_{Q_b} - (\vartheta_\varphi)_A + (\vartheta_\varphi)_A - (\vartheta_\varphi)_{Q_a}\right] .$$

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Hence,

(20) $$\frac{\partial \alpha(\varphi,\lambda)}{\partial \varphi}\, d\varphi = -\frac{1}{GR}\left[\vartheta_{\varphi\varphi}\, d\varphi + \vartheta_{\varphi r}\frac{\partial r}{\partial \varphi}\, d\varphi\right].$$

Along the surface u, the radius r has the relations

(21) $$r = R + H_Q \;,$$

(22) $$\frac{\partial r}{\partial \varphi} = \frac{\partial H_Q}{\partial \varphi} \;.$$

H_Q is the height above the globe of the radius R. (20) and (22) can be combined to

(23) $$\frac{\partial \alpha(\varphi,\lambda)}{\partial \varphi} = -\frac{1}{GR}\left[\vartheta_{\varphi\varphi} + \vartheta_{r\varphi}\frac{\partial H_Q}{\partial \varphi}\right].$$

Obviously, in a similar way, the derivative of β with regard to λ can be found, Fig. 2, (13). Fig. 2 is a cross-section through the surface of the Earth u for the case that φ = const.

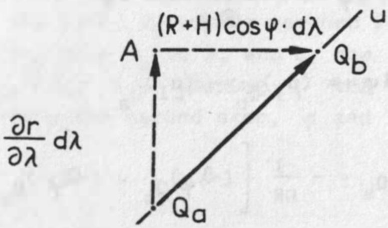

Fig. 2: The replacement of the oblique derivation in the direction of λ by a horizontal and vertical derivation step.

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The subsequent relation results, (13),

$$(24)\quad \frac{\partial \beta(\varphi,\lambda)}{\partial \lambda} = -\frac{1}{GR\cdot \cos\varphi}\left[\vartheta_{\lambda\lambda} + \vartheta_{r\lambda}\frac{\partial H_Q}{\partial \lambda}\right].$$

Now, we return back to (2) and (3). With (12) (13) (23) (24), the expression (2) turns to

$$(25)\quad \Phi(\mu_1,\mu_2) = -\frac{1}{G}\left[\frac{1}{R^2}\vartheta_{\varphi\varphi} + \frac{1}{R^2\cos^2\varphi}\vartheta_{\lambda\lambda} - \frac{\tan\varphi}{R^2}\cdot\vartheta_{\varphi}\right] - \\ -\frac{1}{G}\left[\frac{1}{R^2}\vartheta_{r\varphi}\frac{\partial H_Q}{\partial \varphi} + \frac{1}{R^2\cos^2\varphi}\vartheta_{r\lambda}\frac{\partial H_Q}{\partial \lambda}\right].$$

A comparison of (16) and (25) leads to

$$(26)\quad \Phi(\mu_1,\mu_2) = \Phi_1 + \Phi_2 ,$$

with

$$(27)\quad \Phi_1 = \frac{1}{G}\left[\vartheta_{rr} + \frac{2}{R}\vartheta_r\right] ,$$

$$(28)\quad \Phi_2 = -\frac{1}{G}\left[\frac{1}{R^2}\vartheta_{r\varphi}\frac{\partial H_Q}{\partial \varphi} + \frac{1}{R^2\cos^2\varphi}\vartheta_{r\lambda}\frac{\partial H_Q}{\partial \lambda}\right].$$

H_P is fixed. Thus, (5),

$$(29)\quad \frac{\partial H_Q}{\partial \varphi} = \frac{\partial Z}{\partial \varphi} ,\quad \frac{\partial H_Q}{\partial \lambda} = \frac{\partial Z}{\partial \lambda} .$$

The combination of the equations (1) (28) (29) yields

$$(30)\quad GZ\,\Phi_2 = -\frac{1}{2}\left[\frac{1}{R^2}\vartheta_{r\varphi}\frac{\partial Z^2}{\partial \varphi} + \frac{1}{R^2\cos^2\varphi}\vartheta_{r\lambda}\frac{\partial Z^2}{\partial \lambda}\right].$$

With (1) (26) (27) (28) (30), the equation (31) follows

$$(31)\quad C_1(M) = GZ\cdot\Phi = GZ\left[\Phi_1 + \Phi_2\right].$$

In the solution of the geodetic boundary value problem, the term $C_1(M)$ appears in the integrand of the Stokes integral, (68); [2], page 10, equation (3). Therefore, the following terms have a

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direct impact on the perturbation potential T obtained by the boundary value problem,

$$\Psi = \frac{1}{4\pi R} \iint_{v} C_1(M) \cdot S(p) \cdot dv \,, \tag{32}$$

$$\Psi = \Psi_1 + \Psi_2 \,, \tag{33}$$

$$\Psi_1 = \frac{1}{4\pi R} \iint_{v} GZ \cdot \Phi_1 \cdot S(p) \cdot dv \,, \tag{34}$$

$$\Psi_2 = \frac{1}{4\pi R} \iint_{v} GZ \cdot \Phi_2 \cdot S(p) \cdot dv \,. \tag{35}$$

The sphere v has the radius $R + H_p$.

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3. The term Ψ_1

Ψ_1 is defined by (34). In the integrand of this expression, the term $G \cdot \Phi_1$ appears. With (27), it has the following development,

$$(36) \quad G \cdot \Phi_1 = \vartheta_{rr} + \frac{2}{R} \cdot \vartheta_r \; .$$

In [2] , page 77, equation (274), it was demonstrated that the radial derivative of M can be put equal to the Bouguer anomalies Δg_{Bou} with the reverse sign. Hence,

$$(37) \quad \frac{\partial \vartheta}{\partial r} = \vartheta_r \cong - \Delta g_{Bou} \; .$$

Comparing (36) and (37), it seems to be possible to express $G \cdot \Phi_1$ by the Bouguer anomalies. In this context, it seems to be convenient to introduce the harmonic potential $V = V(r, \varphi, \lambda)$ by

$$(38) \quad \Delta V = 0$$

and by

$$(39) \quad V = r \cdot \vartheta_r \; .$$

The vertical derivative of V has the following relation, [4] pg. 38,

$$(40) \quad \left(V_r\right)_Q = - \frac{1}{R} V_Q + \frac{R^2}{2\pi} \iint_\omega \frac{V_Y - V_Q}{e_{oo}^3} d\omega \; .$$

ω is the unit sphere.
The radial derivation of (39) gives (for $r = R$),

$$(41) \quad V_r = R \cdot \vartheta_{rr} + \vartheta_r \; .$$

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(39) and (41) is inserted into (40).
Hence,

$$(42)\qquad R\cdot\vartheta_{rr} + \vartheta_r = -\vartheta_r + \frac{R^2}{2\pi}\iint_\omega \frac{(R\vartheta_r)_Y - (R\vartheta_r)_Q}{e_{oo}^3}\,d\omega$$

(42), (36), and (37) give

$$(43)\qquad G\cdot\Phi_1 = -\frac{R^2}{2\pi}\iint_\omega \frac{(\Delta g_{Bou})_Y - (\Delta g_{Bou})_Q}{e_{oo}^3}\,d\omega\ .$$

Consequently, (34) takes the following final shape

$$(44)\qquad \Phi_1 = \frac{1}{4\pi R}\iiint_v \left[-Z\frac{R^2}{2\pi}\iint_\omega \frac{(\Delta g_{Bou})_Y - (\Delta g_{Bou})_Q}{e_{oo}^3}\,d\omega\right] S(p)\cdot dv.$$

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4. The term Ψ_2

The expression for Ψ_2 is given by (35). The formula for the integrand of (35) is represented by (30); (30) can be written in the shape of a scalar product. With the vector

$$(45)\quad \underline{\underline{q}}_1 = \begin{pmatrix} q_{1.1} \\ q_{1.2} \end{pmatrix} = \begin{pmatrix} \frac{1}{R}\vartheta_{r\varphi} \\ \frac{1}{R\cos\varphi}\vartheta_{r\lambda} \end{pmatrix},$$

and

$$(46)\quad \underline{\underline{q}}_2 = \begin{pmatrix} q_{2.1} \\ q_{2.2} \end{pmatrix} = \begin{pmatrix} \frac{1}{R}\frac{\partial Z^2}{\partial\varphi} \\ \frac{1}{R\cos\varphi}\frac{\partial Z^2}{\partial\lambda} \end{pmatrix},$$

the relation (30) takes the shape

$$(47)\quad GZ\cdot\Phi_2 = -\frac{1}{2}\cdot\underline{\underline{q}}_1\cdot\underline{\underline{q}}_2 .$$

According to (45) and (46), the vectors $\underline{\underline{q}}_1$ and $\underline{\underline{q}}_2$ can be written as gradients, which are situated in the horizontal plane, (that is the ∇ operator),

$$(48)\quad \underline{\underline{q}}_1 = \nabla(\vartheta_r) ,$$

$$(49)\quad \underline{\underline{q}}_2 = \nabla(Z^2) .$$

For the rearrangement of the integrand of (35), we put (see [1])

$$(50)\quad \underline{\underline{t}} = \underline{\underline{q}}_1\cdot Z^2\cdot S(p) .$$

The multiplication with the nabla operator leads to

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(51) $\nabla \underline{\underline{t}} = \underline{\underline{q}}_1 \cdot Z^2 \cdot \nabla S(p) + \underline{\underline{q}}_1 \cdot \nabla Z^2 \cdot S(p) + \nabla \underline{\underline{q}}_1 \cdot Z^2 \cdot S(p)$.

Here, we have

(52) $\underline{\underline{q}}_1 \cdot \nabla S(p) = \frac{1}{R^2} \vartheta_{rp} \cdot S_p$.

Further, Beltrami's differential parameter of the second order gives

(53) $\nabla \underline{\underline{q}}_1 = \nabla^2 (\vartheta_r) = \Delta_2 (\vartheta_r)$,

with, (16),

(54) $\Delta_2 \vartheta_r = \frac{1}{R^2} \vartheta_{r\varphi\varphi} + \frac{1}{R^2 \cos^2\varphi} \vartheta_{r\lambda\lambda} - \frac{1}{R^2} \tan\varphi \cdot \vartheta_{r\varphi}$.

Inserting (52) (53) (54) into (51), the relation (55) is obtained,

(55) $\underline{\underline{q}}_1 \cdot \underline{\underline{q}}_2 \cdot S(p) = - Z^2 \cdot \left(\Delta_2 \vartheta_r \right) \cdot S(p) - Z^2 \frac{1}{R^2} \vartheta_{rp} \cdot S_p + \nabla \underline{\underline{t}}$.

(35) and (47) gives

(56) $\Psi_2 = - \frac{1}{8\pi R} \iint\limits_v \underline{\underline{q}}_1 \cdot \underline{\underline{q}}_2 \cdot S(p) \cdot dv$.

Thus,

(57) $\Psi_2 = \frac{1}{8\pi R} \iint\limits_v Z^2 \cdot \left(\Delta_2 \vartheta_r \right) \cdot S(p) \cdot dv +$

$+ \frac{1}{8\pi R} \iint\limits_v Z^2 \frac{1}{R^2} \vartheta_{rp} \cdot \frac{\partial S(p)}{\partial p} \cdot dv -$

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$$- \frac{1}{8\pi R} \iint\limits_{v} \nabla \underline{\underline{t}} \cdot dv \quad .$$

The integrands in the first and second integral on the right hand side of (57) are well defined, because we consider a starshaped Earth which has per definitionem finite values for $Z^2 \cdot S(p)$ and for $Z^2 \frac{\partial S}{\partial p}$.

As to the third term on the right hand side of (57), for the investigation of it, the test point P is surrounded by a very small circle c_o of the radius $R \cdot p_o$ .The interior of this circle is v_o and the exterior v_{oo},

(58) $\quad v = v_o + v_{oo} \quad .$

The unit vector of the normal of this circle is $\underline{\underline{n}}^o$, it is heading into the exterior of the circle c_o, Fig. 3.

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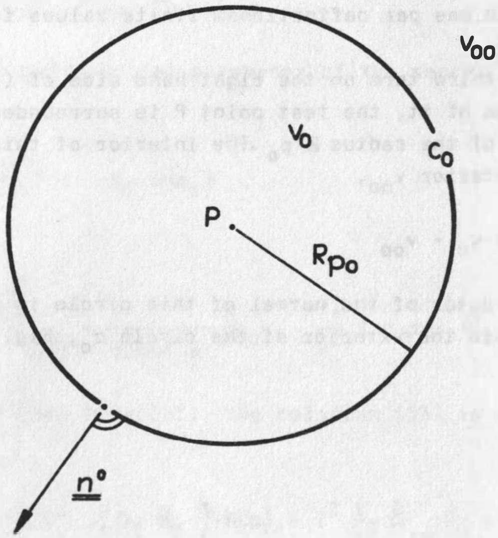

Fig. 3: The Gauss divergence theorem is extended over the area v_{oo} and its boundary c_o.

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The divergence of the vector $\underline{\underline{t}}$ is treated by the Gauss divergence theorem, [4] , [2] pg. 58.

(59) $$\iint_{V_{oo}} (\nabla \cdot \underline{\underline{t}})\, dv = - \int_{c_o} \underline{\underline{t}} \cdot \underline{\underline{n}}^o \cdot dc_o \quad .$$

If p_o tends to zero, $\underline{\underline{t}}$ tends to

(60) $$\underline{\underline{t}} \longrightarrow \underline{\underline{q}}_1 \cdot \tau^2 \cdot (Rp_o)^2 \cdot \frac{2}{p_o} \quad ,$$

with

(61) $$\tau = \frac{Z}{Rp_o} \quad .$$

The amount of $\underline{\underline{q}}_1$ is finite and continuous, (48). The quantity τ is finite because we have a starshaped Earth. Thus, (60) turns to

(62) $$\underline{\underline{t}} \longrightarrow \underline{\underline{q}}_1 \cdot \tau^2 \cdot 2 \cdot R^2 p_o \quad .$$

Hence,

(63) $$|\underline{\underline{t}}| \longrightarrow 0, \text{ if } p_o \longrightarrow 0 \quad .$$

Further, the length of the circle c_o is equal to $2\pi Rp_o$. Consequently, a look on the right hand side of (59) shows that the amount of the integral on this side tends to zero as p_o^2 if p_o tends to zero. Thus, (59),

(64) $$\iint_V (\nabla \underline{\underline{t}})\, dv = 0 \quad .$$

(37), (57), and (64) yield

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(65) $\Psi_2 = \theta_1 + \theta_2$,

with

(66) $$\theta_1 = -\frac{1}{8\pi R}\iiint_V Z^2 \cdot (\Delta_2 \Delta g_{Bou}) \cdot S(p) \cdot dv \ ,$$

(67) $$\theta_2 = -\frac{1}{8\pi R}\iiint_V \left[\frac{Z}{R}\right]^2 \cdot \left(\frac{\partial}{\partial p} \Delta g_{Bou}\right) \cdot \frac{\partial S(p)}{\partial p} \cdot dv.$$

Obviously, the deductions from (45) to (67) involve some simplifications. Of course, certain oblique derivations were substituted by their horizontal derivations. But, these simplifications will have a small effect on the quantity of the term Ψ_2 . These simplifications will not change the order of the quantity of Ψ_2 . In the next paragraph 5, the quantity of the term Ψ_2 comes out to be negligible, (71) (72) (76) (77) (78). Thus, these simplifications in the mathematical deductions from (45) through (67) will falsify the term Ψ_2 by negligible quantities, only. These simplifications in the deductions executed in order to reach (66) and (67) have the same basing philosophy as a simplification in the relation (33) which comes into being by the neglection of the expression (35).
Consequently, the evaluations executed in the next paragraph will yield reliable quantities for the crucial term Ψ_2 .

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5. The quantity of the term Ψ_2

At first, the amount of the term θ_1 is to be evaluated, (66). The solution of the boundary value problem was ([2], pg. 10, eq. (3))

$$(68) \quad T = \frac{1}{4\pi R} \iint_v \left[\Delta g_T + C + C_1(M) \right] \cdot S(p) \cdot dv + \{ \Omega(M) \} .$$

Δg_T are the free-air anomalies and C is the plane terrain reduction of the gravity. The supplementary term $\{\Omega(M)\}$ is explained in [2] .

Comparing (66) and (68), it is evidenced that the expression

$$(69) \quad -\frac{1}{2} Z^2 \cdot (\Delta_2 \quad \Delta g_{Bou})$$

has the character of a free-air anomaly. (69) and (54) lead to

$$(70) \quad -\frac{1}{2} Z^2 \cdot (\Delta_2 \Delta g_{Bou}) = -\frac{1}{2} Z^2 \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{R} \tan\varphi \frac{\partial}{\partial x} \right] \Delta g_{Bou} .$$

dx and dy is the line element in the north-south and in the east-west direction, along the globe.

For the numerical evaluation of (70), the data of a realistic example are for instance: $Z = 1$ km, $\frac{\partial}{\partial x} \Delta g_{Bou} = \frac{50 \text{ mgal}}{100 \text{ km}}$, $\tan\varphi = 10$, $\frac{\partial^2}{\partial x^2} \Delta g_{Bou} = \frac{\partial^2}{\partial y^2} \Delta g_{Bou} = \frac{100 \text{ mgal}}{100 \text{ km} \cdot 100 \text{ km}}$.

With these data, the expression (70) results to be equal to

$$(71) \quad \left| \frac{1}{2} Z^2 \cdot \left(\Delta_2 \quad \Delta g_{Bou} \right) \right| \leq 10 \ \mu\text{gal} .$$

Thus, θ_1 can be neglected generally.

Now, the amount of θ_2 is to be evaluated, (67). For a surface element which has a relative great distance to the test point P, an example with the following parameters is realistic:

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$$R \cdot p = 2000\ \text{km}, \quad \frac{Z}{R} = \frac{1}{6000} \quad ,$$

$$S_p \cong - \frac{2}{p^2} \quad , \quad \frac{\partial}{\partial R \cdot p} \Delta g_{Bou} = \frac{60\ \text{mgal}}{100\ \text{km}} \quad ,$$

$$dv = 100\ \text{km} \times 100\ \text{km} \ .$$

These data lead to the following impact exerted by one compartment

$$(72) \qquad \left| \frac{1}{G} B_2 \right| = 10^{-5}\ \text{cm} \ .$$

In case, we have a number of N = 10 000 of such compartments globally distributed, the total impact will be 0.001 cm. This is a negligible quantity.
But, for a surface element which lies in a close vicinity to the test point P, it is convenient to adapt the formula (67) to this special situation. For small values of p, the surface element takes the form

$$(73) \qquad dv = e \cdot de \cdot dA \quad ,$$

where

$$(74) \qquad e = R \cdot p \ ,$$

and where A is the azimuth.
Considerung (61), the relation (67) takes the following shape adapting it to the case where the p values are small,

$$(75) \qquad B_2 = \frac{1}{4\pi} \iint \tau^2 \left[\frac{\partial}{\partial e} \Delta g_{Bou} \right] \cdot e \cdot de \cdot dA \ .$$

With the following parameters,

$$\tau = \frac{1}{20} \quad , \quad \frac{\partial}{\partial e} \Delta g_{Bou} = \frac{40\ \text{mgal}}{40\ \text{km}} \quad ,$$

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$dA = \frac{\pi}{2}$, $0 \leq e \leq 40$ km ,

(75) yields

(76) $\left| \frac{1}{G} \Theta_2 \right| = 0.02$ cm .

And, in a second example for Θ_2, the data set

$\tau = 1$, $\frac{\partial}{\partial e} \Delta g_{Bou} = \frac{4 \text{ mgal}}{4 \text{ km}}$,

$dA = \frac{\pi}{2}$, $0 \leq e \leq 4$ km ,

leads to

(77) $\left| \frac{1}{G} \Theta_2 \right| = 0.1$ cm.

The relations (72) (76) (77) show that the Θ_2 value can be neglected, always.
Summarizing (71) (72) (76) (77), (65) turns to

(78) $\Psi_2 \cong 0$.

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6. Conclusions

Considering (33) and (78), (79) is obtained,

(79) $\Psi \cong \Psi_1$.

For the computation of Ψ according to (32),

(80) $$\Psi = \frac{1}{4\pi R} \iiint_V C_1(M) \cdot S(p) \cdot dv \quad ,$$

there exist two possibilities. The theoretical model of each of these possibilities has the same precision; this is the main result of the above developments, (78). The first possibility depends on deflections for the potential M, (1) (2) (3),

(81) $$C_1(M) = GZ \cdot \left[\frac{\partial \mu_1}{R \, \partial \varphi} + \frac{1}{R \cdot \cos\varphi} \frac{\partial \mu_2}{\partial \lambda} - \frac{\tan\varphi}{R} \mu_1 \right] .$$

The second way depends on the Bouguer anomalies. The theory of the second way has the same precision as the theory of the first way. We have, (44),

(82) $$C_1(M) = - Z \frac{R^2}{2\pi} \iint_\omega \frac{(\Delta g_{Bou})_Y - (\Delta g_{Bou})_Q}{e_{oo}^3} \cdot d\omega \quad .$$

The term $C_{1.b}$ of [2] (pg. 79, eq. (287)) can always be neglected consequently because of (78).

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D. The Hotine version of the boundary value problem

Contents

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Summary

The boundary value problem of geodesy is considered. The surface of the Earth is the boundary surface. The gravity disturbances serve as the boundary values (Hotine problem). The theory is developed for an error in the height anomalies of not more than about 1 cm.

Zusammenfassung

Das Randwertproblem der Geodäsie wird betrachtet. Die Erdoberfläche ist die Randfläche. Die Schwerestörungen sind die Randwerte (Hotine Problem).
Die Theorie wird entwickelt für einen Fehler in den Höhenanomalien von nicht mehr als etwa 1 cm.

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1. The preferences of the Hotine problem

The refined Stokes solution is well developed by [3], pg. 10, eq. (3),

$$(1) \quad T = \frac{R}{4\pi}\iint_{1} \left[\Delta g_T + C + C_1(M) \right] \cdot S(p) \cdot dl + \left\{ \Omega(M) \right\} .$$

T is the perturbation potential in the test point P at the surface of the Earth u, Δg_T the free-air anomaly, C the plane terrain reduction, $C_1(M)$ is in close relation to the vertical gradient of the refined Bouguer anomalies ([3], pg. 10, eq. (4); see also the previous chapter), the expression S(p) is the Stokes function depending on the spherical distance p to the test point P, 1 represents the unit sphere, and, finally, $\Omega(M)$ is a relative small supplementary term depending on the heights H and on the model potential M,

$$(2) \quad M = T - B \quad ,$$

where B is the gravitational potential of the mountain masses (with the standard density $\vartheta = 2.67$ g cm^{-3}) situated above sea level ([3], pg. 46).
The free-air anomalies are obtained by

$$(3) \quad \Delta g_T = (g)_Q - (g')_{Q_t}$$

where $(g)_Q$ is the real gravity at the running surface point Q, and where $(g')_{Q_t}$ is the standard gravity at the running telluroid point Q_t perpendicular below Q, ([3], pg. 12, eq. (6)); Fig. 1.

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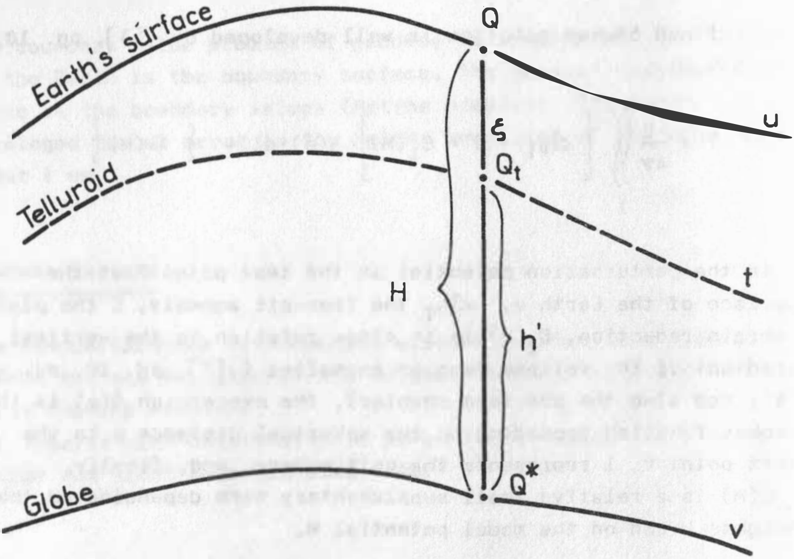

Fig. 1. The telluroid t, the Earth's surface u, the globe v, the normal height h , and the height anomaly ζ .

Now, the term $(g')_{Q_t}$ is in the fore. Considering the precision of this term, we have with 1. order approximation, Fig. 1,

$$(4) \qquad (g')_{Q_t} = (g')_{Q^*} - \frac{2G}{R} h' .$$

Q* is perpendicular below Q at the globe v having the radius R. G is the global mean of the gravity, and h' is the normal height, Fig. 1. An error $\Delta h'$ in the height has the following impact on $(g')_{Q_t}$,

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(5) $\Delta(g')_{Q_t} = -\frac{2G}{R}\,\Delta h'$,

or

(6) $\Delta(g')_{Q_t} = -0.3\ \Delta h'$;

in (6), the left hand side in mgal, and $\Delta h'$ on the right hand side in meters. As long as the distance to the coast is not too great, $\Delta h'$ will not surmount some centimeters. Hence, the left hand side of (6) will be negligible in this case.
But, in case of a great continent with levelling lines of 1000 km length and more, the quantity of $\Delta h'$ can reach one meter. By (6) and (4), an error of 0.3 mgal in the free-air anomaly is the result. In the midst of this continent, we may have a GPS-determined geocentric radius (r_{GPS}) of the point Q with a r.m.s. error of $\pm$ 0.1 meter; than, h' can be obtained by (see Fig. 1) (for a spherical Earth)

(7) $h' = r_{GPS} - R - \zeta$.

From satellite orbit perturbations and by the combination of these satellite methods with terrestrial gravimetric methods, the ζ values are known within about $\pm$ 2 meters, in a global scale, [7]. If this error is denominated by $\Delta\zeta$, (7) gives

(8) $\Delta h' = -\Delta\zeta$,

and with (6), in this case,

(9) $\Delta(g')_{Q_t} = +0.3 \cdot \Delta\zeta$.

In case, $\Delta\zeta$ is equal to $\pm$ 2 m, the free-air anomalies are falsified by + 0.6 mgal, (3) (6) (9). These considerations are valid in case of the Stokes problem, introducing free-air anomalies.

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Now, we turn to the Hotine problem, ([2], pg. 122, eq. (54)). Here, the gravity disturbances δg figure instead of the free-air anomalies.

$$(10) \qquad \delta g = - \frac{\partial T}{\partial r} = g - g' = (g)_Q - (g')_Q \; .$$

In (10), both the gravity values g and g' refer to the same surface point Q. Computing $(g')_Q$ instead of $(g')_{Q_t}$, the normal height h' has to be replaced in (4) by, (Fig. 1),

$$(11) \qquad H = h' + \zeta \; .$$

This fact has the advantage that H can be determined directly from GPS measurements. From (7) and (11), (12) follows

$$(12) \qquad H = r_{GPS} - R \; ,$$

r_{GPS} is known from GPS within about $\pm$ 0.1 meter. R is errorless computed. Thus, H is known within about $\pm$ 0.1 meter, too. From (10) (11) (4) (6), (13) yields in a self-explanatory way

$$(13) \qquad \Delta(\delta g) = 0.3 \cdot \Delta H \; .$$

With ΔH = 0.1 meter, the gravity disturbances δg are falsified by 0.03 mgal only, whereas for the free-air anomalies, the much more great value of 0.6 mgal was found, above. This fact is of cardinal importance, comparing the Hotine integral with the Stokes integral.

Considering a great continent with height determinations by spirit levelling over distances of about 1000 km and more, a strengthening and an improvement of the height values by r_{GPS} values is more effective in case of the Hotine method (δg values) than in case of the Stokes method (Δg_T values).

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2. The identity of Green

The formula (1) leads to the height anomalies ζ expressed in terms of the free-air anomalies, with a theoretical error of not more than about 0.01 meter. Now, it is intended to develop the corresponding formula which expresses the ζ values by the gravity disturbances(ζ with a theoretical error of not more than about 0.01 m, too, (10)).The subsequent derivations will be carried out under the influence of [3].
Referring to [3], pg. 16, eq. (17), the identity of Green gives for the perturbation potential T at the test point P situated on the surface of the Earth u, Fig. 2,

$$(14) \quad T(P) = \frac{1}{2\pi}\iint_u \frac{1}{e(P,Q)} \cdot \frac{\partial T}{\partial n} \cdot du - \frac{1}{2\pi}\iint_u T \cdot \left[\frac{\partial}{\partial n} \frac{1}{e(P,Q)}\right] \cdot du \quad .$$

The meaning of the symbols of (14) is explained further to Fig. 2.

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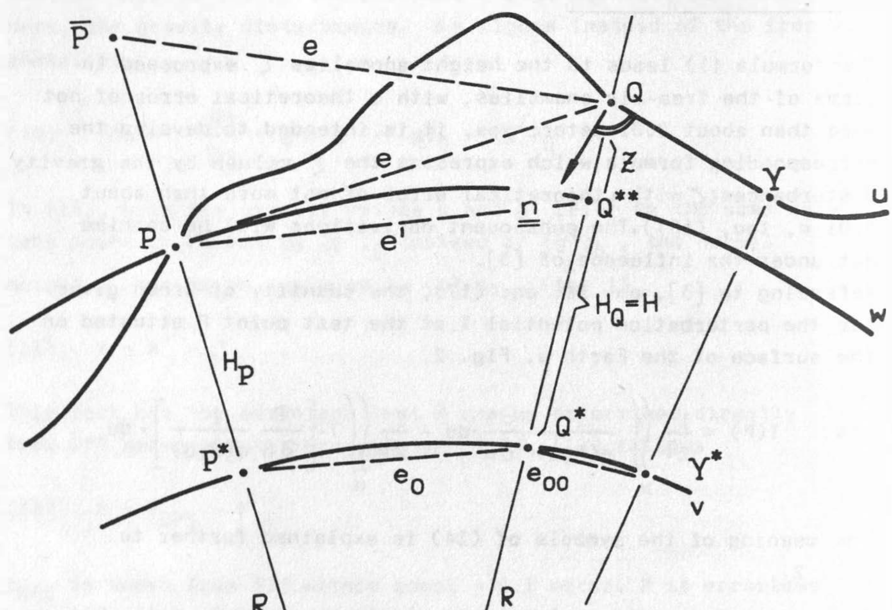

Fig. 2.

u	: Surface of the Earth,
v	: Mean (geocentric) globe in sea level, R is the radius,
w	: Geocentric sphere, $R + H_P$ is the radius,
P	: Fixed test point at the surface of the Earth u,
Q	: A point on u, moving during the integrations which have P as fixed test point,
Y	: A point on u, moving during the integrations which have Q as fixed test point,
P*, Q*, Y*	: The vertical projections of the points P, Q, Y on v,
Q**	: The perpendicular projection of the point Q on w,
$\bar{P}$	: A point perpendicular above the test point P,
e	: Straight distance between P and Q, ($\bar{P}$ and Q),
e', e_o, e_{oo}	: Straight distance between P and Q**, resp. P* and

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Q*, resp. Q* and Y*,

H_P, H_Q : Height of P, Q above the globe v,

Z : The difference of H_Q minus H_P,

The identity of Green of the shape of (14) refers to the real surface of the Earth u. The oblique straight line e, the unit normal vector $\underline{\underline{n}}$ of the surface u, and the surface element du refer to the oblique surface of the Earth u shaped by the topography. All the two integrands on the right hand side of (14) are now multiplied with and divided through the term cos (g', n). $\sphericalangle(g', n)$ is the angle defined by the positive directions of the two vectors $\underline{\underline{g}}'$ and $\underline{\underline{n}}$, taken for points on the surface of the Earth u. $\underline{\underline{g}}'$ is the vector of the standard gravity heading into the interior of the Earth. The vector $\underline{\underline{n}}$ is heading into the interior, too, Fig. 3.

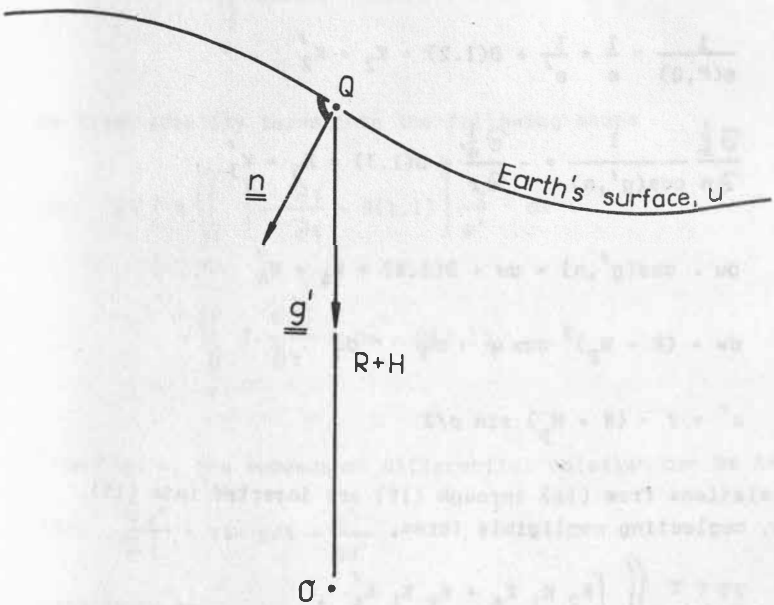

Fig. 3. The vector of the standard gravity $\underline{\underline{g}}'$ and the unit normal vector $\underline{\underline{n}}$ of the Earth´s surface u.

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Along these lines, (14) turns to

$$(15)\quad T(P) = -\frac{1}{2\pi}\iint_u \frac{1}{e(P,Q)} \cdot \frac{\partial T}{\partial n} \cdot \frac{1}{oos(g',n)} \cdot du \cdot \cos(g',n) -$$

$$-\frac{1}{2\pi}\iint_u T\,\frac{\partial\left(\frac{1}{e(P,Q)}\right)}{\partial n} \cdot \frac{1}{\cos(g',n)} \cdot du \cdot \cos(g',n) \quad .$$

Now, the terms in the integrands of (15) are decomposed into their spherical parts and into the residual parts. The relations from (16) through (21) come up,

$$(16)\quad \frac{\partial T}{\partial n}\,\frac{1}{\cos(g',n)} = -\frac{\partial T}{\partial r} + D(1.1) = K_1 + K_1' \;,$$

$$(17)\quad \frac{1}{e(P,Q)} = \frac{1}{e} = \frac{1}{e'} + D(1.2) = K_2 + K_2' \;,$$

$$(18)\quad \frac{\partial \frac{1}{e}}{\partial n}\,\frac{1}{\cos(g',n)} = -\frac{\partial \frac{1}{e'}}{\partial r} + D(1.3) = K_3 + K_3' \;.$$

$$(19)\quad du \cdot \cos(g',n) = dw + D(1.4) = K_4 + K_4' \;.$$

$$(20)\quad dw = (R + H_P)^2 \cos\varphi \cdot d\varphi \cdot d\lambda \;,$$

$$(21)\quad e' = 2 \cdot (R + H_P)\sin p/2 \;.$$

The relations from (16) through (19) are inserted into (15). Hence, neglecting negligible terms,

$$(22)\quad 2\pi T \cong \iint_u \Big[K_2\,K_1\,K_4 + K_2\,K_1\,K_4' +$$

$$+ K_2\,K_1'\,K_4 + K_2'\,K_1\,K_4 + K_2'\,K_1'\,K_4\Big] -$$

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$$- \iint_u T \left[K_3 K_4 + K_3 K_4' + K_3' K_4 \right] .$$

The equations from (16) through (21) are combined with (22); thus, putting

(23) $$D(2.1) = - \iint_w \frac{\partial T}{\partial r} \cdot D(1.2) \cdot dw - \iint_w \frac{\partial T}{\partial r} \frac{1}{e'} \cdot D(1.4) + \iint_w T \cdot \frac{\partial \frac{1}{e'}}{\partial r} \cdot D(1.4) - \iint_w T \cdot D(1.3) \cdot dw + \iint_w D(1.1) \cdot D(1.2) \cdot dw ,$$

the Green identity turns into the following shape

(24) $$2\pi T = \iint_w \left[- \frac{\partial T}{\partial r} + D(1.1) \right] \frac{1}{e'} \cdot dw + \iint_w T \cdot \frac{\partial \frac{1}{e'}}{\partial r} \cdot dw + D(2.1) .$$

From Fig. 4, the subsequent differential relation can be taken,

(25) $$\frac{\partial e'}{\partial r} = \sin p/2 = \frac{e'}{2R'} .$$

(25) leads to

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$$(26)\qquad \frac{\partial \frac{1}{e'}}{\partial r} = -\frac{1}{2e'R'} = -\frac{1}{4R'^2 \cdot \sin p/2}$$

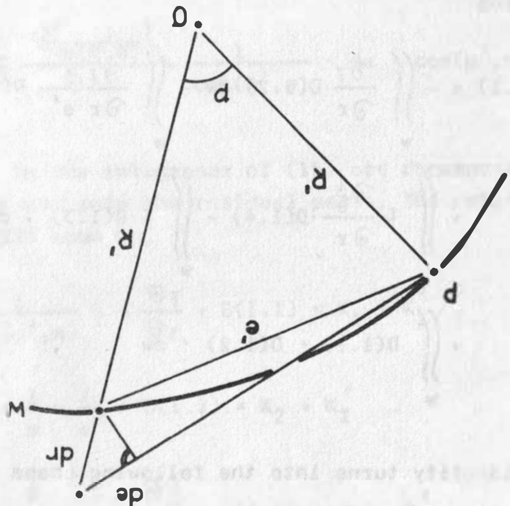

Fig. 4. The derivation of the distance e' with regard to the radius r.

In order to have denotations which are not too different from the corresponding symbols of [3], pg. 29, we put

$$(27)\qquad F(T)_H = D(2.1)$$

Putting

$$(28)\qquad \alpha = -\frac{\partial T}{\partial r} + D(1.1)\ ,$$

$$(29)\qquad \beta = \frac{1}{2\pi} F(T)_H\ ,$$

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(30) $\gamma = \frac{T}{R'}$,

(31) $R' = R + H_p$,

(32) $dw = R'^2 \cos\varphi \cdot d\varphi \cdot d\lambda$,

(33) $dw = R'^2 \sin p \cdot dp \cdot dA$,

(34) $dl = \cos\varphi \cdot d\varphi \cdot d\lambda$,

(24) turns to

(35) $$\gamma = \frac{1}{4\pi} \iint_l \frac{\alpha}{\sin p/2} \cdot dl - \frac{1}{8\pi} \iint_l \gamma \frac{1}{\sin p/2} dl + \beta \frac{1}{R'} .$$

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3. The Hotine integral

The continuous functions α, β, and γ describe values which are distributed along the surface of the Earth u, (28) (29) (30),

$$(36) \quad \alpha = \alpha(\varphi, \lambda) \quad ,$$

$$(37) \quad \beta = \beta(\varphi, \lambda) \quad ,$$

$$(38) \quad \gamma = \gamma(\varphi, \lambda) \quad .$$

Consequently, these functions can be developed in surface spherical harmonics,

$$(39) \quad \alpha = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[\alpha_{1.n.m} \cdot R_{n.m}(\varphi, \lambda) + \alpha_{2.n.m} \cdot S_{n.m}(\varphi, \lambda) \right] \quad .$$

$\alpha_{1.n.m}$ and $\alpha_{2.n.m}$ are the Stokes constants. $R_{n.m}(\varphi, \lambda)$ and $S_{n.m}(\varphi, \lambda)$ are the well-known normalized spherical harmonics of the degree n and of the order m, [3] pg. 18,

$$(40) \quad \iint_v R_{n.m}(\varphi, \lambda) \cdot R_{i.k}(\varphi, \lambda) \cdot dv = \begin{cases} 0 & ; n \neq i \text{ or } m \neq k \text{ or both} \\ 4\pi R^2 & ; n = i,\ m = k \end{cases} \quad ;$$

for $S_{n.m}(\varphi, \lambda)$, a similar relation is valid.

$$(41) \quad dv = R^2 \cdot \cos\varphi \cdot d\varphi \cdot d\lambda = R^2 \cdot dl \quad .$$

As usual, (39) is now written in the following abbreviating form

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$$(42) \quad \alpha = \sum_{n=0}^{\infty} a_n \cdot Y_n (\varphi, \lambda)$$

Further, (37) turns to

$$(43) \quad \frac{\beta}{R'} = \sum_{n=0}^{\infty} c_n \cdot Y_n (\varphi, \lambda) ,$$

$$(44) \quad \gamma = \sum_{n=0}^{\infty} d_n \cdot Y_n (\varphi, \lambda) .$$

In (35), the inverse of sin p/2 appears also. According to the decomposition formula of the spherical harmonics, this inverse has the following development, [3] [5], Fig. 2,

$$(45) \quad \frac{1}{\sin p/2} = \sum_{n=0}^{\infty} \frac{2}{2n+1} \cdot Y_n (\varphi, \lambda)_{P^*} \cdot Y_n (\varphi, \lambda)_{Q^*} .$$

(42) (43) (44) (45) are inserted into (35). Hence, the equation (46) is obtained

$$(46) \quad \sum_{n=0}^{\infty} d_n \cdot Y_n (\varphi, \lambda)_{P^*} = \frac{1}{4\pi} \sum_{n=0}^{\infty} a_n \frac{2}{2n+1} \cdot Y_n (\varphi, \lambda)_{P^*} \cdot 4\pi - \frac{1}{8\pi} \sum_{n=0}^{\infty} d_n \cdot \frac{2}{2n+1} \cdot Y_n (\varphi, \lambda)_{P^*} \cdot 4\pi + \sum_{n=0}^{\infty} c_n \cdot Y_n (\varphi, \lambda)_{P^*} .$$

The orthogonality relations for $Y_n (\varphi, \lambda)$ are, (40),

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$$(47)\quad \iint_l Y_i(\varphi,\lambda)\cdot Y_j(\varphi,\lambda)\cdot dl = \begin{cases} o, & \text{if } i \neq j \\ 4\pi, & \text{if } i = j \end{cases} .$$

(46) and (47) give

$$(48)\quad d_n = a_n \cdot \frac{2}{2n+1} - \frac{1}{2n+1} \cdot d_n + c_n \quad .$$

Thus,

$$(49)\quad o = 2a_n + (2n+1)\cdot c_n - 2\cdot(n+1)\cdot d_n \quad ;$$

$$(n = 0, 1, 2, \ldots) .$$

In (49), the Stokes constants d_n have the character of unknown values, whereas the constants a_n and c_n have to be considered as given quantities. For the computation of D(1.1) in(28)and of $F(T)_H$ in(29), an approximate knowledge of T suffices. This requirement is met since the height anomalies

$$(49a)\quad \zeta = \left(\frac{T}{g'}\right)_Q$$

are known within some meters, considering their global distribution, [7]. We are now confronted with the problem to find a closed analytical relation by which the function developed in terms of the d_n values, (44), is expressed by the functions developed in terms of the a_n and c_n values, (42) (43), observing (49). For a moment, the relation (50) (hereinafter) is supposed to be the solution of the system (49). Then, immediately afterwards, this supposition is verified,

$$(50)\quad \gamma = \frac{1}{4\pi} \iint_l \left[\alpha - \frac{1}{2}\,\frac{\beta}{R'} \right] H(p)\cdot dl + \frac{\beta}{R'} \quad .$$

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H(p) is the Hotine function, ([2], pg. 114, eq. (23); [6] pg. 311).

$$(51)\quad H(p) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \cdot P_n (\cos p) =$$

$$= \operatorname{cosec} p/2 - \ln (1 + \operatorname{cosec} p/2) .$$

The Hotine function comprises the spherical harmonics of all degrees, the degrees n=0 and n=1 included. But, the Stokes function is free of these degrees of the numbers n=0 and n=1.

As to the verification of (50), the Legendre functions $P_n (\cos p)$ of (51) have the following expression, [3] pg. 35, [5] pg. 33,

$$P_n (\cos p) = \frac{1}{2n+1} \cdot \sum_{m=0}^{n} \Big[R_{n.m} (\varphi,\lambda)_{P^*} \cdot R_{n.m} (\varphi,\lambda)_{Q^*} +$$

$$+ S_{n.m} (\varphi,\lambda)_{P^*} \cdot S_{n.m} (\varphi,\lambda)_{Q^*} \Big] . \qquad (52)$$

(52) is inserted into (51). With the here preferred manner of writing, the equation (53) is obtained,

$$H(p) = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot Y_n (\varphi,\lambda)_{P^*} \cdot Y_n (\varphi,\lambda)_{Q^*} . \qquad (53)$$

The equations (42) (43) (44) and (53) are introduced into (50), the subsequent equation follows

$$(54)\quad \sum_{n=0}^{\infty} d_n \cdot Y_n = \sum_{n=0}^{\infty} a_n \cdot \frac{1}{n+1} \cdot Y_n - \sum_{n=0}^{\infty} \frac{1}{2} \cdot c_n \cdot \frac{1}{n+1} \cdot Y_n + \sum_{n=0}^{\infty} c_n \cdot Y_n .$$

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(54) and (47) lead to

$$(55)\quad 0 = 2a_n + (2n+1)\cdot c_n - 2(n+1)\cdot d_n \; .$$

(55) corroborates (49). Thus, (50) is right. (28) (29) (30) are inserted into (50) and the detailed shape of the solution is found,

$$(56)\quad T = \frac{R'}{4\pi} \iint\limits_{l} \left[- \frac{\partial T}{\partial r} + D_T(1.1) - \frac{1}{4\pi} \frac{F(T)_H}{R'} \right] H(p)\cdot dl + \frac{F(T)_H}{2\pi} \; .$$

Comparing (56) with (28), the reader will realize that in (56) the term D_T (1.1) has now the suffix T. This suffix is useful in the further developments, it stresses the fact that D_T (1.1) refers to the pertubation potential T. Later on, in the formula for D_T (1.1), T will be replaced by another potential.

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4. The superposition with the visible mountain masses

In (56), the term D_T (1.1) is rather rugged, even in low mountains, (see [1] , pg. 14: Fig. 2 and eq. (77); the term KG(Δg_T) is equal to D_T (1.1)). This term is smoothed now by the superposition with the visible mountain masses. Here, these masses have the standard density $\vartheta = 2.65$ g cm^{-3}. These masses have the following gravitational potential B, [3] pg. 46,

$$(57) \quad B = f\vartheta \iiint\limits_V \frac{1}{\bar{e}} \cdot dV \quad .$$

f is the gravitational constant, V is the volume element, and $\bar{e}$ represents the straight distance between the running volume element dV and the test point P at the surface of the Earth u. Thus, (57) turns to

$$(58) \quad B = f\vartheta \int\limits_{p=0}^{\pi} \int\limits_{A=0}^{2\pi} \int\limits_{r=R}^{R+H} \frac{1}{\bar{e}} \cdot r^2 \cdot \sin p \cdot dp \cdot dA \; .$$

The potential M is introduced by

$$(59) \quad M = T - B \quad .$$

In (56), T can be substituted by M,

$$(60) \quad M = \frac{R'}{4\pi} \iint\limits_{l} \left[-\frac{\partial M}{\partial r} + D_M \, (1.1) - \frac{F(M)_H}{4\pi R'} \right] H(p) \cdot dl + \frac{F(M)_H}{2\pi} \; .$$

The relation (56) is valid for M, just as for T.
In the mathematical developments in [3] from pg. 52 through 61, or from eq. (176) through (221), it is allowed to substitute the function S(p) by H(p), obviously. In consideration of these circumstances, the Gauss integral theorem turns the integral

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$$(61) \quad J = \frac{R'}{4\pi} \iint_{1} D_M(1.1) \cdot H(p) \cdot dl \quad ,$$

appearing in (60), to

$$(62) \quad J = \frac{1}{4\pi R'} \iint_{w} C_1(M) \cdot H(p) \cdot dw \quad -$$

$$- \frac{1}{4\pi R'^2} \iint_{w} Z \frac{dH(p)}{dp} \frac{1}{R'} \frac{\partial M}{\partial p} dw ;$$

$$(63) \quad Z = H_Q - H_p \quad ,$$

$$(64) \quad dw = R'^2 \cdot dl \quad ,$$

dl is the surface element of the unit sphere.

$$(65) \quad C_1(M) = G Z \cdot \Phi(\mu_1, \mu_2) \quad ,$$

$$(65a) \quad \Phi(\mu_1, \mu_2) = \frac{\partial \mu_1}{R' \cdot \partial \varphi} + \frac{\partial \mu_2}{R' \cos \varphi \cdot \partial \lambda} - \frac{\tan \varphi}{R'} \mu_1 \quad .$$

G is the global mean gravity, μ_1 and μ_2 are the components of the deflection of the vertical in the potential field M + U, where U is the standard potential. As to details about $C_1(M)$, see the previous chapter C of the publication in hand, and further [3], from eq. (176) through eq. (221), replacing S(p) by H(p) in a self-explanatory way. $C_1(M)$ can be expressed in terms of the Bouguer anomalies. (61)(62) and (65) are inserted into (60), the equation (66) is obtained,

$$(66) \quad M = \frac{1}{4\pi R'} \iint_{w} \left[- \frac{\partial M}{\partial r} + C_1(M) \right] H(p) \cdot dw + \Omega_{H.1}(M) \quad ,$$

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with

$$(67)\quad \Omega_{H.1}(M) = -\frac{1}{4\pi R'}\iint\limits_{w} \frac{F(M)_H}{4\pi R'}\cdot H(p)\cdot dw + \frac{F(M)_H}{2\pi} -$$

$$-\frac{1}{4\pi R'^2}\iint\limits_{w} Z\cdot\frac{dH(p)}{dp}\cdot\frac{1}{R'}\cdot\frac{\partial M}{\partial p}\cdot dw.$$

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5. The retransformation back to the potential T

Now, the way back to the perturbation potential T has to be gone. (59) is inserted into (66), yielding

$$(68) \quad T - B = \frac{1}{4\pi R'} \iint_w \left[- \frac{\partial T}{\partial r} + \frac{\partial B}{\partial r} + C_1(M) \right] H(p) \cdot dw + \Omega_{H.1}(M) \quad .$$

B_P is the potential B at the test point P, (58), Fig. 2; and

$\left(\frac{\partial B}{\partial r}\right)_Q$ is the radial derivative of B at the running surface point Q. $(L_1 + L_2)_{P*}$ is the potential of the mountain masses condensed at the globe v, it is taken at the point P*, Fig. 2. $(L_3 + L_4)_{Q*}$ is the corresponding quantity for the radial derivative of B, taken at the point Q*. Thus, [3] pg. 70,

$$(69) \quad B_P = (L_1 + L_2)_{P*} + [B]'' \quad ,$$

$$(70) \quad \left(\frac{\partial B}{\partial r}\right)_Q = (L_3 + L_4)_{Q*} + \left[\frac{\partial B}{\partial r}\right]'' \quad .$$

If we have a spherical boundary surface v with radius R, and if we have a harmonic potential X exterior of v, in this case, the Hotine integral gives

$$(70a) \quad X = - \frac{1}{4\pi R} \iint_v \frac{\partial X}{\partial r} \cdot H(p) \cdot dv,$$

(see [6], pg. 311; [2], pg. 114).

Consequently, the Helmert condensation method gives rigorously

$$(71) \quad (L_1 + L_2)_{P*} = - \frac{1}{4\pi R} \iint_v (L_3 + L_4)_{Q*} \cdot H(p) \cdot dv \quad ,$$

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or

$$(72)\quad -(L_1 + L_2)_{P^*} \cong \frac{1}{4\pi R'} \iint_w (L_3 + L_4)_{Q^*} \cdot H(p) \cdot dw + B \cdot \frac{H_P}{R} ;$$

with

$$(73)\quad \frac{1}{R}\, dv = \frac{1}{R'} \cdot dw \cdot \left(1 - \frac{H_P}{R} \right) .$$

Further,

$$(74)\quad \frac{1}{4\pi R'} \iint_w \left[-\frac{\partial T}{\partial r} \right] \cdot H(p) \cdot dw \cong \frac{1}{4\pi R} \iint_v \left[-\frac{\partial T}{\partial r} \right] \cdot H(p) \cdot dv +$$

$$+ T \frac{H_P}{R} .$$

On the left hand side of (68), we have with (69)

$$(75)\quad T - (L_1 + L_2)_{P^*} - [B]'' .$$

Considering (74) (72) (70), on the right hand side of (68) appears the subsequent expression with always tolerable approximations

$$(76)\quad \frac{1}{4\pi R} \iint_v \left[-\frac{\partial T}{\partial r} \right] \cdot H(p) \cdot dv + T \cdot \frac{H_P}{R} +$$

$$+ \frac{1}{4\pi R'} \iint_w \left\{ (L_3 + L_4)_{Q^*} + \left[\frac{\partial B}{\partial r} \right]'' \right\} \cdot H(p) \cdot dw +$$

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$$+ \frac{1}{4\pi R} \iiint_V C_1(M) \cdot H(p) \cdot dv + \Omega_{H.1}(M).$$

According to (68), (75) is equal to (76). Thus, accounting for (10) (72),

$$(77) \quad T = \frac{1}{4\pi R} \iiint_V \left[\delta g + C + C_1(M) \right] H(p) \cdot dv + \Omega_H(M) \quad ,$$

with the topographical supplement

$$(78) \quad \Omega_H(M) = \Omega_{H.1}(M) + M \frac{H_P}{R} + [B]'' + f \vartheta \iiint_V \left(\frac{H_Q}{R} \right)^2 \cdot H(p) \cdot dv \cdot$$

δg are the gravity disturbances, C is the plane terrain reduction of the gravity, (see [1], from pg. 36 through 39),

$$(79) \quad \left[\frac{\partial B}{\partial r} \right]'' = C + \delta C - \frac{2}{R} [B]'' \; ,$$

$$(80) \quad \delta C \cong \delta_4 C = 4 \pi f \vartheta H_Q \frac{H_Q}{R} \; .$$

The third term on the right hand side of (79) will not surmount 10 µgal, ([1],pg. 36).
Hence

$$(81) \quad \left[\frac{\partial B}{\partial r} \right]'' \cong C + 4\pi f \vartheta H_Q \frac{H_Q}{R} \; .$$

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6. The topographical supplements for test points in high mountains

The equation (77) describes the perturbation potential T in terms of the gravity disturbances δg; the theoretical error of (77) will be smaller than about 1 cm in the height anomalies ζ, if the computations will be executed carefully. (77) is of universal applicability, may the test point P be situated in high mountains, in the lowlands, or on the oceans.

As to the terms on the right hand side of (77), after $C_1(M)$ was discussed thoroughly in the last chapter, the description of the way how to reach $\Omega_H(M)$ is left over for the author. (78) is the formula for $\Omega_H(M)$. The computation of the second term on the right hand side of (78) happens with (58) and (59) by means of

$$(82) \quad M\frac{H_P}{R} = \frac{H_P}{R} T - f\vartheta \frac{H_P}{R} \int_{p=0}^{\pi}\int_{A=0}^{2\pi}\int_{r=R}^{R+H} \frac{1}{e} r^2 \sin p \cdot dp \cdot dA \quad .$$

[B]" is the third term on the right hand side of (78). The formula for [B]" is developed in [1] pg. 36 and further in [2], from page 25 through page 33. In nearly all cases, (if G is the global mean gravity), the amount of [B]"/G can be forgotten because it is smaller than 1 cm, an exception perhaps in mountains crossable by roped party only. The computation of the fourth term on the right hand side of (78) is simple, it requires no comment. But, the computation of the first term on the right hand side of (78), i.e. $\Omega_{H.1}(M)$, needs a detailed description. This term has the formula (67), depending on $F(M)_H$. $F(M)_H$ is defined by (27), exchanging T by M,

$$(83) \quad F(M)_H = D_M(2.1) \quad .$$

From the developments in [3], from eq. (74) through eq. (78), or from the eq. (225) through (225h), the subsequent expression

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yields,

$$(84)\quad F(M)_H = f_1(M) + f_2(M)_H + \sum_{i=3}^{8} f_i(M) \quad .$$

The individual terms on the right hand side of (84) are as follows,

$$(85)\quad f_1(M) = \iint_w \Delta g_M \frac{Z}{R}\left[2 - \frac{1}{y+y^2}\right]\frac{1}{e'}\cdot dw \quad ,$$

$$(86)\quad f_2(M)_H = \iint_w \frac{M}{R}\frac{Z}{R}\left[3 - \frac{2}{y+y^2}\right]\frac{1}{e'}\cdot dw \quad ,$$

$$(87)\quad f_3(M) = \iint_w \frac{M}{R}\cdot\frac{v_1}{R}\cdot dw \quad ,$$

$$(88)\quad f_4(M) = -\iint_w \frac{\partial M}{R\cdot\partial p}\cdot\frac{1}{R}\cdot\frac{(\cos p/2)^2}{\sin p}\, b_7\cdot dw \quad ,$$

$$(89)\quad f_5(M) = -\iint \Delta g_M \frac{x^2}{y+y^2}\cdot de'\cdot dA \quad ,$$

$$(90)\quad f_6(M) = \iint \frac{M}{R}\left[-\frac{2x^2}{y+y^2} + v_3\right]\cdot de'\cdot dA \quad ,$$

$$(91)\quad f_7(M) = \iint \frac{\partial M}{\partial e'}\cdot(v_2 - b_{11})\cdot de'\cdot dA \quad ,$$

$$(92)\quad f_8(M) = -\iint GZ\cdot \Phi(x^*\cdot\mu_1,\ x^*\cdot\mu_2)\cdot de\cdot dA \quad .$$

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A is the azimuth, counted clockwise. In the expressions for f_1, f_2, f_3, f_4, the integrations cover whole the globe. But, in the integrals for f_5, f_6, f_7, and f_8, being of interest in case of high mountain test points only, the integration has to be extended over the surroundings of the test point P only, up to a distance of not more than about 30 km or 100 km. Δg_M is equal to the Bouguer anomaly, in sufficient approximation, [1] pg. 48.

(92a) $$\Delta g_M \cong \Delta g_{Bouguer} \quad .$$

Calculating $\Omega_{H.1}(M)$ by (67) and (84), the term $f_4(M)$ appearing in $\frac{1}{2\pi} F(M)_H$ by (88) (in the second expression on the right hand side of (67)) should be combined with the third term on the right hand side of (67). Both these terms should be melted into one another, which will bring a great relief to the computations.

The above equations contain the following abbreviations, [3] pg. 30 and 31,

(93) $$x = \frac{Z}{e'} \quad ,$$

(94) $$x' = 1 + x^2 + \frac{Z}{R'} \quad ,$$

(95) $$y^2 = 1 + x^2 \quad ,$$

(96) $$x^* = \left[x^2 + \frac{e' x}{R'} \right] \frac{1}{x' + (x')^{1/2}} \quad ,$$

(97) $$v_1 = \frac{1}{2} (x + \text{arsinh } x) \quad ,$$

(98) $$v_2 = - \frac{x}{y} + \text{arsinh } x + (\sin p/2) \left[1 - \frac{3}{y} + 2y \right] ,$$

$$(-\infty < x < +\infty ,\ e' < 1000 \text{ km}),$$

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$$\text{(99)} \quad v_3 = 1 + \frac{1}{2} y - \frac{3}{2y} + \frac{1}{2} x^2 \left[- \frac{1}{y} + \left(\frac{1}{y}\right)^3 \right] +$$

$$+ x^3 \cdot \left(\frac{1}{y}\right)^3 \cdot \sin p/2 + \frac{1}{2} x^4 \cdot \left(\frac{1}{y}\right)^3$$

$$(- \infty < x < + \infty , e' < 1000 \text{ km}),$$

$$\text{(100)} \quad b_7 = \text{arsinh } x \quad ,$$

$$\text{(101)} \quad b_{11} = x \cdot x^* \quad .$$

Some of the above expressions have the following series developments valid for small values of x,

$$\text{(101a)} \quad x^2 \ll 1 \quad .$$

[3] eq. (A 327a) gives

$$\text{(101b)} \quad v_1 = x - \frac{1}{12} x^3 + - \ldots ,$$

[3] eq. (A 334) gives

$$\text{(101c)} \quad v_2 = \frac{1}{3} x^3 + - \ldots ,$$

[3] eq. (A 345) gives

$$\text{(101d)} \quad v_3 = x^2 + - \ldots ,$$

[3] eq. (A 320) gives

$$\text{(101e)} \quad b_7 = x - \frac{1}{6} x^3 + - \ldots ,$$

[3] eq. (84) gives

$$\text{(101f)} \quad b_{11} \cong \frac{1}{2} x^3 + - \ldots .$$

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The universal formula (77),(with (78) and the expressions from (84) through (101)),should have an exclusive field of application, only. This sole and exclusive field of application will be the area of test points situated in high mountains. In all the other cases (and this are by far the most cases having test points in low mountains, in the lowlands, and on the oceans),the application of (85) through (92) will be eccentric. In these cases, the computation by (85) through (92) means to be a procedure that does go too far, because in the lowlands many parts of (85) through (92) are very very small; they can be cancelled saving much work.
Hence, it is convenient to adapt the formulas (77), (78), (84) through (101) to the case where the test points are situated in the lowlands, in the Mittelgebirge, or on the oceans.

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7. The topographical supplements for test points in the lowlands

The transition from the universal formula (77) to this special lowland formula is carried out by putting the higher powers of x, (93), equal to zero, i.e. x^2, x^3, ..., .By this transition, the term $\Omega_H(M)$ of the relations (77) (78) turns to $\Omega^*_H(M)$. Consequently, the lowland formula for T has the following shape (102),

$$(102) \qquad T = \frac{1}{4\pi R} \iint_V \left[\delta g + C + C_1(M) \right] \cdot H(p) \cdot dv + \Omega^*_H(M) \; .$$

In the lowland version (102), in the term $\Omega^*_H(M)$, the expression [B]" figuring yet in the universal expression (78) can be neglected, [1] pg. 35 and 36, [2] from pg. 18 through pg. 33. Thus,

$$(103) \qquad \Omega^*_H(M) = \Omega^*_{H.1}(M) + M \frac{H_P}{R} + f\vartheta \iint_V \left(\frac{H_Q}{R}\right)^2 \cdot H(p) \cdot dv \; .$$

With (67), $\Omega^*_{H.1}(M)$ has the subsequent expression

$$(104) \qquad \Omega^*_{H.1}(M) = -\frac{1}{4\pi R'} \iint_W \frac{F^*(M)_H}{4\pi R'} \cdot H(p) \cdot dw + \frac{1}{2\pi} F^*(M)_H -$$

$$- \frac{1}{4\pi (R')^2} \iint_W Z \cdot \frac{dH(p)}{dp} \cdot \frac{1}{R'} \cdot \frac{\partial M}{\partial p} \cdot dw \; .$$

The expression for $F^*(M)_H$ of (104) is obtained modifying the formulas from (84) through (92) by putting the terms x^2, x^3, ... equal to zero. Thus, the terms from (89) through (92) can be cancelled. In (85), we have the transition

$$(105) \qquad 2 - \frac{1}{y+y^2} \longrightarrow \frac{3}{2} \; ,$$

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and in (86) and (87), inserting (101b),

$$(106)\quad \frac{Z}{R}\left[3-\frac{2}{y+y^2}\right]\frac{1}{e'}+v_1\,\frac{1}{R}\longrightarrow 3.\frac{Z}{R}.\frac{1}{e'}\;.$$

A similar modification happens with (88) accounting for (101e). Thus, the equations from (84) to (92) turn to the following lowland version

$$(107)\quad F^*(M)_H=\sum_{i=1}^{3} f_i^*(M)_H\;,$$

$$(108)\quad f_1^*(M)_H=\iint_w \Delta g_M\,\frac{Z}{R}\,\frac{3}{2}\,\frac{1}{e_o}\cdot dw\;,$$

$$(109)\quad f_2^*(M)_H=\iint_w \frac{M}{R}\,\frac{Z}{R}\,\frac{3}{e_o}\cdot dw\;,$$

$$(110)\quad f_3^*(M)_H=-\iint_w \frac{\partial M}{R\cdot\partial p}\,\frac{Z}{4R^2}\cdot\frac{\cos p/2}{(\sin p/2)^2}\cdot dw\;,$$

$$(111)\quad e_o=2\cdot R\cdot\sin p/2\;.$$

With (92a), and with [1] pg. 48, the term Δg_M can be replaced by the Bouguer anomalies of the definition of [5] from pg. 130 through 133, plane terrain correction of the gravity is applied calculating the Bouguer anomalies.

$$(112)\quad \Delta g_M\cong\Delta g_{Bouguer}\;.$$

Inserting the equations from (107) through (110) into (104), the following lowland version of $\Omega^*_{H.1}(M)$ is obtained; [3] eq.

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(230) and (272) and (273); and [4], eq. (29) through (33), and eq. (37).

(113) $$\Omega^*_{H.1}(M) = -\frac{1}{(4\pi R)^2}\iint_w F^*(M)_H \cdot H(p)\cdot dw + \frac{1}{2\pi}\iint_w \Delta g_M \frac{Z}{R}\frac{3}{2}\frac{1}{e_o}\cdot dw + \frac{1}{2\pi}\iint_w \frac{M}{R}\frac{Z}{R}\frac{3}{e_o}.dw - \frac{1}{8\pi R^2}\iint_w \frac{\partial M}{R\cdot\partial p}.Z\cdot\left[\frac{\cos p/2}{(\sin p/2)^2} + 2\frac{dH(p)}{dp}\right]\cdot dw.$$

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8. The application of the gravimetrically obtained height anomalies for the interpolation between the GPS derived height anomalies

Hence, by (102) (103) and (113), the explicit formulas of the lowland version is obtained. It is the lowland version for the computation of the T values, or for the calculation of the height anomalies ζ ,

(114) $$\zeta = \left(\frac{T}{g'}\right)_P ,$$

in terms of the gravity disturbances δg.

As to the practical application of (102), (or the high mountain version (77)), in many cases, this formula is used for the interpolation of the ζ values between the ζ_{GPS} values obtained from the GPS derived geocentric radii, r_{GPS}, (7),

(115) $$\zeta_{GPS} = r_{GPS} - R - h' + \Psi(\varepsilon) ,$$

$\Psi(\varepsilon)$ is a correction for the flattening of the Earth. The more detailed formulation of (115) is

(116) $$\zeta_{GPS} = r_{GPS} - r_E - h' + \Gamma .$$

r_E is the radius of the mean Earth ellipsoid E for the geocentric latitude φ of the GPS station on the surface of the Earth, (more precise: The geocentric latitude of the surface GPS station after its vertical projection down to the ellipsoid)

(117) $$r_E = a_E \left[1 - \frac{1}{2} e_E^2 \sin^2\varphi + \frac{1}{2} e_E^4 \left(- \sin^2\varphi + \frac{3}{4} \sin^4\varphi\right) \right] .$$

a_E resp. b_E is the semi-major axis (resp. semi-minor axis) of the mean Earth ellipsoid. e_E is defined by

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(118) $$e_E^2 = \frac{a_E^2 - b_E^2}{a_E^2} \quad .$$

(117) can be found in the text books.
h' is the normal height, in (116). The correction term Γ accounting for the flattening of the ellipsoid can be taken from: Arnold, K.; Das Geoid aus Beobachtungen der Satellitenaltimetrie. Veröff. Zentralinst. Physik d. Erde, Nr. 7, Potsdam, 1972, pg. 19, eq. (98).

(119) $$\Gamma = \frac{1}{8}\, e_E^4\, \frac{a_E}{r_{GPS}}\, (h' + \zeta)\, \sin^2 2\varphi \quad .$$

In the braces of (119), an approximative value of ζ is required merely.
As long as the distances between the ζ_{GPS} values are not more than about 500 km, the first and the third term on the right hand side of (103) will vary as a linear function between these ζ_{GPS} values, probably. Thus, the first and the third term on the right hand side of (103) will, probably, be absorbed by the procedure of the linear interpolation. The linear variations of these two terms between the ζ_{GPS} values will be taken into account automatically by the procedure of the interpolation. Thus, in the lowlands, for this interpolation procedure working between the points with ζ_{GPS} values, it will possibly suffice to calculate the gravimetrical ζ values simply by the subsequent formula (120), along the lines between two GPS stations,

(120) $$\frac{1}{4\pi R g'} \iint_v \left[\delta g + C + C_1(M) \right] H(p)\, dv + M \frac{H_P}{R} \quad .$$

But, only in the lowlands, the form (120) can be convenient to simplify the interpolation of the gravimetrically obtained ζ values between the GPS derived ζ values. In the high mountains, for this interpolation procedure, we have to take the ζ values of (77) and (78). The fourth term of (78) will

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be linear over ranges of 500 km, probably. Thus, it is neglected here. The following formula can suffice for the interpolation of the ζ values, possibly, in high mountains,

$$(121)\quad \frac{1}{4\pi R g'} \iiint_V \left[\delta g + C + C_1(M) \right] H(p)\, dv + $$

$$+ \; \Omega_{H.1}(M) + M \frac{H_p}{R} + \left[B\right]'' .$$

For the interpolation of the ζ values over ranges of about 500 km, in (121), $\Omega_{H.1}(M)$ is computed by (67). Here, in the expression for $F(M)_H$, the terms linear over 500 km can be split off. Thus, in context with the relation (121), we can put,(in (67)), possibly, the expression (122) instead of $F(M)_H$, approximately, for the interpolation procedure over 500 km ranges,(84)(107),

$$(122)\quad \left[f_1(M) - f_1^*(M)_H \right] +$$

$$+ \left[f_2(M)_H + f_3(M) - f_2^*(M)_H \right] +$$

$$+ \left[f_4(M) - f_3^*(M)_H \right] +$$

$$+ f_5(M) + f_6(M) + f_7(M) + f_8(M) .$$

(122) is quasi the expression of $F(M)_H$ minus $F^*(M)_H$,(122) is free of the constituents which variate linearly over ranges not longer than about 500 km. Sure, the expressions (120) (121) (122) come into question only within this above discussed interpolation procedure, (see (84) (107)).

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E. Recent crustal movements on Iceland and the accompanying density changes in the interior

Contents

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Summary

Recent crustal movements give rise to changes of the heights, of the gravity values, and of the gravity potential. The vertical derivative of this deformation potential is expressed in terms of the changes of the height and of the gravity. This vertical derivative depends on the density changes which accompany the recent crustal movements. These density changes consist of two parts: The first part is a surface layer of the real density and of a thickness which is equal to the height changes. Thus, the first part has beforehand given parameters. The second part consists of the density changes in the interior of the Earth. Along these lines, it is possible to find an empirically given signal function for these density changes in the interior. These density changes can be found in terms of the quantities of this signal function along the lines of the gravity methods of the geophysical prospecting.

Zusammenfassung

Rezente Erdkrustenbewegungen reflektieren in Änderungen der Höhen, der Schwerewerte und des Schwerepotentials. Die vertikale Ableitung dieses Deformationspotentials kann dargestellt werden als Funktion von den Änderungen der Höhe und der Schwere. Andererseits kann diese vertikale Ableitung dargestellt werden als Funktion von den Dichteänderungen im Erdinnern, die im Zuge der rezenten Erdkrustenbewegungen entstehen. Diese Dichteänderungen bestehen aus 2 Teilen. Der erste Teil ist eine Schicht an der Erdoberfläche; sie hat die Dichte des Oberflächengesteins und ihre Mächtigkeit ergibt sich aus den Höhenänderungen. Der zweite Teil besteht aus den Dichteänderungen im Erdinneren. Für diese Dichteänderungen kann eine Signalfunktion angegeben werden, die empirisch gegeben ist. Mit Hilfe der Methoden der gravimetrischen Lagerstättenforschung können diese Dichteänderungen im Erdinneren als Funktion von den Werten dieser Signalfunktion gefunden werden.

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1. Introduction

In many test-areas and along many test-lines, the changes of the heights and of the gravity values caused by recent crustal movements are detected by levellings and by gravity measurements. As to the geophysical interpretation of these measurements, it is intended here to develop a comprehensive and satisfactory theory. Till now, the height changes are discussed, separately. In other cases, the gravity changes are discussed separately accounting for the reduction on account of the height changes (applying the free-air gradient or the free-air gradient supplemented by the effect of the Bouguer plate). Then, the reduced or the non-reduced gravity values are divided through the height changes, and, finally, the thus obtained quotient is computed. But in the literature, there is not a satisfactory quantitative discussion about the value of this quotient which is influenced by the accompanying density changes in the interior of the Earth. The latter question is the subject here to be treated.

2. Theoretical foundations

Along the surface of the Earth σ, the perturbation potential T depends on the free-air gravity anomalies Δg_T by the following expression, K. Arnold (1986)(1987b)(1989a,b), (lowland version)

$$(1) \quad T = \frac{1}{4\pi R} \iint_v [\Delta g_T + C + C_1(M)]\, S(\psi)\, dv + \{\Omega^*(M)\} .$$

The braces denote that the harmonics of zero and first degree are split off. In (1), we have (for test points in the lowlands)

$$(2) \quad \Omega^*(M) = M \frac{H_P}{R} + \frac{1}{4\pi R} \iint_v [4\pi f \varrho_0 H_Q \frac{H_Q}{R} - \frac{2B H_Q}{R^2}]\, S(\psi)\, dv +$$

$$+ \frac{1}{2\pi} \iint \Delta g_M \frac{3}{2} \frac{Z}{R} \frac{1}{e_0}\, dv + \frac{1}{2\pi} \iint \frac{M}{R} \frac{Z}{R} \frac{1}{e_0}\, dv -$$

$$- \frac{1}{8\pi R^2} \iint \frac{\partial M}{R\, \partial\psi} Z [\frac{\cos \frac{\psi}{2}}{(\sin \frac{\psi}{2})^2} + 2 \frac{dS(\psi)}{d\psi}]\, dv .$$

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v denominates the globe with the radius R. H_P resp. H_Q is the height of the test point P resp. of the moving integration point Q. f is the gravitational constant, ρ_o is the standard density ($\rho_o = 2.67\ g\ cm^{-3}$). $S(\psi)$ is the Stokes function, ψ the spherical distance. We have

(3) $Z = H_Q - H_P$,

(4) $e_o = 2R \sin \frac{\psi}{2}$,

(5) $M = B - T$,

(6) $\Delta g_M = - \frac{\partial M}{\partial r} - \frac{2}{r} M$.

Δg_M can be replaced by the Bouguer anomaly in sufficient approximation, (see (7), $\Delta g^* \cong \Delta g_M$).
B is the potential of the mountain masses (of standard density ρ_o) situated above the surface of the globe v. C is the plane terrain reduction of the gravity; $C_1(M)$ has the following relation, K. Arnold (1989a,b),

(7) $C_1(M) \cong - Z \frac{1}{2\pi} \iint_v \frac{(\Delta g^*)_Y - (\Delta g^*)_Q}{(e_{oo})^3} dv$;

(see also chapter C of this volume).
Δg^* is the Bouguer anomaly which is described by W.A.Heiskanen and H. Moritz (1967). The relation (1) is valid as long as the test-point P is not situated in high mountains, K. Arnold (1989a).
By the recent crustal movements which happen during the epoch Φ situated between the time values t_1 and t_2, the T value changes by

(8) $D = T_2 - T_1$.

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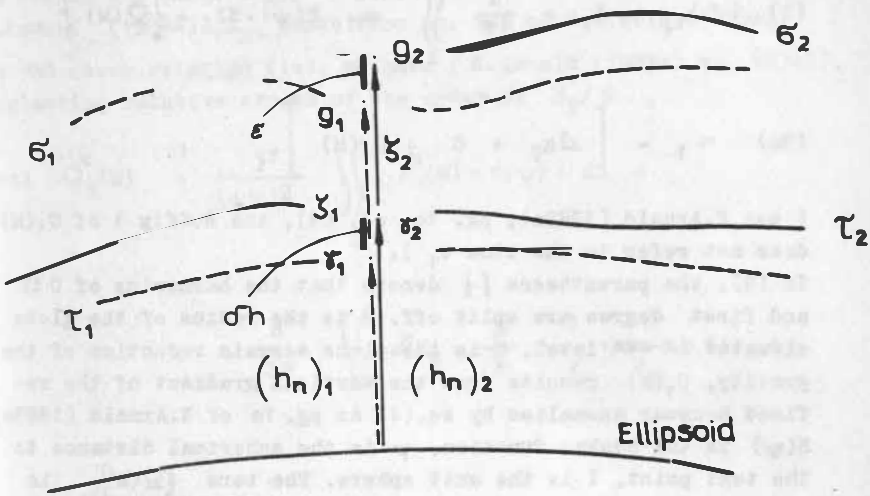

Fig. 1.: The shifts of the telluroid τ and of the Earth's surface σ; the changes of the normal gravity γ, of the observed gravity g, and of the normal heights h_n. The epoch covers the time from t_1 to t_2.

Thus, at the beginning of the here considered epoch Φ, at the time t_1, the perturbation potential T at the Earth's surface can be computed in terms of the free-air anomalies Δg_T which are located on this surface and which are measured at this time t_1 (i.e. $(\Delta g_T)_1$). This computation happens by means of the following universal formula (9) which can be regarded as the solution of the geodetic boundary value problem. The formula (9) is valid for test points in the lowlands, in the Mittelgebirge, and in the high mountains, too.

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$$(9) \quad (T)_{t_1} = T_1 = \frac{R}{4\pi} \iint_1 \alpha_1 \cdot S(\psi) \cdot dl + \{\Omega(M)\}_{t_1} ,$$

$$(9a) \quad \alpha_1 = \left[\Delta g_T + C + C_1(M) \right]_{t_1} ;$$

(see K.Arnold (1989a), pg. 1o, eq. (3), the suffix 1 of $C_1(M)$ does not refer to the time t_1).

In (9), the parentheses $\{ \}$ denote that the harmonics of 0 th and first degree are split off. R is the radius of the globe situated in sea level, C is the plane terrain reduction of the gravity, $C_1(M)$ results from the vertical gradient of the refined Bouguer anomalies by eq.(4) on pg. 1o of K.Arnold (1989a). $S(\psi)$ is the Stokes function, ψ is the spherical distance to the test point, 1 is the unit sphere. The term $\{\Omega(M)\}_{t_1}$ is given by eq.(268) and (224) on page 75 and 62 of K.Arnold (1989a). These equations represent the universal formula for $\Omega(M)$ which is valid for test points in high mountains , too.

In case, the test point is situated in low mountains or in the lowlands or on the oceans, the universal supplementary term $\{\Omega(M)\}$ can be replaced by the simple term $\{\Omega^*(M)\}$ which can be computed more easily than the universal version $\{\Omega(M)\}$ using eq. (272) (273) (23o) (266) given on pg. 76, 66, and 74 of K.Arnold (1989a).

Hence, the transformation from the one version to the other version can be described by (see (1)),

$$(9b) \quad \Omega(M) \longrightarrow \Omega^*(M) .$$

These above cited relations give

$$(1o) \quad \Omega^*(M) = \Omega_1^*(M) + M \frac{H_P}{R} + [B]'' + \frac{R}{4\pi} \iint_1 C_2 \cdot S(\psi) \cdot dl.$$

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H_P is the height of the test point, $[B]''$ can be neglected in nearly all cases (see the first 3 lines of pg. 76 of K.Arnold (1989a)). C_2 comes from eq. 266 of K.Arnold (1989a).

In the above relation (1o), we have (K. Arnold (1989a) eq. (23o)), neglecting relative errors of the order of H_P/R,

$$(11)\quad \Omega_1^*(M) = \frac{3}{(4\pi)^2} \iint_1 F^*(M) \cdot S(\psi) \cdot dl + \frac{R^2}{2\pi} \iint_1 \Delta g_M \cdot \frac{Z}{R} \cdot \frac{3}{2} \cdot \frac{1}{e_o} \cdot dl + \frac{R^2}{2\pi} \iint_1 \frac{M}{R} \cdot \frac{Z}{R} \cdot \frac{1}{e_o} \cdot dl - \frac{1}{8\pi} \iint_1 \frac{\partial M}{R\,\partial\psi} \cdot Z \cdot \alpha_2 \cdot dl .$$

Here is

$$(11a)\quad \alpha_2 = \frac{\cos\psi/2}{(\sin\psi/2)^2} + 2\frac{d\,S(\psi)}{d\,\psi} .$$

$F^*(M)$ comes from eq. (227) of K.Arnold (1989a). The quantity of the right hand side of (11) will be dominated by the 2 nd, 3rd and 4th term on this side. Δg_M can be replaced by the

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Bouguer anomalies, in good approximation, (see K.Arnold (1986) pg. 48). Z is the height difference relative to the test point. Further,

(12) $$e_o = 2 \cdot R \cdot \sin \psi/2 \quad .$$

The spatial position of a point in the exterior of the body of the Earth is given by the placement vector $\underline{\underline{x}}$. A point specially situated on the surface of the Earth has the placement vector $\underline{\underline{\bar{x}}}$.

Hence, considering the placement of a certain point on the surface of the Earth at the beginning of the epoch , $\underline{\underline{\bar{x}}}_1$, the potential T_1 (at the time t_1) on the left hand side of (9) can be represented by

(13) $$T_1(\underline{\underline{\bar{x}}}_1) \quad ;$$

here, the two suffixes 1 refer to the time t_1 .

For the end of the epoch, at the time t_2 , the perturbation potential T has the analogous expression

(14) $$T_2(\underline{\underline{\bar{x}}}_2) \quad .$$

Here, $\underline{\underline{\bar{x}}}_1$ and $\underline{\underline{\bar{x}}}_2$ refer to the same physical particle, the first vector refers to the time t_1 and the second one to the time t_2 . The shift from $\underline{\underline{\bar{x}}}_1$ to $\underline{\underline{\bar{x}}}_2$ happens by the recent crustal movements. (9) and (14) lead to

(15) $$(T)_{t_2} = T_2 = T_2(\underline{\underline{\bar{x}}}_2) = \frac{R}{4\pi} \iint\limits_{l} \alpha_3 \cdot S(\psi) \cdot dl + \left\{ \Omega(M) \right\}_{t_2} \quad .$$

(15a) $$\alpha_3 = \left[\Delta g_T + C + C_1(M) \right]_{t_2} \quad .$$

Further, for a fixed spatial position $\underline{\underline{x}}$ which is not shifted by the recent crustal movements, we have for the time t_1

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(16) $$T_1(\underline{\underline{x}}) = \left[T(\underline{\underline{x}}) \right]_{t_1} .$$

Similarly, for the time t_2, in the same fixed spatial point $\underline{\underline{x}}$,

(17) $$T_2(\underline{\underline{x}}) = \left[T(\underline{\underline{x}}) \right]_{t_2} .$$

Consequently, the change the potential T undergoes at the fixed point $\underline{\underline{x}}$ during the epoch between t_1 and t_2 has the subsequent relation

(18) $$D(\underline{\underline{x}}) = T_2(\underline{\underline{x}}) - T_1(\underline{\underline{x}}) .$$

$D(\underline{\underline{x}})$ is a harmonic potential function in the exterior of the body of the Earth, likewise as $T_1(\underline{\underline{x}})$ and $T_2(\underline{\underline{x}})$. $D(\underline{\underline{x}})$ fulfills the Laplace differential equation.
Now, the solution of the geodetic boundary value problem (which is represented by (1) (9) and (15)) is to be applied to the potential $D(\underline{\underline{x}})$, (18).
In this context, and to be as precise as possible, we introduce now the surface $\overline{\underline{\underline{x}}_o}$, which is defined in the following way :
In case, the new geocentric radius of the surface of the Earth (for the time t_2) is greater than the old one (for the time t_1),

(19) $$r_2 > r_1 ,$$

on this condition, the radius r_2 describes the surface $\overline{\underline{\underline{x}}_o}$. But, in case we have

(20) $$r_1 > r_2 ,$$

on this condition, even the old radius r_1 describes the surface $\overline{\underline{\underline{x}}_o}$.
With these peculiar definitions, the space exterior to the surface described by the vector $\overline{\underline{\underline{x}}_o}$ is free of masses. The

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difference potential D is a harmonic function in the exterior of the surface $\bar{\underline{\underline{x}}}_o$. D fulfills the Laplace differential equation in the exterior of $\bar{\underline{\underline{x}}}_o$.

Thus, it is possible to understand the potential D as a function which can be introduced into the solution of the boundary value problem, likewise as $T_1(\underline{\underline{x}})$ and $T_2(\underline{\underline{x}})$, (9) (15). However, here we should observe the fact that the radial derivative of D (i.e. $\partial D / \partial r$) has no correlation with the topographical heights. This fact is in clear contrast to the peculiarities of the free-air anomalies Δg_T (appearing in (9) and (15))which have a distinct correlation with the heights. Hence, applying the solution of the boundary value problem to the potential D, it is not necessary to work with the superposition of the potential T and the potential B of the visible mountain masses, (5). This superposition procedure transforms the rugged term $D_T(1.1)$ or $C_1(T)$ or $KG(\Delta g_T)$ into the smoothed term $C + C_1(M)$; (9) (15) , (see K.Arnold (1989a) chapters 5, 7, and 8 , and K.Arnold (1986) pg. 14). This discussed ruggedness of the free-air anomalies (and of these 3 expressions depending on them) comes into being by these correlations with the height.

Therefore, we can desist from an application of the formulas of the type of (9) or (15). Here, we can prefer the relations developed in K.Arnold (1989a), eq. (114), pg. 36. With D as a substitute for T, we obtain ($D_T(1.1) \rightarrow D^*_T(1.1)$)

(21) $$\{D\} = \{D(\bar{\underline{\underline{x}}}_o)\} = \frac{R'}{4\pi} \iint_l \alpha_4 \cdot S(\psi) \cdot dl + \frac{\{F(D)\}}{2\pi} ;$$

(21a) $$\alpha_4 = \Delta g_D + \dot{D}^*_D(1.1) + \frac{3}{4\pi} \cdot \frac{F(D)}{R'} ,$$

(22) $$\Delta g_D = - \frac{\partial D}{\partial r} - \frac{2}{r} D .$$

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For the second term in the integral of (21), the following development is known, K.Arnold (1989a) from pg. 52 through pg. 61.

$$(22a)\qquad \frac{R'}{4\pi}\iint_{l} D_D^{*}(1.1)\cdot S(\psi)\cdot dl \;=$$

$$= \frac{1}{4\pi R'}\iint_{w} C_1(D)\cdot S(\psi)\cdot dw \;-$$

$$- \frac{1}{4\pi (R')^2}\iint_{w} Z\,\frac{d\,S(\psi)}{d\psi}\,\frac{1}{R'}\,\frac{\partial D}{\partial \psi}\cdot dw\,.$$

The symbol w denotes the globe with the radius R',

$$(22b)\qquad R + H = R'\,.$$

H is the height above the globe the surface of which is situated in sea level. Z is the height difference : Running point height minus test point height.
Neglecting relative errors of the order of

$$(22c)\qquad \frac{H}{R}\,,$$

and inserting (22a) into (21), the following relation for the difference potential D yields

$$(22d)\qquad \{D\} = \{D(\underline{\underline{\bar{x}}}_o)\} = \frac{R}{4\pi}\iint_{l}\alpha_5\cdot S(\psi)\cdot dl + \Psi\,,$$

here is

$$\alpha_5 = \Delta g_D + C_1(D) + \frac{3}{4\pi}\cdot\frac{F(D)}{R}\,,$$

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and

$$(22e) \qquad \Psi = -\frac{1}{4\pi}\iint_{1} Z\cdot\frac{d\,S(\psi)}{d\psi}\cdot\frac{1}{R}\cdot\frac{\partial D}{\partial\psi}\cdot dl + \frac{\{F(D)\}}{2\pi} .$$

$C_1(D)$ is explained by eq. (4) on pg. 1o of K.Arnold (1989a) ,

$$(22f) \qquad C_1(D) \cong - Z\cdot\frac{R^2}{2\pi}\iint_{1}\frac{1}{e_{oo}^2}\cdot\alpha_6\cdot dl ;$$

$$\alpha_6 = (\Delta g_D)_Y - (\Delta g_D)_Q ,$$

as to Δg_D , see eq. (23g) which follows later in this chapter , (see also Fig.2 on pg. 15 of K.Arnold (1989a) , see also chapter C of this volume in hand).

In most cases, the test point of (22d) for which D is to be computed, this point is situated in the low mountain areas or in the lowlands, but not in the high mountain ranges. In this case, not the universal expression for F(D) is recommanded to be applied. This universal formula is given by K.Arnold (1989a) pg. 63, from eq. (225) through (225h), replacing M by D .

In case, the test point is not situated in high mountains, the much more simple form $F^*(D)$ should be preferred in place of F(D) .

Thus, in (22d) and (22e),we substitute

$$F(D) \longrightarrow F^*(D) .$$

Referring to the relations from eq. (227) through (228) on pg. 65 of K.Arnold (1989a) , the following equation(22g) is found; it expresses the lowland term $F^*(D)$ by three global integrals. Thus, $F^*(D)$ is a very smoothed function. The subsequent equation (22g) expresses $F^*(D)$ in terms of Δg_D , of the potential D and its radial derivative; in this context, approximative values for Δg_D , for D and its radial derivative are required in the integrands on the right hand side of (22g) , only. Δg_D is obtained by (23g).

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$$(22g) \qquad F^{*}(D) = \iint_{w} \Delta g_D \frac{Z}{R} \cdot \frac{3}{2} \cdot \frac{1}{e_o} \cdot dw \; + $$

$$+ \iint_{w} \frac{D}{R} \cdot \frac{Z}{R} \cdot \frac{1}{e_o} \cdot dw \; - $$

$$- \iint_{w} \frac{\partial D}{R \, \partial \psi} \cdot \frac{Z}{4R^2} \cdot \frac{\cos \psi/2}{(\sin \psi/2)^2} \cdot dw \; .$$

Now, it is necessary to speak of the anomaly Δg_D ,(22), which appears in the integrand of the relation (22d). In the first term on the right hand side of (22), the derivative $\partial D/\partial r$ appears. It is the radial derivative of the potential $D(\underline{\underline{x}})$ taken for the points at the surface described by $\underline{\underline{\bar{x}}}_o$, (18) (19) (2o). The value D stands for the change of the perturbation potential T during the time interval between the times t_1 and t_2 . At the time t_2 , the measurements happen on the surface $\underline{\underline{\bar{x}}}_o$ if (19) is right. Thus, the measured gravity values are

$$(23) \qquad (g)_{t_2} = g_2 \quad , \text{ on the surface } \underline{\underline{\bar{x}}}_o \; ,$$

where g_2 is the measured gravity on the surface $\underline{\underline{\bar{x}}}_o$ at the time t_2 .

In the identical spatial point (with the same spatial co-ordinates) for which the relation (23) is valid for the time t_2 , the gravity at the time t_1 is described by

$$(23a) \qquad (g)_{t_1} \quad , \text{ on the surface } \underline{\underline{\bar{x}}}_o \; .$$

$(g)_{t_1}$ is not a directly measured quantity on the surface $\underline{\underline{\bar{x}}}_o$. The quantity of $(g)_{t_1}$ on the surface $\underline{\underline{\bar{x}}}_o$ has to be

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computed in terms of the measured gravity values obtained on the old surface of the Earth which does exist at the time t_1 . Here, the derivation of the concerned computation procedure may base on the vertical free-air gradient of the standard gravity. This procedure can be followed in sufficient approximation, at least in this context. For a more precise procedure, we have to go over to the vertical free-air gradient of the real gravity, (see K.Arnold (1989), pg. 214, eq. (75a) (76) (77)).

During the time interval $t_2 - t_1$, the measurement station at the Earth's surface undergoes a vertical spatial shift by the amount ε ,Fig.1,

$$\text{(23b)} \qquad \varepsilon = \frac{1}{G} D + \delta h \ ,$$

(see K.Arnold (1986), pg. 2o9, eq. (49)). G is the global mean of the gravity, δh is the change of the normal heights obtained by levellings. The first term of (23b) , D/G, stands for the change of the height anomalies ζ .

Hence, by the standard value of the free-air gradient of the gravity, (23a) (23b), the following relation is found

$$\text{(23c)} \qquad (g)_{t_1} = g_1 - \frac{2G}{R}\left[\delta h + \frac{D}{G}\right], \text{ on the surface } \underline{\underline{\bar{x}}}_o \ .$$

The g_1 value of (23c) is the measured gravity on the old surface found at the time t_1 . Consequently, in a self-explanatory way,

$$\text{(23d)} \qquad -\frac{\partial D}{\partial r} = (g)_{t_2} - (g)_{t_1} , \text{ on the surface } \underline{\underline{\bar{x}}}_o \ .$$

$$\text{(23e)} \qquad g_2 - g_1 = \delta g \ ,$$

$$\text{(23f)} \qquad -\frac{\partial D}{\partial r} = \delta g + \frac{2G}{R}\delta h + \frac{2}{R} D \ .$$

δh is the change of the measured normal heights, δg that

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of the measured gravity. The relation (23f) is inserted into (22), the following relation yields in sufficient approximation

$$\text{(23g)} \qquad \Delta g_D = \delta g + \frac{2G}{R}\,\delta h \quad .$$

The relation (23g) is valid not only for (19). It is easily proved that the relation (23g) is valid for whole the surface $\bar{\underline{\underline{x}}}_o$, for (2o) in the same way as for (19).
Now, (23g) is inserted into (22d). The subsequent relation follows,

$$\text{(23h)} \qquad \{D\} = \{D(\bar{\underline{\underline{x}}}_o)\} = \frac{R}{4\pi}\iint_l \alpha_7 \cdot S(\psi)\cdot dl + \Psi \;;$$

$$\alpha_7 = \delta g + \frac{2G}{R}\,\delta h + C_1(D) + \frac{3}{4\pi}\,\frac{F(D)}{R} \quad .$$

This is the formula for the D potential expressed in terms of the changes of the gravity and height . Computing the D potential by (23h), the test point is situated on the surface of the Earth; or - to be more precise - on the surface $\bar{\underline{\underline{x}}}_o$. The same is valid for the boundary values appearing in (23h).
The spherical simplification of the formula (23h) is the integral of Strang van Hees,

$$\text{(23i)} \qquad D \cong \frac{R}{4\pi}\iint_l \left[\delta g + \frac{2G}{R}\,\delta h\right] S(\psi)\cdot dl \;.$$

A shift of the values of

$$\text{(23j)} \qquad D \;,\; \frac{\partial D}{\partial r} \;,\; \text{and} \;\; \Delta g_D$$

from the old surface of the Earth to the new surface (which is moved by the recent crustal movements) has a negligible

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effect on these values of (23j). This fact can be demonstrated easily, now.
For instance, the potential D consists of constituents of the following type,

(23k) $$\Lambda = \Lambda(\underline{\underline{x}}) = \frac{1}{r^{n+1}} \cdot \varkappa_n \cdot Y_n(\varphi,\lambda) \quad .$$

r,φ,λ are geocentric polar co-ordinates, $\varkappa_n$ are the Stokes constants, and Y_n is a spherical harmonic of the degree n. In (23k), a shift of the test point $\underline{\underline{x}}$ in the vertical direction by the amount of δr reflects in a certain change of the Λ value. The following change is obtained

(24) $$\delta\Lambda = -(n+1)\frac{1}{r^{n+2}} \cdot \varkappa_n \cdot Y_n(\varphi,\lambda) \cdot \delta r \quad .$$

(23k) is inserted into (24), the relation (25) yields,

(24a) $$\delta\Lambda \cong -(n+1) \cdot \frac{1}{R} \cdot \Lambda \cdot \delta r \quad .$$

In case, the wave length of the globally distributed Λ values is denominated by L, the assigned degree n is obtained by the following rule of thumb,

(24b) $$n = \frac{2\pi R}{L} \quad .$$

Thus, we find

(24c) $$\frac{n}{R} = \frac{2\pi}{L} \quad .$$

Here, in our example, the value

(24d) $$L = 10 \text{ km}$$

is a convenient choice. The relations (24d) and (24b) give

(24e) $$n = 2\pi\frac{R}{L} \cong 3600 \cong n+1 \quad .$$

Hence, in (24a), n + 1 can be replaced by n. Inserting (24c)

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(24e) into (24a) , the following relation for $\delta\Lambda$ results,

(24f) $$\frac{\delta\Lambda}{\Lambda} \cong - 2\pi \frac{\delta r}{L} .$$

With δr = o.oo1 km, and accountig for (24d), the relation (24f) turns to

(25) $$\frac{\delta\Lambda}{\Lambda} \cong - 6\cdot 1o^{-4} .$$

(25) proves that a vertical shift of the test point by the amount of o.oo1 km reflects in the D value by a negligible impact. The same property can be found for the terms of $\partial D/\partial r$ and Δg_D of (23j), too.

The relations (22) and (23g) lead to

(25a) $$- \frac{\partial D}{\partial r} - \frac{2}{r} D = \delta g + \frac{2G}{R} \delta h .$$

The potential D has the following series development (25b) which is uniform convergent in the exterior of the surface $\bar{\underline{x}}_o$, K.Arnold (1986) (1987a,b). This series convergence was proved considering the problem from different sides and along different ways; all these deliberations corroborate the fact that the series development (25b) is uniform convergent in the mass free exterior of the gravitating body. (25b) is a representation of the potential D valid in the exterior.

(25b) $$D = \sum_{n=2}^{\infty} \frac{1}{r^{n+1}} \cdot \varkappa_n \cdot Y_n(\varphi,\lambda) .$$

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will have a horizontal extent of not more than about 1ooo km x 1ooo km . Consequently, it will be allowed to put the inequality (25c) which states a lower bound for the quantity of the degree n,

(25c) $$n > 20 .$$

A look on (23k) and (24a) gives the inequality (25d) , acc- counting for (25c) ,

(25d) $$\left| \frac{\partial D}{\partial r} \right| \gg \left| \frac{2}{r} D \right| ,$$

and with (25a) and (25d) ,

(25e) $$- \frac{\partial D}{\partial r} = \frac{\partial D}{\partial \nu} \cong \delta g + \frac{2G}{R} \delta h .$$

$\partial D / \partial \nu$ is the downward derivative of the potential D, it is taken in the direction vertically downwards into the interior of the Earth. The equation (25e) represents this downward derivative of D in terms of the measured quantities of δg and δh .

Now, we finish these theoretical preliminaries. We go over to a consideration of the potential D . This step is recomm- mended in order to prepare this potential D for the further numerical evaluations. In view of the further intentions, it is convenient to divide the potential D into 2 parts : The potential D_b of a surface layer and the potential D_g of the density changes in the interior.
Thus,

(26) $$D = D_b + D_g .$$

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with

$$(27) \quad D_b = f \iint_{\sigma} \varrho \frac{1}{e} \varepsilon \, d\sigma \quad ,$$

$$(28) \quad D_g = f \iiint_{V} \delta\varrho \frac{1}{e} dV \quad .$$

ϱ is the real density along the Earth's surface, e is the straight distance, ε is the vertical shift of the Earth's surface (Fig. 1), and V is the volume of the body of the Earth. The derivation of (27) in the vertical direction of $d\nu$ leads to (29) using the jump relation for this derivation, O.D. Kellogg (1929),

$$(29) \quad \frac{\partial D_b}{\partial \nu} = 2\pi f \varrho \varepsilon + \frac{1}{2R} D_b \quad ,$$

and with (23), (ε can be put equal to δh in sufficient approximation, K. Arnold (1985)(1986)),

$$(30) \quad \frac{\partial D_b}{\partial \nu} \cong 2\pi f \varrho \varepsilon \cong 2\pi f \varrho \cdot \delta h \quad .$$

The relations(25e),(26), and (30) give

$$(31) \quad \frac{\partial}{\partial \nu} D_g = \frac{\partial}{\partial \nu} D - \frac{\partial}{\partial \nu} D_b = \delta g + \left(\frac{2G}{R} - 2\pi f \varrho\right) \delta h \quad .$$

Approximating ϱ by the standard density $\varrho_0 = 2.67 \text{ g cm}^{-3}$, (31) turns to

$$(32) \quad \frac{\partial}{\partial \nu} D_g = \delta g + 0.1967 \, \delta h \quad ;$$

(The gravity in mgal, the heights in meters).

$\frac{\partial}{\partial \nu} D_g$ is a signal function for the density changes $\delta\varrho$ in the interior, (28).

The validity of (31) and (32) can be corroborated in a trivial way. The right hand side of (31) and (32) is the difference of the new gravity at the new surface minus the value of the old gravity reduced from the old surface to the new surface. This reduction happens by the free-air reduction and by the Bouguer plate reduction, whereat the effect of the shift of the level surface can be neglected (or the change of the height anomalies).

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3. The density changes along the main profile of 100 km length

The main profile on Iceland crosses the rift zone and has a length of about 100 km. In 1975 and in 1980, along this profile, precise measurements of the heights and of the gravity were carried out. The levellings have a standard deviation of ± 1.5 mm/km. The gravity values are measured within ± 6 μgal by relative gravity meters. Thus, the changes of the heights and of the gravity values are found precisely. The reference point of the levellings lies at an undisturbed coastal place,(a height change by 1 cm reflects in the gravity by 2 μgal). A comprehensive review of these measurements can be found in: Zeitschrift f. Vermessungswesen 114 (1989), Tectonophysics 71 (1981), J. of Geophysics 47 (1980). By (32), the δg values and the δh values measured along this main profile allow to compute the signal function $\partial D_g / \partial \nu$ along this profile, Kanngieser (1982), Torge (1989). Considering the course of the signal function in Fig. 2, it is obvious that the general level of these values is lowered down during the epoch from 1975 to 1980. It is lowered down to the quantity of - 9 μgal; this number has a standard deviation of about ± 1.3 μgal averaging over 150 values of the signal function. Thus, the subsidence of the level of the signal function is significant; it cannot be explained only by a change of the gravity at the reference point.

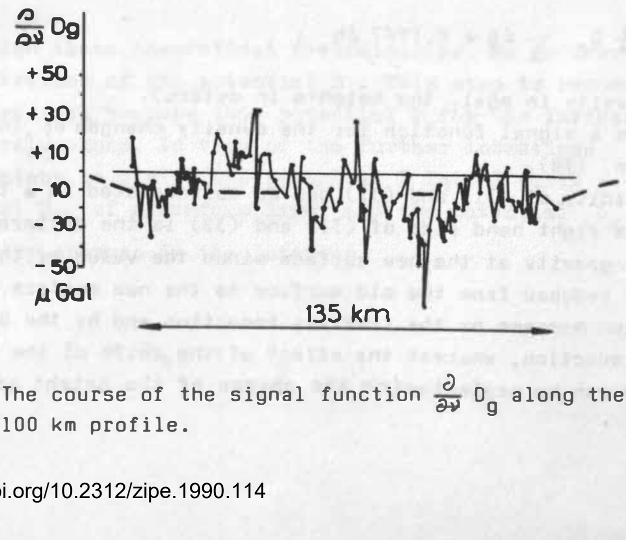

Fig. 2. The course of the signal function $\frac{\partial}{\partial \nu} D_g$ along the 100 km profile.

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As to the interpretation of this subsidence, the well-developed methods of gravimetrical prospection come now into the fore. The potential D_g can be expressed in terms of mass changes δm in the interior of the body of the Earth,

$$(33)\quad D_g(P_k) = f \sum_{i=1}^{N} \delta m_i \frac{1}{e(P_k, K_i)} .$$

e is here the distance between the test point P_k and the place K_i of the point mass δm_i. The following 4 lines are self-explanatory,

$$(34)\quad \frac{\partial}{\partial \nu} D_g(P_k) = f \sum_{i=1}^{N} \delta m_i (z_k - \bar{z}_i) \frac{1}{e^3(P_k, K_i)} ,$$

$$(35)\quad \underline{p} = \underline{A}\, \underline{q} ,$$

$$(36)\quad \underline{q} = \underline{A}^{-1} \underline{p} ,$$

$$(37)\quad \underline{q} = \{\delta m_i\} , \quad \underline{p} = \left\{ \left(\frac{\partial}{\partial \nu} D_g\right)_{P_k} \right\} .$$

z_k is the vertical co-ordinate of P_k, $\bar{z}_i$ that of the point K_i.

Returning back to the interpretation of the values of $\frac{\partial}{\partial \nu} D_g$ shown by Fig. 2, the gravitating scources which cause these values can be represented by a Bouguer plate of 7 km thickness, (7 km is about the width of the lithosphere in the area of Iceland).

A lowering down of the $\frac{\partial}{\partial \nu} D_g$ values by the quantity of - 9 μgal is equivalent to a lowering down of the density of this Bouguer layer (of 7 km thickness) by the quantity of $\delta\rho = -3.4 \times 10^{-5}\, gcm^{-3}$.

In this context, the dynamic of the spreading movement of the lithosphere in the area of Iceland is of interest. A diminution of the density of the masses in the lithosphere plate by $-3.4 \times 10^{-5}\, g\, cm^{-3}$ can have its cause in a horizontal extension of this plate. This extension has to happen in the direction of the main profile of 100 km length, i.e. the direction perpendicular to the rifts.

There are two opinions about this driving mechanism. They are described by Jacoby et al. (1980): "What is the driving mechanism of the rifting event? Is magma squeezed in gravitationally (buoyantly) pushing the sides into compression or is regional tension

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from plate divergence released in fissures tearing open and making space for the magma? The regional deformation of the area can be interpreted either way."

Our above gravimetric investigations about the signal function $\frac{\partial}{\partial \nu} D_g$ led to a diminution of the density along the profile. Thus, the evalution of our signal function is in favour of a long-distance extension of the lithosphere plate. Thus, our signal function is able to discriminate between the different geophysical models.

In this context, it is of interest that the extension of the main profile of 100 km was determined by terrestrial geodetic distance measurements, Möller (1989): "... whole the test area having an east-west range of about 110 km has merely an extension of not more than 2 m...".

This quantity leads to a density change by about $\delta\varrho = -2 \cdot 10^{-5} g\,cm^{-3}$, sure. Both the values of the density change are in a relative good agreement, (i.e. the value obtained gravimetrically by $\frac{\partial}{\partial \nu} D_g$, and the value obtained by terrestrial geodetic distance measurements).

In the above investigations about the lowering down of the signal function $\frac{\partial}{\partial \nu} D_g$ along the 100 km profile, the reference points for the heights and for the gravity were considered to be stable. The stability of the heights can be controlled within some millimeters by water-gauge observations in a satisfactory way. The stability of the gravity level can be checked by absolute gravity measurements, a precision of about $\pm$ 1 μgal is announced to come.

4. The density changes within the test area of 10 km x 14 km size

Now, we consider a test area of the extension of 10 km x 14 km. The eastern and the southern part of it covers the hot spots of the Kraflar caldera and of the Namafjall area. In the pronounced uplift phase of 1978, the changes of the heights δh and that of the gravity δg are determined precisely by measurements. The first measurement campaign was in January 1978 and the final one was in June 1978. During this time,some seismic events and eruptions occured in this area. These δg and δh values allow to compute the signal function $\frac{\partial}{\partial \nu} D_g$ by the formula (32).

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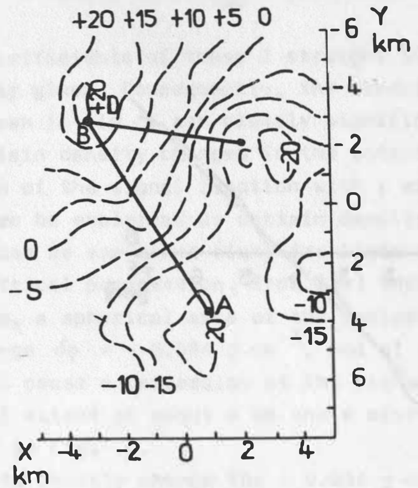

Fig. 3. The course of the signal function within the 10 km x 14 km test area.

Fig. 3 shows the course of our signal function within the 10 km x 14 km test area. Falk (1988), Kanngieser (1985).
The signal function of Fig. 3 has a smoothed shape because a smoothing operator was applied. In the areas of the hot spots, the signal function $\frac{\partial}{\partial \nu} D_g$ has two minima of about - 20 μgal. In the north-western part, the test area has a maximum of about ± 20 μgal. In the Fig. 3, the course of 2 profiles is plotted. Fig. 4 and Fig. 5 show the course of the signal function along these two profiles.
The course of the quantities of the signal function along these two profiles was approximated by straight lines respectively, applying the method of least squares.
The parameters of these 2 straight lines and the concerned standard deviation are as follows, taking the signal function in μgal:
Profile A - B,

(38) $\frac{\partial}{\partial \nu} D_g = (+5.8 \pm 0.1)\, E_{km} - (24 \pm 0.6)$,

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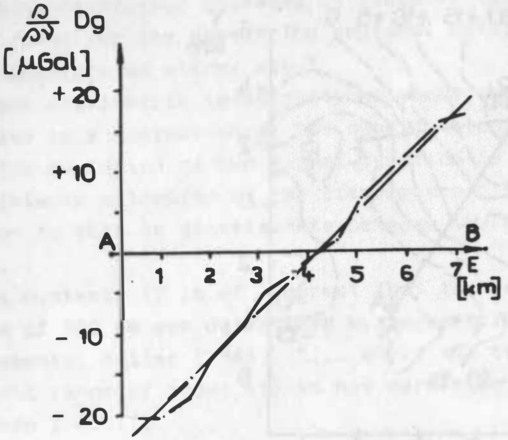

Fig. 4. The profile A - B

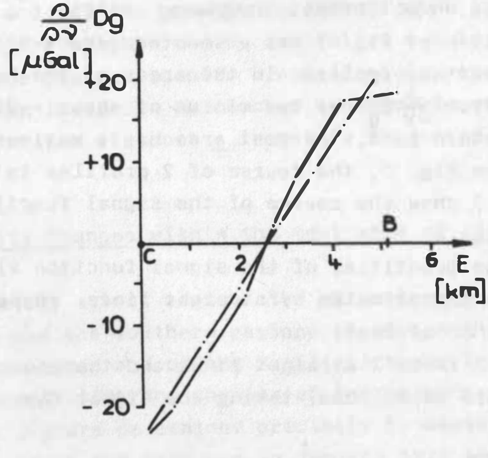

Fig. 5. The Profile C - B

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(39) $\frac{\partial}{\partial \nu} D_g = (+9.1 \pm 0.6)\, E_{km} - (23 \pm 1.8)$.

Thus, the coefficients of these 2 straight lines are clearly significantly given. Consequently, the structures of the signal function shown in Fig. 3 are clearly significant, proving clearly that certain density changes in the interior have to exist.
A depression of the signal function with a minimum value of about - 20 μgal can be explained by certain density changes in the interior. It can be explained along the lines of the methods of the gravimetrical prospection, from (34) through (37).
For instance, a spherical mass of the radius ϑ = 1 km, of the density change $\delta\rho$ = - 0.006 g cm^{-3}, and of a center in a depth of 3 km will cause a depression of the signal function having a horizontal extent of about 4 km and a minimum of about -20 μgal, as figuring in Fig. 3.
This absolute density change (by - 0.006 g cm^{-3}) means a relative density change by - 0.006/2.67, being equal to - $2 \cdot 10^{-3}$.
This quantity of the relative density change corresponds to a horizontal extension of the upper layers of the Earth by about 4 m over a distance of 2 km. Extensions of such an amount are determined by terrestrial geodetic distance measurements in this rift area, indeed, Möller (1989): "... the great extension quantities in the rift zone amounting up to 4 m ...". (This is valid for the period 1977 - 1980).

5. The relation of δg to δh

Several authors finish the discussion of the measured δg and δh values by quoting the relation of δg to δh. For instance, Hagiwara found for the Izu peninsula, H.G. Wenzel (1989),

(40) $\frac{\delta g}{\delta h} = -0.3$ mgal m^{-1} ,

leading to the following quantity of our signal function, (32),

(41) $\frac{\partial}{\partial \nu} D_g = -0.1 \cdot \delta h$.

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For Iceland, we have with W. Torge (1989),

(42) $-0.43 \text{ mgal m}^{-1} < \frac{\delta g}{\delta h} < -0.12 \text{ mgal m}^{-1}$;

hence, for the lower limit of (42),

(43) $\frac{\partial}{\partial \nu} D_g = -0.23 \cdot \delta h$,

and for the upper bound of (42)

(44) $\frac{\partial}{\partial \nu} D_g = +0.08 \cdot \delta h$.

As an extreme quantity, W.Torge found

(45) $\frac{\delta g}{\delta h} = +1.3 \text{ mgal m}^{-1}$;

thus,

(46) $\frac{\partial}{\partial \nu} D_g = +1.5 \cdot \delta h$.

For $\delta h = 1$ m, the relation (46) leads to the relative great value of

(47) $\frac{\partial}{\partial \nu} D_g = 1.5$ mgal .

This latter amount of our signal function can be interpreted by the gravitational effect of a sphere of 1 km radius, having a homogeneous density of 0.45 g cm^{-3}, and having a center point situated in a depth of 3 km. In this case, we have possibly an inflow of magma into an empty or into a widening chamber. Consequently, the evaluation of the δg and δh values should not stop after the first step which leads to the values of only $\delta g / \delta h$. A second step should follow computing the signal function (31)(32) which allows to calculate plausible values for the density changes in the interior.

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6. The mass conservation law

Finally, a discussion of the mass conservation law is of importance. In this context, this law has the following shape introducing tolerable approximations, W.A. Heiskanen and H. Moritz (1967), O. D. Kellog (1929), (without the Earth rotation term)

$$(48) \quad \delta M = 0 = \frac{1}{4\pi f} \iint_V \frac{\partial}{\partial \nu} D \cdot dv .$$

Of course, the mass change δM during the period Φ has to be equal to zero.
The relations (25e) and (48) lead to

$$(49) \quad 0 = \iint_V [\delta g + \frac{2G}{R} \delta h] \, dv ,$$

relating the global integral over δg and that over δh,

$$(50) \quad \iint_V \delta g \, dv = - \frac{2G}{R} \iint_V \delta h \cdot dv .$$

This is a condition which is to be observed considering a recent crustal movement phenomenon.
The coefficient - 2G/R is the free-air gradient being equal to - 0.3 mgal m^{-1}.
For instance, applying the above developments about the mass conservation law on the fennoscandian land uplift, we have for this area by empirical means, H.G. Wenzel (1989),

$$(51) \quad \frac{\delta g}{\delta h} = - 0.19 \text{ mgal } m^{-1} .$$

(32) and (51) give

$$(52) \quad \frac{\partial}{\partial \nu} D_g = 0 .$$

The above relation (52) shows that our signal function is equal to zero for the area of the fennoscandian land uplift. Thus, it is very probable that there are no great density changes in the interior.
Consequently, the mass conservation law demands that the masses of the central uplift, $\rho \delta h$, distributed over the Earth's surface

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($\delta h > 0$) have to be compensated by the mass defects of a surrounding belt of subsidence, ($\delta h < 0$). For the concerned surfical mass distribution ($\rho \cdot \delta h$), we have the following constraint

$$(53) \quad 0 = \delta M = \iint_V \rho \cdot \delta h \cdot d\sigma \quad .$$

7. Results

In a refinement of the here discussed geodynamic model, the first step should be to replace the standard density ρ_0 of the surface layer by the real density on the surface of the Earth.
The usual interpretation method which finishes the discussion of the gravity changes and height changes by quoting the relation $\delta g : \delta h$ only, this is not an optimal one. The information content of the measurements is not exhausted fully; this method means to stop halfway.
In any case, it is better to add a second step, computing the signal function (31)(32) in terms of the δg and δh values and determining plausible quantities for the density changes in the interior. This second step should not be missed. The estimation of the density changes should happen in close collaboration with geophysicists and geologists.
The above investigations show that it is possible to find significantly certain parameters characterizing the time-dependent density variations which appear in the interior and which are caused by the procedures of recent crustal movements.

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