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Local coupling and conversion of surface waves due to Earth's rotation. Part 1: theory

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SUMMARY

Earth's rotation affects wave propagation to first order in the rotation through the Coriolis force. The imprint of rotation on wave motion has been accounted for in normal mode theory. By extending the theory to propagating surface waves we account for the imprint of rotation as a function of propagation distance. We describe the change in phase velocity and polarization, and the mode conversion of surface waves by Earth's rotation by extending the formalism of Kennett for surface wave mode conversion due to lateral heterogeneity to include the Coriolis force. The wavenumber of Rayleigh waves is changed by Earth's rotation and Rayleigh waves acquire a transverse component. The wavenumber of Love waves in not affected by Earth's rotation, but Love waves acquire a small additional Rayleigh wave polarization. In contrast to different Rayleigh wave modes, different Love wave modes are not coupled by Earth's rotation. We show that the backscattering of surface waves by Earth's rotation is weak. The coupling between Rayleigh waves and Love waves is strong when the phase velocities of these modes are close. In that regime of *resonant coupling*, Earth's rotation causes the difference between the Rayleigh wave and Love wave phase velocities that are coupled to increase through the process of level-repulsion.

Key words: Theoretical seismology; Wave propagation; Surface waves and free oscillations; Wave scattering and diffraction; Structure of the Earth.

1 INTRODUCTION

The rotation of the Earth affects wave propagation to first order through the Coriolis force. The effect of rotation on body waves in anisotropic media was accounted for by Schoenberg & Censor (1973). To first order in the rotation rate, *P* waves are not affected while the polarization of *S* waves changes in the same way as a Foucault pendulum responds to rotation (Snieder *et al.* 2016). Multiply reflected *ScS* waves under Japan show a change in the polarization induced by rotation that agrees with the theory of Snieder *et al.* (2016).

The imprint of Earth's rotation on normal modes has been extensively studied. Using Rayleigh's principle the change of normal mode frequencies was determined first (Backus & Gilbert 1961; MacDonald & Ness 1961; Pekeris *et al.* 1961). The theory was extended to include the coupling of normal modes by rotation (Dahlen 1968, 1969; Luh 1974; Park & Gilbert 1986). When the spheroidal and toroidal modes are not degenerate, the rotational coupling between these modes types is weak (Dahlen 1968; Luh 1974), but for spheroidal and toroidal modes that are nearly degenerate, one needs to account for the coupling between these mode types, and the normal modes on a rotating Earth are a mix of spheroidal and toroidal motion (Dahlen 1969).

The description of wave motion on a rotating Earth with normal mode theory is elegant and complete, but one needs to sum over normal modes to extract the properties of body waves and surface waves. Normal modes are—by definition—global, and to localize the wavefield for a particular source and receiver one must use the addition theorem to sum over the angular order m, while the surface waves can be retrieved by applying Poisson's sum formula to the sum over degree l (Dahlen 1979; Romanowicz & Roult 1986; Snieder & Romanowicz 1988).

In order to circumvent the need to sum over all angular orders and degrees to obtain the surface wave motion, we study the imprint of rotation on surface waves by formulating the theory as a problem of coupled surface waves. This simplifies the theory considerably, and it also makes it possible to study how the surface waves are affected by rotation as a function of propagation distance. In a normal mode formulation the dependence of the wave motion on propagation distance is implicit, but in our theory this dependence is explicitly accounted for by a differential equation that depends on propagation distance.

Backus (1962) studied the effect of rotation on a propagating surface wave. He showed in a rotating Cartesian coordinate system that (1) the Rayleigh wave has a transverse component, (2) the Love wave has a radial and vertical component, (3) the phase velocity of Love waves

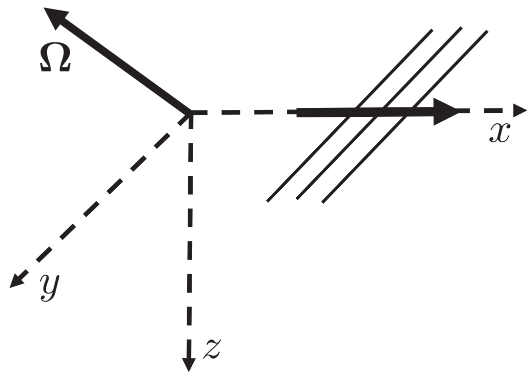

Figure 1. Geometry of the coordinate system where the x-axis is aligned with the direction of propagation of the plane wave, and the z-axis points down. In this coordinate system the rotation vector Ω can have any direction.

is not perturbed by the Coriolis force but the phase velocity of Rayleigh waves is and (4) the group velocity has a component perpendicular to the direction of propagation. We refer to the change in phase velocity and/or polarization as *mode coupling*. In contrast to mode coupling, we use use the words *mode conversion* for the process where a mode in (partly) converted to another mode which then propagates independently in space or time.

In his treatment, Backus (1962) assumed that for a given frequency, the surface wave propagates with one fixed wavenumber k. This means that he considered the propagation of a single surface wave mode. Because of this restriction, his theory does not account for the rotational conversion between different surface wave modes. Tromp (1994) provides the theory of surface wave propagation on a spherical Earth by using the WKBJ approximation. His theory accounts for the change in phase velocity and polarization of surface waves, but not for the conversion between different surface wave modes.

To account for the conversion between different surface wave types, we extend the theory of Kennett (1984) for surface wave mode conversion due to heterogeneity along the path of propagation to include rotation. This treatment accounts for the rotational coupling between Love and Rayleigh waves, as well as for the conversion between the different modes of each wave type.

We revisit the theory of Kennett (1984) in Section 2, we extend the theory in Section 3 to include the rotation of the coordinate system along an arbitrary rotation vector Ω, and we derive the coupling coefficients between modes due to rotation. In Section 4 we present the change in the wave number of Love and Rayleigh waves due to rotation and show that the theory predicts the same change in wavenumber as in the theory of Backus (1962). To show the evolution of surface waves as they propagate in the presence of rotation we show in Section 5 what happens when only one mode is excited at the source and how other surface wave types emerge due to rotation. In Section 6 we show how the fundamental Love and Rayleigh waves are coupled by rotation, and show that the coupling is different depending on whether the phase velocities of the waves are the same (degeneracy), or whether they are different. We also show that rotation increases the difference of the phase velocities of these modes through the process of level-repulsion. The theory is presented in a Cartesian system, but we extend the theory to a spherical geometry in a heuristic way in Section 7. In appendices we derive details of the theory developed in this study. In a companion paper (Sens-Schönfelder *et al.* 2020) we show numerical examples that indicate the importance of Earth's rotation on long-period surface waves.

2 THE FORMALISM OF KENNETT (1984)

The theory of Kennett (1984) for the coupling of surface wave modes considers surface waves propagating as plane waves in the x-direction in the frequency domain for a fixed angular frequency ω (Fig. 1). The three components of the displacement are ordered in a vector $\mathbf{w} = (u_x, u_y, u_z)^T$. The theory of Kennett (1984) uses the following definition of the vectors $\mathbf{w}$ and $\mathbf{t}$ for the displacement and stress components

$$\mathbf{w} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \quad , \quad \mathbf{t} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{pmatrix} . \tag{1}$$

Note that in contrast to the treatment of Aki & Richards (2002), who consider the stress components σ_{xz}, σ_{ys} and σ_{zz}, the formalism of Kennett uses the stress components that have at least one x component. As shown in eq. (2.1) of Kennett (1984), the stress and displacement vector satisfy the a linear differential equation in x:

$$\partial_x \begin{pmatrix} \mathbf{w} \\ \mathbf{t} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{w} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{ww} & \mathbf{A}_{wt} \\ \mathbf{A}_{tw} & \mathbf{A}_{tt} \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \mathbf{t} \end{pmatrix} , \tag{2}$$

where, for example, the 3×3 submatrix $\mathbf{A}_{wt}$ gives the contribution of $\mathbf{t}$ to the derivative $\partial_x \mathbf{w}$ of the displacement.

The surface wave modes in a reference medium, which in our case is a non-rotating system have a displacement vector $\mathbf{w}_r^0(k_r, z)$ and a stress vector $\mathbf{t}_r^0(k_r, z)$, where the index r labels the modes and k_r is the wavenumber of that mode. In a non-rotating system this mode varies as $\exp(ik_r x)$ with the x-coordinate. Kennett (1984) expresses in his eq. (2.7) the displacement and stress in a general earth model where the waves propagate along the x-axis as a superposition of modes propagating in the positive and negative x-directions:

$$\mathbf{w} = \sum_r \left(c_r^+(x)e^{ik_r x}\mathbf{w}_r^0(k_r, z) + c_r^-(x)e^{-ik_r x}\mathbf{w}_r^0(-k_r, z)\right) ,$$
$$\mathbf{t} = \sum_r \left(c_r^+(x)e^{ik_r x}\mathbf{t}_r^0(k_r, z) + c_r^-(x)e^{-ik_r x}\mathbf{t}_r^0(-k_r, z)\right) . \tag{3}$$

In the reference model the coefficients $c_r^\pm$ are constant, but lateral heterogeneity and the Coriolis force lead to an x-dependence of these coefficients.

Lateral heterogeneity leads to a perturbation $\Delta\mathbf{A}$ of the matrix $\mathbf{A}$, defined in eq. (2), and we derive in Section 3 the perturbation $\Delta\mathbf{A}$ due to rotation. When $\Delta\mathbf{A} = 0$, the modal coefficients $c_r^\pm$ are constant, but as shown in expression (2.19) of Kennett (1984) the modal coefficient are not constant and they are coupled through the following differential equations when $\Delta\mathbf{A}$ is non-zero:

$$\partial_x c_q^+ = \sum_r \left(iK_{qr}\bar{e}_q e_r c_r^+ + iL_{qr}\bar{e}_q\bar{e}_r c_r^-\right) ,$$
$$\partial_x c_q^- = -\sum_r \left(i\overline{L}_{qr}e_q e_r c_r^+ + i\overline{K}_{qr}e_q\bar{e}_r c_r^-\right) , \tag{4}$$

where

$$e_q = e^{ik_q x} \quad \text{and} \quad \bar{e}_q = e^{-ik_q x} . \tag{5}$$

The coupling matrices L and K are given by expressions (2.16) and (2.17) of Kennett (1984). We show in Section 3 that for the special case where only $\Delta\mathbf{A}_{tw}$ is non-zero the coupling matrices are given by

$$K_{qr} = -\int_0^\infty \overline{\mathbf{w}}_q^0 \cdot \Delta\mathbf{A}_{tw} \cdot \mathbf{w}_r^0 \mathrm{d}z ,$$
$$L_{qr} = -\int_0^\infty \overline{\mathbf{w}}_q^0 \cdot \Delta\mathbf{A}_{tw} \cdot \overline{\mathbf{w}}_r^0 \mathrm{d}z . \tag{6}$$

According to Kennett (1984), the coupling matrices $\overline{K}$ and $\overline{L}$ and the modes $\overline{\mathbf{w}}^0$ follow by letting the modes propagate in the opposite direction, that is, by replacing $k \to -k$. The bar over any quantity, for example $\overline{\mathbf{w}}^0$, denotes that this quantity if for a leftward moving surface wave. Using the notation of Aki & Richards (2002) for the modal eigenfunctions, the surface wave modes are given by

$$\mathbf{w}^0 = \begin{pmatrix} 0 \\ l_1 \\ 0 \end{pmatrix} \quad \text{(Love Waves)}, \quad \mathbf{w}^0 = \begin{pmatrix} r_1 \\ 0 \\ ir_2 \end{pmatrix} \quad \text{(Rayleigh Waves)} . \tag{7}$$

Turning a rightward moving wave into a leftward moving wave amounts to replacing $k \to -k$. Expression (7.24) of Aki & Richards (2002) shows that the Love waves depend only on k^2, hence the eigenfunction for Love waves does not change when the direction of propagation is reversed. Inspection of eq. (7.28) of Aki & Richards (2002) shows that when r_1 and r_2 are Rayleigh wave eigenfunction for wavenumber k, then r_1 and $-r_2$ are eigenfunctions for wavenumber $-k$. This means that

$$\overline{\mathbf{w}}^0 = \begin{pmatrix} 0 \\ l_1 \\ 0 \end{pmatrix} \quad \text{(Love Waves)}, \quad \overline{\mathbf{w}}^0 = \begin{pmatrix} r_1 \\ 0 \\ -ir_2 \end{pmatrix} \quad \text{(Rayleigh Waves)} . \tag{8}$$

A comparison of eqs (7) and (8) shows that

$$\overline{\mathbf{w}}_q^0 = \mathbf{w}_q^{0*} , \tag{9}$$

where the asterisk denotes complex conjugation.

The modes satisfy the normalization condition (2.8) of Kennett (1984). We show in Appendix A that this corresponds to the normalization

$$4\omega U I_1 = 1 , \tag{10}$$

where U is the group velocity and I_1 is defined in expressions (7.66) and (7.74) of Aki & Richards (2002):

$$I_1 = \frac{1}{2}\int_0^\infty \rho l_1^2 dz \quad \text{(Love Waves)} ,$$
$$I_1 = \frac{1}{2}\int_0^\infty \rho\left(r_1^2 + r_2^2\right) dz \quad \text{(Rayleigh Waves)} , \tag{11}$$

with ρ being the mass density.

3 DERIVATION OF THE MATRIX ΔA DUE TO THE CORIOLIS FORCE

In this section we derive the perturbation of the matrix $\Delta\mathbf{A}$ due to the Coriolis force and we use the Fourier convention $f(t) = \int F(\omega)\exp(-i\omega t)\mathrm{d}\omega$. The rotation vector Ω introduces a Coriolis force $-2\rho\Omega\times\mathbf{v}$, with $\mathbf{v}$ the particle velocity, and a centrifugal force $-\Omega\times(\Omega\times\mathbf{r})$, with $\mathbf{r}$ the position vector. Since we only consider the first-order effect of rotation, we ignore the centrifugal force, and hence the equation of motion is, in the frequency domain, given by

$$-\rho\omega^2\mathbf{u} = (\nabla\cdot\sigma) + 2i\rho\omega\Omega\times\mathbf{u}\ , \tag{12}$$

where σ is the stress tensor. Following Kennett (1984) we align the x-axis with the direction of horizontal wave propagation and assume plane-wave solutions e^{ikx}, hence all y-derivatives vanish: $\partial_y = 0$, and the equation of motion (12) is in component form:

$$\begin{aligned} -\rho\omega^2 u_x &= \partial_x\sigma_{xx} + \partial_z\sigma_{zx} + 2i\rho\omega\left(\Omega_y u_z - \Omega_z u_y\right)\ , \\ -\rho\omega^2 u_y &= \partial_x\sigma_{xy} + \partial_z\sigma_{zy} + 2i\rho\omega\left(\Omega_z u_x - \Omega_z u_x\right)\ , \\ -\rho\omega^2 u_z &= \partial_x\sigma_{xz} + \partial_z\sigma_{zz} + 2i\rho\omega\left(\Omega_x u_y - \Omega_y u_x\right)\ . \end{aligned} \tag{13}$$

For an isotropic medium these expressions need to be supplemented with stress–strain relations, when the y-derivatives vanish these are given by

$$\begin{aligned} \sigma_{xx} &= (\lambda+2\mu)\partial_x u_x + \lambda\partial_z u_z\ , \\ \sigma_{xy} &= \mu\partial_x u_y \\ \sigma_{xz} &= \mu(\partial_x u_z + \partial_z u_x)\ , \\ \sigma_{zy} &= \mu\partial_z u_y \\ \sigma_{zz} &= \lambda\partial_x u_x + (\lambda+2\mu)\partial_z u_z\ , \end{aligned} \tag{14}$$

where λ and μ are the Lamé parameters. Following the steps taken by Kennett (1984) we eliminate σ_{zy} and σ_{zz}. Expressions (13) and (14) can then be written as the following six coupled equations

$$\begin{aligned} \partial_x u_x &= -\frac{\lambda}{\lambda+2\mu}\partial_z u_z + \frac{1}{\lambda+2\mu}\sigma_{xx}\ , \\ \partial_x u_y &= \frac{1}{\mu}\sigma_{xy}\ , \\ \partial_x u_z &= -\partial_z u_x + \frac{1}{\mu}\sigma_{xz}\ , \\ \partial_x\sigma_{xx} &= -\rho\omega^2 u_x - \partial_z\sigma_{xz} - 2i\rho\omega(\Omega_y u_z - \Omega_z u_y)\ , \\ \partial_x\sigma_{xy} &= (-\rho\omega^2 - \partial_z\mu\partial_z)u_y - 2i\rho\omega(\Omega_z u_x - \Omega_x u_z)\ , \\ \partial_x\sigma_{xz} &= (-\rho\omega^2 - \partial_z\zeta\partial_z)u_z - \partial_z\left(\frac{\lambda}{\lambda+2\mu}\sigma_{xx}\right) - 2i\rho\omega(\Omega_x u_y - \Omega_y u_x)\ , \end{aligned} \tag{15}$$

with $\zeta = 4\mu(\lambda+\mu)/(\lambda+2\mu)$. For the vectors $\mathbf{w} = (u_x, u_y, u_z)^T$ and $\mathbf{t} = (\sigma_{xx}, \sigma_{xy}, \sigma_{xz})^T$ as above, expressions (15) can be written in the form of eq. (2), where the matrix $\mathbf{A}$ is a 6×6 matrix that combines eqs (2.5) and (2.6) of Kennett (1984) and the perturbation $\Delta\mathbf{A}$ that is due to the Coriolis force. The terms containing the rotation vector Ω in the system (15) are given by

$$\Delta\mathbf{A}_{ww} = \Delta\mathbf{A}_{wt} = \Delta\mathbf{A}_{tt} = 0\ , \tag{16}$$

and

$$\Delta\mathbf{A}_{tw} = 2i\rho\omega\begin{pmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{pmatrix}\ . \tag{17}$$

This matrix is all that is needed to compute the coupling coefficients for the surface wave modes. Inserting expression (17) into eq. (6) gives for the surface wave modes propagating in the same direction

$$\begin{aligned} K_{qr}^{LL} &= 0\ , \\ K_{qr}^{RL} &= 2\omega\int_0^\infty \rho\left(-i\Omega_z r_1^q + \Omega_x r_2^q\right) l_1^r\,\mathrm{d}z\ , \\ K_{qr}^{LR} &= 2\omega\int_0^\infty \rho l_1^q\left(i\Omega_z r_1^r + \Omega_x r_2^r\right)\mathrm{d}z\ , \\ K_{qr}^{RR} &= -2\omega\Omega_y\int_0^\infty \rho\left(r_2^q r_1^r + r_1^q r_2^r\right)\mathrm{d}z\ . \end{aligned} \tag{18}$$

The coupling coefficients for modes propagating in opposite directions are given by

$$
\begin{aligned}
L_{qr}^{LL} &= 0 \,, \\
L_{qr}^{RL} &= 2\omega \int_0^\infty \rho \left(-i\Omega_z r_1^q + \Omega_x r_2^q\right) l_1^r dz \,, \\
L_{qr}^{LR} &= 2\omega \int_0^\infty \rho l_1^q \left(i\Omega_z r_1^r - \Omega_x r_2^r\right) dz \,, \\
L_{qr}^{RR} &= -2\omega\Omega_y \int_0^\infty \rho \left(r_2^q r_1^r - r_1^q r_2^r\right) dz \,.
\end{aligned}
\tag{19}
$$

In these expressions, K_{qr}^{RL} gives, for example, the coupling of a Rayleigh wave q to a Love wave mode r moving in the same direction. The coupling coefficients have a number of properties:

(i)Love waves do not couple to other Loves waves:

$$\mathbf{K}^{LL} = \mathbf{L}^{LL} = 0 \,. \tag{20}$$

The reason for this if that for Love waves the Coriolis force $2i\rho\omega\Omega \times \mathbf{u}$ is perpendicular to the y-direction, and it therefore has a vanishing projection on the mode to which the Love wave couples.

(ii)Love wave and Rayleigh waves couple to each other through the components Ω_x and Ω_z. This coupling vanishes when the rotation vector is in the transverse direction.

(iii)Rayleigh waves are coupled through Ω_y, that is the component of the rotation vector in the transverse direction. The coupling depends on products such as $r_1^q r_2^r$, hence on the coupling of the motion in the radial and vertical directions.

(iv)The coupling matrix for coupling between Love and Rayleigh waves propagating in the same direction is unitary:

$$K_{qr}^{RL} = \left(K_{rq}^{LR}\right)^* \,. \tag{21}$$

(v)The coupling matrix for coupling between Love and Rayleigh waves propagating in opposite directions is anti-unitary:

$$L_{qr}^{RL} = -\left(L_{rq}^{LR}\right)^* \,. \tag{22}$$

(vi)A Rayleigh wave mode is not backscattered into the same Rayleigh wave mode because

$$L_{qq}^{RR} = 0 \,. \tag{23}$$

(vii)We show in Appendix B that

$$\overline{K}_{qr} = K_{qr}^* \quad \text{and} \quad \overline{L}_{qr} = L_{qr}^* \,. \tag{24}$$

(viii)The coefficients that couple Love and Rayleigh waves are, in general, complex, but

$$K_{qr}^{RR} \quad \text{and} \quad L_{qr}^{RR} \quad \text{are real} \,. \tag{25}$$

4 THE CHANGE IN WAVENUMBER DUE TO ROTATION

In this section we derive the change in wavenumber for Love and Rayleigh waves due to rotation. Inserting the definition (5) into the differential eq. (4) for the modal coefficients gives

$$
\begin{aligned}
\partial_x c_q^+ &= \sum_r \left(iK_{qr}e^{-i(k_q-k_r)x}c_r^+ + iL_{qr}e^{-i(k_q+k_r)x}c_r^-\right) \,, \\
\partial_x c_q^- &= -\sum_r \left(iL_{qr}^* e^{i(k_q+k_r)x}c_r^+ + iK_{qr}^* e^{i(k_q-k_r)x}c_r^-\right) \,,
\end{aligned}
\tag{26}
$$

where we used the identity (24) to replace $\overline{K}_{qr}$ and $\overline{L}_{qr}$ by K_{qr}^* and K_{qr}^*, respectively. We next make the following change of variables for the modal coefficients

$$g_q^+(x) = e^{ik_qx}c_q^+(x) \quad \text{and} \quad g_q^-(x) = e^{-ik_qx}c^-(x) \,. \tag{27}$$

Inserting this into eq. (26) and splitting the sum over modes for K_{qr} into terms with $r = q$ and $r \neq q$ gives

$$
\begin{aligned}
\partial_x g_q^+ &= +i\left(k_q + K_{qq}\right)g_q^+ + \sum_{r\neq q} iK_{qr}g_r^+ + \sum_r iL_{qr}g_r^- \,, \\
\partial_x g_q^- &= -i\left(k_q + K_{qq}^*\right)g_q^- - \sum_r iL_{qr}^* g_r^+ - \sum_{r\neq q} iK_{qr}^* g_r^- ; .
\end{aligned}
\tag{28}
$$

In the absence of coupling to modes propagating in the same direction (the sums $\sum_{r\neq q}$), and of backscattering (the terms containing L), the coefficient for the right going wave satisfies $\partial_x g_q^+ = +i\left(k_q + K_{qq}\right)g_q^+$. This means that the wavenumber for mode q is in the presence of rotation given by

$$k_q^\Omega = k_q + K_{qq} \ , \tag{29}$$

hence K_{qq} is the wavenumber perturbation for mode q. A similar argument applies to the left going wave in expression (28) that has a wavenumber $-k_q - K_{qq}^*$. According to expression (20) K_{qq} vanishes for Love waves, while according to expression (25) for Rayleigh waves K_{qq} is real, hence the wavenumber for a leftward propagating mode is in both cases given by $-k_q - K_{qq}$.

This means that the wavenumber of Love waves is not perturbed by rotation

$$k_L^\Omega = k_L \quad \text{(Love waves)} \ , \tag{30}$$

while the perturbed wavenumber of the Rayleigh waves is given by

$$k_R^\Omega = k_R + K_{RR} \quad \text{(Rayleigh waves)} \ , \tag{31}$$

with, according to expression (18),

$$K_{RR} = -2\omega\Omega_y \int_0^\infty \rho r_1 r_2 \mathrm{d}z \ . \tag{32}$$

This expression is subject to the normalization condition (8). Dividing expression (32) by $4\omega U I_1$, which is equal to 1, and using eq. (11) for I_1 shows that the wavenumber perturbation of Rayleigh waves due to rotation is given by

$$\delta k_R = -\frac{2\Omega_y}{U} \frac{\int_0^\infty \rho r_1 r_2 \mathrm{d}z}{\int_0^\infty \rho \left(r_1^2 + r_2^2\right) \mathrm{d}z} \ . \tag{33}$$

This perturbation depends only on the horizontal component Ω_y of the rotation vector perpendicular to the direction of propagation, because this is the only component that of the rotation vector that couples the Rayleigh wave polarization vector to itself. The component Ω_y depends, for a fixed rotation vector, on the direction of wave propagation, hence the wavenumber perturbation depends on the direction of wave propagation. As a result, the phase velocity of Rayleigh waves is anisotropic.

Backus (1962) treated the problem of a surface wave mode with a fixed wavenumber k and derived that the perturbation $\delta\omega$ of the frequency for a Rayleigh wave is in that case given by

$$\delta\omega = -2\Omega_y \frac{\int_0^\infty \rho r_1 r_2 \mathrm{d}z}{\int_0^\infty \rho \left(r_1^2 + r_2^2\right) \mathrm{d}z} \ , \tag{34}$$

where we rephrased his notation in the notation of our paper. The corresponding perturbation in the wavenumber at a fixed frequency can be obtained from

$$\delta k = \frac{\partial k}{\partial\omega}\delta\omega = \frac{1}{U}\delta\omega = -\frac{2\Omega_y}{U} \frac{\int_0^\infty \rho r_1 r_2 \mathrm{d}z}{\int_0^\infty \rho \left(r_1^2 + r_2^2\right) \mathrm{d}z} \ , \tag{35}$$

where expression (34) is used in the last identity. The wavenumber perturbation (35) of Backus (1962) thus agrees with the wavenumber perturbation (33) derived here.

For future use we insert the definition (29) for the perturbed wavenumber in into expression (28), this gives

$$\begin{aligned} \partial_x g_q^+ &= +i k_q^\Omega g_q^+ + \sum_{r \neq q} i K_{qr} g_r^+ + \sum_r i L_{qr} g_r^- \ , \\ \partial_x g_q^- &= -i k_q^\Omega g_q^- - \sum_r i L_{qr}^* g_r^+ - \sum_{r \neq q} i K_{qr}^* g_r^- \ , \end{aligned} \tag{36}$$

where k_q^Ω is given by expression (30) or (31) depending on whether mode q is a Love wave or Rayleigh wave. Using definition (27) and expressions (3) the displacement and stress vectors are given by

$$\begin{aligned} \mathbf{w} &= \sum_r \left(g_r^+(x)\mathbf{w}_r^0(k_r, z) + g_r^-(x)\mathbf{w}_r^0(-k_r, z)\right) \ , \\ \mathbf{t} &= \sum_r \left(g_r^+(x)\mathbf{t}_r^0(k_r, z) + g_r^-(x)\mathbf{t}_r^0(-k_r, z)\right) \ . \end{aligned} \tag{37}$$

5 APPROXIMATE SOLUTION FOR ONE INCOMING MODE

In this section, we consider the special case where at $x = 0$ only one right going surface wave is present, and all other left going and right going surface wave modes vanish. We label the mode that is non-zero at $x = 0$ with the modal index 0, which can be either a Love or a Rayleigh wave, and it may be the fundamental mode or a higher mode. This means that

$$g_0^+(x = 0) = 1 \quad , \quad g_0^-(x = 0) = 0 \quad \text{and} \quad g_q^\pm(x = 0) = 0 \quad (q \neq 0) \ , \tag{38}$$

where we assume in this section that the mode q is not the mode that is non-zero at $x = 0$.

We use an approximate solution to the eqs (36) and use a similar procedure as used in section 18.2 of Merzbacher (1970) for the evolution of modal coefficients for a quantum system. For eq. (36) for $g_0^+(x)$ we set the coefficients $g_r^\pm$ in the right-hand side equal to their value at $x = 0$, hence we set $g_r^\pm = 0$ in the top equation of (36) and obtain

$$\partial g_0^+ = i k_0^\Omega g_0^+ \ . \tag{39}$$

With the boundary condition (38) this gives has the solution

$$g_0^+(x) = e^{i k_0^\Omega x} \ . \tag{40}$$

We use this solution in the right-hand side of the system (36) and set all other coefficients $g_r^\pm$ equal to zero. This gives for modes $q \neq 0$:

$$\begin{aligned} \partial_x g_q^+ &= +i k_q^\Omega g_q^+ + i K_{q0} e^{i k_0^\Omega x} \ , \\ \partial_x g_q^- &= -i k_q^\Omega g_q^- - \sum_r i L_{q0}^* e^{i k_0^\Omega x} \ . \end{aligned} \tag{41}$$

Subject to the boundary condition $g_q^\pm(x = 0) = 0$, these equations have for $k_q^\Omega \neq k_0^\Omega$ the solution

$$g_q^+(x) = \frac{K_{q0}}{k_0^\Omega - k_q^\Omega} \left(e^{i k_0^\Omega x} - e^{i k_q^\Omega x} \right) \quad , \ (k_q^\Omega \neq k_0^\Omega) \ , \tag{42}$$

and

$$g_q^-(x) = -\frac{L_{q0}^*}{k_0^\Omega + k_q^\Omega} \left(e^{i k_0^\Omega x} - e^{-i k_q^\Omega x} \right) \ . \tag{43}$$

The mode conversions for rightward propagating surface waves are proportional to $K_{q0}/(k_0^\Omega - k_q^\Omega)$ which is a small dimensionless number for most mode pairs. The conversion to leftward propagating surface waves is proportional to $L_{q0}^*/(k_0^\Omega + k_q^\Omega)$, which is also a small number. Since $1/(k_0^\Omega + k_q^\Omega) < 1/(k_0^\Omega - k_q^\Omega)$, the conversion to backward propagating modes is weaker than to forward propagating modes. Note that both $g_q^+(x)$ and $g_q^-(x)$ are oscillatory functions of space, this is due to the fact that the mode conversions by rotation take place all along the path.

The modal coefficient $g_q^\pm$ is multiplied with the displacement $\mathbf{r}_q^0$ of the modes in their contribution to the displacement. It this follows from expression (42) that the forward propagating mode q gives a contribution

$$\mathbf{w}(x, z) = \frac{K_{q0}}{k_0^\Omega - k_q^\Omega} \left(e^{i k_0^\Omega x} - e^{i k_q^\Omega x} \right) \mathbf{w}_q^0(z) \ . \tag{44}$$

The contribution of the first term in brackets is a wave that propagates with the wavenumber k_0^Ω of the incident mode 0 but that has displacement determined by the converted mode q. This term accounts for the change in the displacement of the incident surface wave due to the Coriolis coupling to mode q. This term, for example, accounts for the transverse component that a propagating Rayleigh wave acquires due to the Coriolis force, this is an example of *mode coupling* by rotation. The contribution of the second term is a surface wave that propagates with the wavenumber k_q^Ω of the converted mode q and that also has the displacement of this converted mode. This term thus accounts for the true *mode conversions* from the incoming mode to a converted mode. The theory thus accounts for both mode coupling and mode conversion. For mode coupling the change of the wavenumber is caused by the coupling of a mode to itself and the change in polarization is due to the coupling to a different mode, while mode conversion is due to the coupling to a different mode that propagates independently after excitation.

Since for surface waves there are different wave types (Love and Rayleigh waves) the wavenumbers k_q^Ω and k_0^Ω can be equal. This does happen when one of the waves is a Rayleigh wave while the other is a Love wave, and at certain frequencies both waves may have the same phase velocity. In that case, $k_q^\Omega = k_0^\Omega$ and the solution (42) for $g_q^0(x)$ still holds. In the limit $k_q^\Omega \to k_0^\Omega$ the solution for $g_q^+(x)$ is given by

$$g_q^+(x) = i K_{q0} \, x e^{i k_0^\Omega x} \quad , \ (k_q^\Omega = k_0^\Omega) \ . \tag{45}$$

In this case the modes are degenerate and the solution is proportional to the propagation distance x. This is an example of *secular growth* (Bender & Orszag 1978) which is a result of the approximations made, but such unbounded growth with x cannot be physical because it is not compatible with energy conservation. The secular growth is an artifact caused by ignoring the energy loss of the excited mode due to mode conversions in the approximation (39). We return to the case of degenerate modes in Section 6.2.

6 COUPLED RAYLEIGH AND LOVE WAVES

We consider in this section the situation where we launch a fundamental mode Rayleigh wave at $x = 0$ and consider the coupling to a fundamental mode Love wave. In that case the excited wave has wavenumber $k_0^\Omega = k_R^\Omega$, and the wave excited by the Coriolis coupling is the fundamental mode Love wave with wavenumber k_L. Here k_R^Ω is the wavenumber of the fundamental mode Rayleigh wave on a rotating system Earth, and k_L the wavenumber of the fundamental mode Love wave. Since the wavenumber of the Love wave is not affected by rotation, see eq. (30), we omit the superscript Ω for the Love wave.

We showed in the previous section that rotation causes the backscattered surface waves to be weaker than the forward scattered surface waves by a factor $(k_0^\Omega - k_q^\Omega)/(k_0^\Omega + k_q^\Omega)$. This means that for the case of coupling of the fundamental mode Rayleigh and Love waves

$$\left|\frac{\text{backscattered waves}}{\text{forward scattered waves}}\right| \propto \left|\frac{k_R^\Omega - k_L}{k_R^\Omega + k_L}\right| \ll 1 \, , \tag{46}$$

where the last inequality follows from the fact that the difference of the wavenumber of the fundamental mode Rayleigh and Love waves is much less than their sum. We therefore analyse in this section the system (36) where we restrict the modes to the rightward propagating fundamental Love and Rayleigh waves, that is we ignore g_q^- and limit the sum over modes to the fundamental Rayleigh and Love waves. This reduces the system (36) to

$$\begin{aligned} \partial_x g_R^+ &= i k_q^\Omega g_R^+ + i K_{RL} g_L^+ \, , \\ \partial_x g_L^+ &= i K_{RL} g_R^+ + i k_L g_L^+ \, , \end{aligned} \tag{47}$$

where K_{RL} and K_{LR} are the coupling coefficients of eq. (18) for the fundamental mode Rayleigh and Love waves. Since we only consider rightward propagating modes, indicated by the superscript + we suppress this superscript in the following.

The system (47) can also be written as

$$\partial_x \mathbf{g} = \mathbf{M}\mathbf{g} \, , \tag{48}$$

with

$$\mathbf{M} = \begin{pmatrix} k_R^\Omega & K_{RL} \\ K_{LR} & k_L \end{pmatrix} \quad \text{and} \quad \mathbf{g} = \begin{pmatrix} g_R \\ g_L \end{pmatrix} . \tag{49}$$

The condition that we excite a Rayleigh wave with unit strength at $x = 0$, but no Love wave, implies that we solve the system (48) subject to the initial condition

$$\mathbf{g}(x = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{50}$$

The system (48) is satisfied by solutions of the form

$$\mathbf{g} = \mathbf{v} e^{i\lambda x} \, , \tag{51}$$

where λ is an eigenvalue of $\mathbf{M}$ and $\mathbf{v}$ the corresponding eigenvector. The eigenvalues λ_+ and λ_- of $\mathbf{M}$ are given by

$$\lambda_\pm = \frac{1}{2}(k_R^\Omega + k_L) \pm \frac{1}{2}\sqrt{(k_R^\Omega - k_L)^2 + 4 K_{RL} K_{RL}} \, . \tag{52}$$

Because of the identity (21)

$$K_{RL} = K_{LR}^* \, , \tag{53}$$

we define

$$|K| = |K_{RL}| = |K_{LR}| \, . \tag{54}$$

With this definition the eigenvalues are given by

$$\lambda_\pm = \frac{1}{2}(k_R^\Omega + k_L) \pm \frac{1}{2}\sqrt{(k_R^\Omega - k_L)^2 + 4|K|^2} \, , \tag{55}$$

These eigenvalues have the physical dimension of wavenumber, and the corresponding phase velocities follow from $c_\pm = \omega/\lambda_\pm$. We show an example of the perturbation in the phase velocities of Rayleigh and Love waves in Fig. 2. The model and parameters we use are described in Appendix C. This model is a simplified model for the degeneracy of the spheroidal and toroidal modes in the Earth, because the dispersion branches of spheroidal and toroidal intersect at only one frequency, whereas the dispersion branches for spheroidal and toroidal modes in the real Earth have several nearby crossing points (Masters *et al.* 1983).

Because of selection rules for the modes on a spherical Earth, the Coriolis force couples spheroidal modes with angular degree l with toroidal modes of degree $l \pm 1$ (Dahlen & Tromp 1998). The coupling between these mode types thus involves an increment $\delta l = \pm 1$ in angular degree. As shown in Section 19.6 of Snieder & van Wijk (2015), spherical harmonics of order l span $l + 1/2$ wavelengths across the sphere, this is a consequence of the Bohr–Sommerfeld quantization rule, that is given by expression (70) of Dahlen & Henson (1985). A mode with angular order l thus has an equivalent wavenumber

$$k = \frac{l + 1/2}{a} \, , \tag{56}$$

where a denotes the Earth's radius. An increment $\delta l = \pm 1$ thus corresponds to a change in wavenumber $\delta k = 1/a$. Transforming from a spherical geometry to a Cartesian geometry corresponds to the limit $a \to \infty$. In that limit the selection rule $\delta l = \pm 1$ corresponds to $\delta k = 0$. This is consistent with the treatment of the simplified example where the degeneracy is determined by the condition that the wavenumbers of

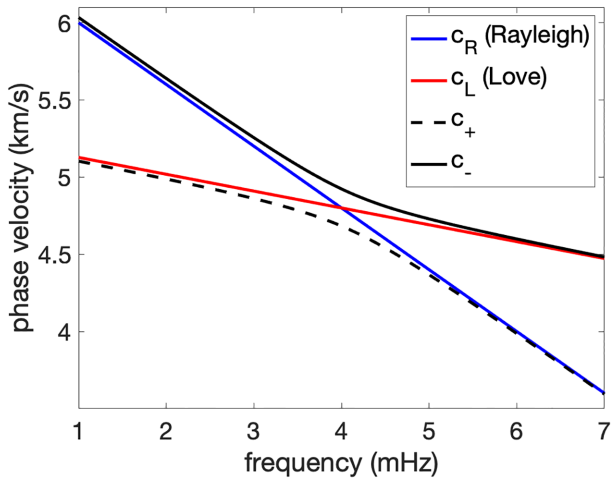

Figure 2. The phase velocity of the Rayleigh wave (blue line) and Love waves (red line) in a non-rotating system, and the phase velocities $c_\pm = \omega/\lambda_\pm$ that correspond to the eigenvalues $\lambda_\pm$ of expression (52).

the Love and Rayleigh waves are equal. Note that the Coriolis coupling of modes on a spherical Earth accounts for *global* mode coupling, while the coupling described in this paper describes the *local* coupling of propagating surface waves.

The used parameters are chosen to give insight in the behavior of the modes in a rotating system, but are not necessarily representative for the real Earth. The unperturbed phase velocities for Rayleigh and Love waves are chosen to be linear functions of frequency, and the phase velocities are equal at a frequency $f = 4$ mHz. The phase velocities for the modes on a rotating system follow from the eigenvalues $\lambda_\pm$ of eq. (52). Note that the difference in the perturbed phase velocities $c_\pm$ in Fig. 2 is larger than the difference in the phase velocities of the unperturbed Rayleigh and Love waves. This phenomenon is called *level repulsion* which is a feature of two-level systems (Frank & von Brentano 1994; Novotny 2010). For Earth's normal modes, there is a frequency repulsion between nearly degenerate spheroidal and toroidal modes due to Earth's rotation (Masters *et al.* 1983; Park & Gilbert 1986). A similar analysis as shown in this section has been formulated for normal modes by Rieger & Park (2014).

The phase velocity c_- (solid black line in Fig. 2) is for frequencies $f < 3$ mHz close to the phase velocity of the Rayleigh wave, while for frequencies $f > 5$ mHz it is close to the phase velocity of the Love wave. This raises the question, what is the nature of these modes in a Cartesian coordinate system for frequencies close to the degenerate frequency $f = 4\,mHz$? To answer this questions we analyse the eigenvectors of $\mathbf{M}$ defined in eq. (49) that correspond to the eigenvalues $\lambda_\pm$. These eigenvectors are given by

$$\mathbf{v}_+ = \begin{pmatrix} \frac{1}{2}(k_R^\Omega - k_L) + \frac{1}{2}\sqrt{(k_R^\Omega - k_L)^2 + 4|K|^2} \\ K_{LR} \end{pmatrix} , \tag{57}$$

and

$$\mathbf{v}_- = \begin{pmatrix} -K_{RL} \\ \frac{1}{2}(k_R^\Omega - k_L) + \frac{1}{2}\sqrt{(k_R^\Omega - k_L)^2 + 4|K|^2} \end{pmatrix} . \tag{58}$$

They are, of course, defined up to an arbitrary multiplicative constant. To address the question what the modes of the rotating system look like, we show the absolute value of the components of the normalized eigenvector $\mathbf{v}_-$ in Fig. 3. For frequencies $f < 3$ mHz the mode with phase velocity c_- consists mostly of a Rayleigh wave, but there is a small Love wave component that describes the small transverse motion that the Rayleigh wave has on a rotating system (Backus 1962). For frequencies $f > 5$ mHz the mode consists mostly of a Love wave with a small additional Rayleigh wave component. The latter component accounts for the radial and vertical motion that a Love wave acquires on a rotating system (Backus 1962). At the resonant frequency $f = 4$ mHz the Rayleigh wave and Love wave contributions are of equal magnitude. At resonance, the mode thus consists of an equal mix of Rayleigh wave and Love contributions, and the Rayleigh wave and Love wave are fully coupled, even when Earth is rotating slowly ($\Omega \ll \omega$). The same behavior for nearly degenerate spheroidal and toroidal modes has been described by Dahlen (1969), but his theory is more complicated than the theory we present because it involves the the coupling of many modes with different degree l and order m. In this section, we considered the resonant coupling between the fundamental mode Rayleigh and Love waves. In the Earth, one may also have resonant coupling that involve higher modes.

We next return to the problem where only a Rayleigh wave is excited at $x = 0$. The general solution $\mathbf{g}(x)$ is given by

$$\mathbf{g}(x) = A\mathbf{v}_+ e^{i\lambda_+ x} + B\mathbf{v}_- e^{i\lambda_- x} , \tag{59}$$

where the constants A and B are determined by the initial condition (50). Although these constants can be solved for, the resulting expressions are unwieldy and difficult to interpret. To gain more insight in the nature of the surface modes, and to illustrate how the wave field evolves

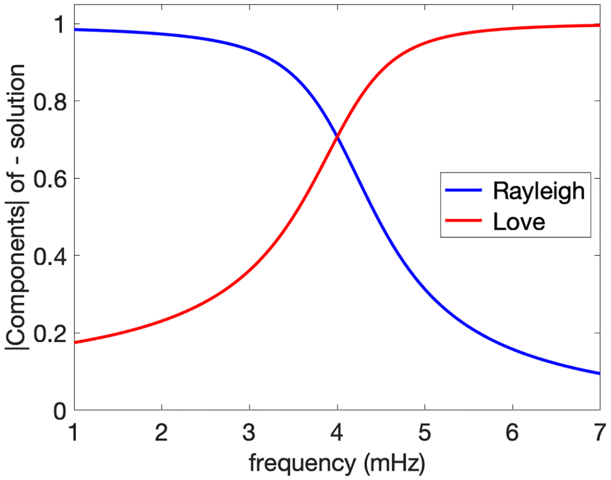

Figure 3. The absolute value of Rayleigh wave and Love wave components of the normalized eigenvector $\mathbf{v}_-$ defined in eq. (58).

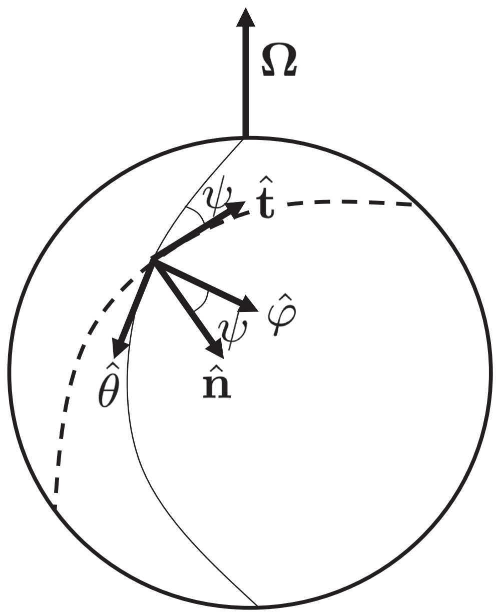

Figure 4. Geometry of variables in a spherical Earth. The thin lines shows a meridian, the dashed line the path of propagation. The unit vectors $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ are tangential and normal to the path, respectively, while the unit vectors $\hat{\theta}$ and $\hat{\varphi}$ point in the south and east directions, respectively.

when only a Rayleigh wave is launched, we treat two limiting cases in the next subsections; the case where the modes are far from degeneracy and the case where the modes are degenerate.

6.1 For modes far from degeneracy

We study in this section the special case where the Love and Rayleigh waves are far from degeneracy in the sense that

$$(k_R^\Omega - k_L)^2 \gg |K|^2 \ . \tag{60}$$

This means that the difference in wavenumbers is much larger than the coupling coefficient. In this regime the square-root in the expressions of the previous section can be replaced by

$$\sqrt{(k_R^\Omega - k_L)^2 + 4|K|^2} \approx (k_R^\Omega - k_L) + \frac{2|K|^2}{k_R^\Omega - k_L} \ , \tag{61}$$

and the eigenvalues given in expression (52) are

$$\lambda_+ = k_R^\Omega + \frac{|K|^2}{k_R^\Omega - k_L} \quad \text{and} \quad \lambda_- = k_L - \frac{|K|^2}{k_R^\Omega - k_L} \ . \tag{62}$$

The eigenvalues are thus close to the wavenumber of Rayleigh and Love waves, respectively, with a weak additional perturbation due to the coupling between these wave types. The corresponding eigenvectors follow from expressions (57) and (58) by using the approximation (61). Ignoring the small terms $|K|/(k_R^\Omega - k_L)$, the eigenvectors are given by

$$\mathbf{v}_+ = \begin{pmatrix} k_R^\Omega - k_L \\ \\ K_{LR} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_- = \begin{pmatrix} -K_{RL} \\ \\ k_R^\Omega - k_L \end{pmatrix} . \tag{63}$$

We use these eigenvalues and eigenvectors in eq. (59) and apply the initial condition (50) gives $A = 1/(k_R^\Omega - k_L)$ and $B = -K_{LR}/(k_R^\Omega - k_L)^2$. Inserting these values into eq. (59) and using the eigenvalues and eigenvectors of eqs (52) and (53) gives

$$\mathbf{g}(x) = \begin{pmatrix} 1 \\ \\ \dfrac{K_{LR}}{k_R^\Omega - k_L} \end{pmatrix} \exp i \left(k_R^\Omega + \frac{|K|^2}{k_R^\Omega - k_L} \right) x + \frac{K_{LR}}{k_R^\Omega - k_L} \begin{pmatrix} \dfrac{K_{RL}}{k_R^\Omega - k_L} \\ \\ -1 \end{pmatrix} \exp i \left(k_L - \frac{|K|^2}{k_R^\Omega - k_L} \right) x \ . \tag{64}$$

To interpret this expression we use eq. (37) to relate the solution $\mathbf{g}(x)$ to the displacement $\mathbf{w} = g_R \mathbf{w}_R^0 + g_L \mathbf{w}_L^0$, hence

$$\begin{aligned} \mathbf{w}(x) = & \left(\mathbf{w}_R^0 + \frac{K_{LR}}{k_R^\Omega - k_L} \mathbf{w}_L^0 \right) \exp i \left(k_R^\Omega + \frac{|K|^2}{k_R^\Omega - k_L} \right) x \\ & - \frac{K_{LR}}{k_R^\Omega - k_L} \left(\mathbf{w}_L^0 - \frac{K_{RL}}{k_R^\Omega - k_L} \mathbf{w}_R^0 \right) \exp i \left(k_L - \frac{|K|^2}{k_R^\Omega - k_L} \right) x \ , \end{aligned} \tag{65}$$

where $\mathbf{w}_R^0$ and $\mathbf{w}_L^0$ are the strain vectors of the modes for Rayleigh wave and Love wave modes, respectively, defined in eq. (8). The terms in the first line propagate with a wavenumber close to the wavenumber k_R^Ω of the Rayleigh waves, hence this is a quasi-Rayleigh wave. Note that this quasi-Rayleigh wave has a transverse component given by the term $\mathbf{w}_L^0$ in the first line. This transverse component in proportional to $K_{LR}/(k_R^\Omega - k_L)$, hence by virtue of the assumption (60) this transverse component is small. The second line in expression (65) describes a mode that propagates with a wavenumber close to the wavenumber k_L of Love waves, hence it is a quasi-Love wave. Note that the displacement in the second line has components in the radial and vertical directions, as described by the term $\mathbf{w}_R^0$. As for the quasi-Rayleigh wave this displacement is small by virtue of the term $K_{RL}/(k_R^\Omega - k_L)$. Note that the quasi-Love wave is weak because of the pre-factor $K_{LR}/(k_R^\Omega - k_L)$ is the second line. The reason that the quasi-Love wave is weak is that at $x = 0$ only a Rayleigh wave of strength 1 is excited, the excitation by quasi-Love wave is due to the Coriolis coupling only, not by the initial used initial conditions.

The transverse polarization of the quasi-Rayleigh wave is also predicted by the theory of Backus (1962). His theory does, however, not predict the presence of the quasi-Love wave because Backus (1962) assumed that the wave field propagates with one fixed wavenumber. That assumption precludes a conversion to modes with a different wavenumber.

The wavenumber of the quasi-Love wave appears in the exponent in the second line of expression (65) and is given by

$$\tilde{k}_L^\Omega = k_L - \frac{|K|^2}{k_R^\Omega - k_L} \quad \text{(quasi-Love waves)} \ . \tag{66}$$

The wavenumber of the quasi-Rayleigh wave is given in the exponent of the first line of expression (65). With the definition (31) this wavenumber is given by

$$\tilde{k}_R^\Omega = k_R + K_{RR} + \frac{|K|^2}{k_R^\Omega - k_L} \quad \text{(quasi-Rayleigh waves)} \ . \tag{67}$$

Note that the shift in the wavenumber of the quasi Rayleigh and Love waves is proportional to $\pm|K|^2 = \pm K_{RL} K_{LR}$. Since, according to expression (18), K_{RL} and K_{LR} depend on both the Rayleigh wave and Love wave eigenfunctions, the perturbation of the wavenumber of Rayleigh waves depends on the Love wave eigenfunctions, and the perturbation of the Love eigenfunction depends on the Rayleigh wave eigenfunctions.

The terms $\pm|K|^2/(k_R^\Omega - k_L)$ in expressions (66) and (67) increase the difference between the wavenumbers of the quasi Love and Rayleigh waves. This can be seen as follows. First suppose that $k_R^\Omega > k_L$. In that case $|K|^2/(k_R^\Omega - k_L) > 0$, hence according to expressions (66) and (67) the wavenumber of the quasi-Rayleigh wave increases and the wavenumber of the quasi-Love wave decreases, and because we assumed that $k_R^\Omega > k_L$ the difference in these wavenumbers increases. On the other hand, when $k_R^\Omega < k_L$, then $|K|^2/(k_R^\Omega - k_L) < 0$, and the wavenumber of the quasi-Rayleigh is reduced by the term $|K|^2/(k_R^\Omega - k_L)$ while the wavenumber of the quasi-Love wave increases. Because we assumed now that $k_R^\Omega < k_L$, the difference in the wavenumber also increases. So regardless of whether k_R^Ω is larger or smaller than k_L, the Coriolis coupling increases the difference between the wavenumbers of the quasi-Love and quasi-Rayleigh waves through the terms $\pm|K|^2/(k_R^\Omega - k_L)$. Since $k = \omega/c$, the increase in the difference between the wavenumbers of Love and Rayleigh waves corresponds to an increase of the difference between the phase velocities of these modes as well. The associated level-repulsion is shown in the numerical example of Fig. 2.

6.2 For degenerate modes

We next turn to the analysis of the expressions (55)–(58) for the case that the Rayleigh and Love waves are degenerate in the sense that

$$k_R^\Omega = k_L = k\ . \tag{68}$$

In this case the eigenvalues are given by

$$\lambda_\pm = k \pm |K|\ , \tag{69}$$

and the eigenvectors by

$$\mathbf{v}_+ = \begin{pmatrix} |K| \\ \\ K_{LR} \end{pmatrix}\quad , \quad \mathbf{v}_- = \begin{pmatrix} -K_{RL} \\ \\ |K| \end{pmatrix}\ . \tag{70}$$

Inserting the eigenvalues and eigenvectors into expression (59) and applying the initial condition (50) that only a Rayleigh wave is excited at $x = 0$, gives $A = 1/(2|K|)$ and $B = -K_{LR}/(2|K|^2)$. Inserting these results into eq. (59) gives

$$\mathbf{g}(x) = \begin{pmatrix} \cos(|K|x) \\ \\ i\sqrt{K_{LR}/K_{RL}}\sin(|K|x) \end{pmatrix} e^{ikx}\ . \tag{71}$$

Using expression (37) this solution corresponds to the displacement

$$\mathbf{w} = \cos(|K|x)\, e^{ikx}\mathbf{w}_R^0 + i\sqrt{\frac{K_{LR}}{K_{RL}}}\sin(|K|x)\, e^{ikx}\mathbf{w}_L^0\ , \tag{72}$$

where the modes $\mathbf{w}_R^0$ and $\mathbf{w}_L^0$ of Rayleigh and Love waves are defined in eq. (8).

Expression (72) states that at $x = 0$ only the Rayleigh wave is excited because the cosine is equal to 1 and the sine vanishes. However, at a distance $|K|x = \pi/2$ the cosine is equal to zero and the sine is equal to 1. This means that at this distance the displacement is given by the Love wave mode only. The solution (72) this describes a wave field that is a pure Rayleigh wave at distances for which $|K|x = n\pi$, with n an integer, while the wave field is a pure Love wave at distances for which $|K|x = (n + 1/2)\pi$. This transfer of energy between the Rayleigh and Love modes is similar to two identical pendulums that are coupled by a spring. At some times all the energy is in one pendulum and the other is at rest, while at other moments all the energy is in the other pendulum. A Foucault pendulum shows the same behavior (Pérez & Pujol 2015). Such a pendulum can be seen as two coupled identical modes, one that swings in the x-direction, and one that swings in the y-direction. The Foucault pendulum may be started oscillating in the x-direction, but after the while Earth has rotated and all the motion is in the y-direction. Rayleigh and Love waves that are in resonance and that are coupled by the Coriolis force show the same behavior of a transfer of energy between the two modes. The transfer of energy between spheroidal and toroidal motion has been observed in field data by Park (1990).

We analysed in Section 5 the approximate evolution of modes when one mode is excited at $x = 0$, and derived in expression (45) that a mode that is degenerate with the mode that is excited at $x = 0$ displays a secular growth in the sense that the modal coefficients are proportional to x. The Love wave coefficient g_L in the solution (71) is given by $g_L(x) = i\sqrt{K_{LR}/K_{RL}}\sin(|K|x)\, e^{ikx}$. A first-order Taylor expansion in x of this expression gives

$$g_L(x) = iK_{LR}x + O(x^2)\ , \tag{73}$$

which agrees with the secular growth of the solution (45). The secular growth thus is valid only for propagation distances that are sufficiently small ($|K|x \ll \pi/2$).

The Love wave in the solution (72) becomes only appreciable for distance $|K|x \approx \pi/4$. Since the Earth is attenuative, one may not be able observe the alternating conversions between resonant surface wave modes because the surface waves may have attenuated to the ambient noise levels at this propagation distance.

7 WKBJ SOLUTIONS ON A SPHERICAL EARTH

In this section we generalize the treatment of Section 5 where one mode is excited in a Cartesian geometry to a spherical Earth. According to expression (8.40) of Aki & Richards (2002) the following changes apply when moving from plane waves in a Cartesian coordinate system to the response to a point source on a spherical Earth.

(i)The amplitude of the waves varies due to geometrical spreading as $1/\sqrt{\sin\Delta}$, where Δ is the angular distance.
(ii)A point source, instead of a plane wave, gives a phase change $\exp(i\pi/4)$.
(iii)Each time the wave moves through a caustic at the source or its antipode, the wave field acquires a phase shift $\exp(-i\pi/2)$. When the wave passes through M caustics the phase shift is $\exp(-iM\pi/2)$ (Tromp 1994).

(iv)As the surface wave propagates along its path, the relative orientation of the rotation vector to the path changes.

We generalize the treatment of Section 5 to a spherical Earth in a qualitative way. The treatment of Section 5 is based on the propagation of plane waves, these solutions are not subject to geometrical spreading. Here we introduce amplitude variations due to geometrical spreading on a spherical Earth in an ad-hoc fashion. For brevity we only consider the case of non-degenerate modes.

In this section we ignore the coupling to backward propagating waves because, as we have shown in Section 5, this coupling is relatively weak. Hence we delete the superscripts + in the following. Instead of the distance x we use the angular distance Δ to the launching point of the incident mode as integration variable. This distance is given by $\Delta = x/a$, where a is the Earth's radius. Making this change of variables and ignoring the coupling to backward propagating waves, we modify eq. (36) to

$$\frac{\partial g_q}{\partial \Delta} = iak_q^{\Omega} g_q + \sum_{r \neq q} iK_{qr} a g_r \ . \tag{74}$$

As in eq. (38) we consider the case where only one mode, denoted with the index 0, is originally excited. Ignoring the coupling to other modes, the incident mode then satisfies expression (74) without the mode-coupling term

$$\frac{\partial g_0}{\partial \Delta} = iak_0^{\Omega}(\Delta) g_0 \ . \tag{75}$$

Note that the orientation of the rotation vector relative to the curved path on the sphere changes with location, hence the wavenumber $k_0^{\Omega}(\Delta)$ also changes with location. We show in appendix D that for a point with co-latitude θ, longitude φ, and azimuth of the path of propagation ψ, the components of the rotation vector are given by

$$\begin{aligned} \Omega_x &= \Omega \cos\psi \sin\theta \ , \\ \Omega_y &= -\Omega \sin\psi \sin\theta \ , \\ \Omega_x &= -\Omega \cos\theta \ , \end{aligned} \tag{76}$$

where the azimuth ψ is measured clockwise from north. These components are to be used in the coupling coefficients of expression (18), and by virtue of eq. (29), in the wavenumbers of the surface waves.

Eq. (75) can be integrated to give

$$g_0(\Delta) = A \exp\left(i \int_0^{\Delta} ak_0^{\Omega} d\Delta' \right) \ , \tag{77}$$

where A is an integration constant. This equation does, however, not account for amplitude variations on the sphere that are caused by the divergence or convergence of rays on a spherical Earth. Hence we generalize eq. (77) in an ad-hoc way to

$$g_0(\Delta) = A(\Delta) \exp\left(i \int_0^{\Delta} ak_0^{\Omega} d\Delta' \right) e^{i\gamma} \ , \tag{78}$$

where $A(\Delta)$ denotes amplitude variations along the propagation path. The factor $e^{i\gamma}$ accounts for the phase shift caused by point source and the number of times the surface wave has traversed a caustic at the source position or its antipode. The phase γ is given by Tromp (1994)

$$\gamma = \frac{\pi}{4} - \frac{M\pi}{2} \ . \tag{79}$$

On a laterally homogeneous spherical Earth, the amplitude of a surface wave caused by point source is given by

$$A(\Delta) = \frac{1}{\sqrt{\sin \Delta}} \ , \tag{80}$$

but for the moment we leave $A(\Delta)$ arbitrary.

For the conversions to another mode q we use, as in Section 5, the unperturbed solution (77) for the mode coupling in the right-hand side of eq. (74) and ignore the coupling to other converted modes, so that g_q satisfies

$$\frac{\partial g_q}{\partial \Delta} = iak_q^{\Omega} g_q(\Delta) + iaK_{q0} g_0(\Delta) \ . \tag{81}$$

We next use the substitution

$$g_q(\Delta) = \exp\left(i \int_0^{\Delta} ak_q^{\Omega} d\Delta' \right) f_q(\Delta) \ , \tag{82}$$

and insert the solution (77) for $g_0(\Delta)$ into expression (81), so that $f_q(\Delta)$ satisfies

$$\frac{\partial f_q}{\partial \Delta} = iB_q(\Delta) \exp\left(i \int_0^{\Delta} a(k_0^{\Omega} - k_q^{\Omega}) d\Delta' \right) \ , \tag{83}$$

with

$$B_q(\Delta) = aK_{q0}(\Delta) A(\Delta) e^{i\gamma} \ . \tag{84}$$

It should be remembered that K_{q0}, A, and the wavenumbers k_0^Ω and k_q^Ω depend on the location Δ along the propagation path. At the source, mode q is not excited, hence $g_q(\Delta = 0) = 0$, and hence $f_q(\Delta = 0) = 0$. Integrating expression (83) with this initial condition gives

$$f_q(\Delta) = i \int_0^\Delta B_q(\Delta'') \exp\left(i \int_0^{\Delta''} a(k_0^\Omega - k_q^\Omega) d\Delta' \right) d\Delta'' \; . \tag{85}$$

We first consider the case where $B(\Delta)$ is a smooth function on the scale of the wavenumbers considered. By taking the derivative of the exponential, expression (85) can be written as

$$f_q(\Delta) = \int_0^\Delta \frac{B_q(\Delta'')}{a(k_0^\Omega - k_q^\Omega)} \frac{d}{d\Delta''} \exp\left(i \int_0^{\Delta''} a(k_0^\Omega - k_q^\Omega) d\Delta' \right) d\Delta'' \; . \tag{86}$$

Applying integration by parts gives

$$f_q(\Delta) = \left[\frac{B_q(\Delta'')}{a(k_0^\Omega - k_q^\Omega)} \exp\left(i \int_0^{\Delta''} a(k_0^\Omega - k_q^\Omega) d\Delta' \right) \right]_{\Delta''=0}^{\Delta''=\Delta}$$
$$- \int_0^\Delta \frac{d}{d\Delta''} \left(\frac{B_q(\Delta'')}{a(k_0^\Omega - k_q^\Omega)} \right) \exp\left(i \int_0^{\Delta''} a(k_0^\Omega - k_q^\Omega) d\Delta' \right) d\Delta'' \; . \tag{87}$$

The wavenumbers k_0^Ω and k_q^Ω vary smoothly with position, and when the amplitude $A(\Delta)$, and hence $B_q(\Delta)$ vary smoothly with position as well, the last term in expression (87) is small compared to the first term. We thus ignore the last term, and using expressions (82) and (84) the excitation of the converted wave is given by

$$g_q(\Delta) = \left\{ \frac{A(\Delta) K_{q0}(\Delta)}{(k_0^\Omega - k_q^\Omega)(\Delta)} \exp\left(i \int_0^\Delta a k_0^\Omega d\Delta' \right) - \frac{A(0) K_{q0}(0)}{(k_0^\Omega - k_q^\Omega)(0)} \exp\left(i \int_0^\Delta a k_q^\Omega d\Delta' \right) \right\} e^{i\gamma} \; . \tag{88}$$

The equation generalizes expression (42) in the WKBJ approximation to a spherical Earth. As discussed under expression (44), in the Cartesian geometry considered in Section 5 and in expression (88) the converted wave that propagates in the forward direction consists of two contributions: a change in displacement of the unconverted wave with a contribution $\mathbf{w}_q^0$ that propagates with wavenumber k_0^Ω, and a converted mode that propagates with wavenumber k_q^Ω.

The derivation of the solution (88) was based on the premise that B_q is a smooth function of distance, and hence that the amplitude $A(\Delta)$ is a smooth function of distance. This condition is not satisfied when the surface wave propagates through the original source position ($\Delta = 2n\pi$), or its antipode ($\Delta = (2n+1)\pi$). In both situations $\sin\Delta = 0$ and according to expression (80) the amplitude A, and hence B_q, has a singularity. Using expressions (82), (84), and (85), the coefficient of the converted wave can be written as

$$g_q(\Delta) = i \int_0^\Delta a K_{q0}(\Delta'') A(\Delta'') \exp\left(i \int_0^{\Delta''} a k_0^\Omega d\Delta' + i \int_{\Delta''}^{\Delta} a k_q^\Omega d\Delta' \right) d\Delta'' e^{i\gamma} \; . \tag{89}$$

This expression describes a wave that propagate as the original mode with wavenumber k_0^Ω from the source ($\Delta = 0$) to the location Δ'', where it is converted to mode q. The remainder of the path, from location Δ'' to the endpoint Δ, is covered with the wavenumber k_q^Ω of the converted mode. The integration over Δ'' indicates that the mode conversion occurs everywhere along the path. The contribution from a point Δ'' is proportional to $K_{q0}(\Delta'')A(\Delta'')$. The first term gives the conversion coefficient at each point, the second term account for the amplitude for the original wave at that point. When the path traverses a caustic, hence at the original source location or its antipode, $A(\Delta'')$ is infinite. The integration over Δ'' will thus be dominated by the caustics along the path; these points appear as excitation points for the converted waves as we show in Sens-Schönfelder *et al.* (2020).

8 DISCUSSION

The theory presented here takes into account the rotational coupling of different surface wave modes, but it also accounts for the imprint of rotation on each surface wave mode in the same way as in the theory of Backus (1962). Specifically, the wavenumber of Love waves is not perturbed, expression (30), while the perturbation of the wavenumber of Rayleigh waves in eq. (33) agrees with the perturbation derived by Backus (1962). As shown in expression (20), rotation does not couple Love waves to Love waves. As shown in eq. (65) rotation causes the Rayleigh waves to have transverse component while the Love wave motion acquires radial and vertical components (Backus 1962). Our theory thus accounts for *mode coupling*. In addition, our theory describes *mode conversion* of surface wave modes by rotation.

Expression (18) predicts that rotation does not couple Love waves to Love waves, while expression (30) states that the wavenumber of Love waves is not perturbed by rotation. This seems to contradict the fact that rotation causes a splitting of toroidal modes (Backus & Gilbert 1961; Dahlen 1968). In their expression (42), Backus & Gilbert (1961) show that after summing over angular degrees m, a toroidal multiplet of angular order l moves westward with an angular velocity $\dot{\varphi} = \Omega/l(l+1)$. Denoting the radius of the Earth by a, this corresponds to a westward phase velocity perturbation

$$\delta c = a\dot{\varphi} = a\Omega/l(l+1) \; . \tag{90}$$

For large angular orders $l(l+1) \approx (l+1/2)^2 = k^2a^2 = \omega^2 a^2/c^2$, where expression (56) is used in the second identity. Inserting this in expression (90) gives

$$\delta c = \Omega c^2/\omega^2 a\ . \tag{91}$$

The Cartesian coordinate system used in this work can be seen as a spherical coordinate system in the limit $a \to \infty$. In this limit the phase velocity perturbation of toroidal modes due to rotation vanishes: $\delta c \to 0$. This shows that the phase velocity perturbation of toroidal modes due to rotation is not just due to rotation; it depends on the combination of Earth's rotation and the sphericity of the Earth.

According to expression (46) the rotational coupling is stronger for surface wave modes propagating in the same direction than for surface wave modes propagating in opposite directions, and according to eq. (44) the coupling is strongest when the surface wave modes have near-equal wavenumbers. The rotational coupling between Love and Rayleigh is of a different nature depending whether the phase velocities of the modes are different or the same (degeneracy).

When the wavenumbers are different, in the sense of the inequality (60), the Love and Rayleigh wave modes propagate with their own phase velocity, but their polarizations are mixed. In this case the rotation causes the difference in the phase velocities to increase, see Fig. 2, through a process called level-repulsion, which is a general feature of coupled two-level systems (Park & Gilbert 1986; Frank & von Brentano 1994; Novotny 2010).

For degenerate Rayleigh and Love waves the situation is completely different. According to eq. (72) the Rayleigh wave motion is after a sufficiently long propagation converted completely to Love wave motion and vice versa. This situation is analogous to two identical pendulums coupled by a spring in which the motion is periodically transferred from one pendulum to the other or to a Foucault pendulum in which the oscillation is transferred from one direction to the perpendicular direction and back. The periodic transfer of motion between Rayleigh waves and Love waves in eq. (72) describes the same behavior. In practice, the propagation distances needed to observe this behavior is likely to be so large that attenuation damps the surface wave before this behavior is observed. We show in a companion paper (Sens-Schönfelder *et al.* 2020) that at the arrival time of the R2 Rayleigh wave with a period of about 250 s, the transverse motion is about 10 per cent of the motion of a Rayleigh wave.

We stress that the treatment of rotational coupling of surface waves on a spherical Earth in Section 7 is heuristic. The main difference between the treatment of Earth's rotation in a Cartesian coordinate system and on the sphere is that on a sphere geometrical spreading causes the amplitude of the surface wave modes to vary and that the orientation of the rotation vector changes relative to the direction of wave propagation on a sphere. The change in the relative orientation of the rotation vector on the sphere can be accounted for with a WKBJ approach as suggested earlier (Backus 1962; Tromp 1994). We account for the variation in geometrical spreading in a heuristic way in expression (89). In this approximation the conversion of surface wave modes at a point on the sphere depends on the product of the mode-coupling coefficient at that point and the amplitude of the mode that is being coupled. A combination of our theory with the treatment of Tromp (1994) is needed to more accurately describe surface mode conversion on a spherical Earth.

In part 2 of this work (Sens-Schönfelder *et al.* 2020) we use numerical modeling on a spherical Earth with and without rotation to study the strength of the rotational coupling of surface waves and to test the predictions made in this paper.

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APPENDIX A: DERIVATION OF THE NORMALIZATION CONDITION (11)

The normalization condition (2.8) of Kennett (1984) when applied to the same mode is given by

$$i\int_0^\infty \left(\mathbf{w}^0\cdot\bar{\mathbf{t}}^0-\mathbf{t}^0\cdot\bar{\mathbf{w}}^0\right)\mathrm{d}z=1\,. \tag{A1}$$

For brevity we leave out the mode index, but this relation holds for every mode separately. We express the normalization condition in the notation of the modes and integrals of modes defined in sections 7.2 and 7.3 of Aki & Richards (2002). The displacement modes are given in expressions (7) and (8). The Love wave traction is in the notation of Aki & Richards (2002) given by

$$\sigma_{xy}=ik\mu u_x=ik\mu l_1\,. \tag{A2}$$

The Rayleigh wave tractions follow from

$$\begin{aligned}\sigma_{xx}&=ik(\lambda+2\mu)u_x+\lambda\partial_z u_z=ik(\lambda+2\mu)r_1+i\lambda\partial_z r_2\,,\\ \sigma_{xz}&=ik\mu u_z+\mu\partial_z u_x=-k\mu r_2+\mu\partial_z r_1\,.\end{aligned} \tag{A3}$$

This means that the traction vector of expression (1) is in the notation of Aki & Richards (2002) given by

$$\mathbf{t}^0=\begin{pmatrix}0\\ ik\mu l_1\\ 0\end{pmatrix}\ \text{(Love Waves)},\quad \mathbf{t}^0=\begin{pmatrix}ik(\lambda+2\mu)r_1+i\lambda\partial_z r_2\\ 0\\ -k\mu r_2+\mu\partial_z r_1\end{pmatrix}\ \text{(Rayleigh Waves)}\,, \tag{A4}$$

and using the reasoning under expression (7), which amounts to replacing $k\to -k$ and $r_2\to -r_2$ when going from $\mathbf{t}^0$ to $\bar{\mathbf{t}}^0$, the traction vector $\bar{\mathbf{t}}^0$ is given by

$$\bar{\mathbf{t}}^0=\begin{pmatrix}0\\ -ik\mu l_1\\ 0\end{pmatrix}\ \text{(Love Waves)},\quad \bar{\mathbf{t}}^0=\begin{pmatrix}-ik(\lambda+2\mu)r_1-i\lambda\partial_z r_2\\ 0\\ -k\mu r_2+\mu\partial_z r_1\end{pmatrix}\ \text{(Rayleigh Waves)}\,. \tag{A5}$$

Using these expressions for the traction vector, and expressions (7) and (8) for the displacement, reduces the normalization condition (A1) for Love waves to

$$2k\int_0^\infty \mu l_1^2\mathrm{d}z=1\,. \tag{A6}$$

In the notation of expression (7.66) of Aki & Richards (2002), eq. (A6) can be written as $4kI_2=1$. According to eq. (7.70) of Aki & Richards (2002), $I_2=cUI_1$, with U the group velocity and I_1 given by eq. (11). Using the relation $kc=\omega$, the normalization condition for Love waves is thus given by expression (10).

For Rayleigh waves the normalization condition (A1) reduces with expressions (7), (8), (A4) and (A5) to

$$2k\int_0^\infty\left((\lambda+2\mu)r_1^2+\mu r_2^2\right)\mathrm{d}z+2\int_0^\infty\left(\lambda r_1\partial_z r_2-\mu r_2\partial_z r_1\right)\mathrm{d}z=1\,. \tag{A7}$$

In the notation of expression (7.74) of Aki & Richards (2002) this condition can be written as

$$4kI_2+2I_3=1\,. \tag{A8}$$

According to eq. (7.76) of Aki & Richards (2002), $I_2+I_3/2k=cUI_1$, with I_1 defined in eq. (11). This means that the normalization condition (A8) for Rayleigh waves is also given by expression (10).

APPENDIX B: DERIVATION OF THE IDENTITY (24)

The coupling matrices $\overline{K}_{qr}$ and $\overline{L}_{qr}$ follow from K_{qr} and L_{qr} by interchanging the modes $\overline{\mathbf{w}}^0$ and $\mathbf{w}^0$ (Kennett 1984). Thus, with expression (6)

$$\overline{K}_{qr} = -\int_0^\infty \mathbf{w}_q^0 \cdot \Delta\mathbf{A}_{tw} \cdot \overline{\mathbf{w}}_r^0 dz \ . \tag{B1}$$

Using the identity (9), K_{qr}, from expression (6) and $\overline{K}_{qr}$ are given by

$$K_{qr} = -\int_0^\infty \mathbf{w}_q^{0*} \cdot \Delta\mathbf{A}_{tw} \cdot \mathbf{w}_r^0 dz \ , \tag{B2}$$

and

$$\overline{K}_{qr} = -\int_0^\infty \mathbf{w}_q^0 \cdot \Delta\mathbf{A}_{tw} \cdot \mathbf{w}_r^{0*} dz \ . \tag{B3}$$

According to expression (17) the matrix $\Delta\mathbf{A}_{tw}$ is imaginary, hence $\Delta\mathbf{A}_{tw}^* = -\Delta\mathbf{A}_{tw}$. Using this in the complex conjugate of eq. (B2) shows that $\overline{K}_{qr} = K_{qr}^*$. The same reasoning applies to the coupling matrix $\overline{L}_{qr}$, hence the identity (24) holds.

APPENDIX C: THE MODEL USED FOR THE FIGS 2 AND 3

We assume that the phase velocity c_R and c_L for Rayleigh waves and Love waves are both linear functions of angular frequency. At a degenerate angular frequency ω_d the phase velocities have a common value c_d. The derivative of the phase velocity for each mode is given by $\partial c/\partial\omega = c^2(c^{-1} - U^{-1})/\omega$, where U is the group velocity. Assuming that this slope is constant, and is given by its value at the degenerate frequency ω_d, the phase velocity is given by

$$c(\omega) = c_d \left(1 + \left(1 - \frac{c_d}{U_d}\right)\left(\frac{\omega - \omega_d}{\omega_d}\right)\right) \ , \tag{C1}$$

where U_d is the group velocity at the resonant frequency. In the model we used $c_d = 4.8\ \mathrm{km\,s^{-1}}$, while for the Rayleigh wave $U_d = 3.6\ \mathrm{km\,s^{-1}}$ and for the Love wave $U_d = 4.4\ \mathrm{km\,s^{-1}}$.

We assume that the coupling coefficient K_{RL} is real. According to expression (18) this is the case when $\Omega_z = 0$. According to eq. (18) the coupling coefficient has a pre-factor ω, hence we use that

$$K_{RL}(\omega) = \gamma k_d \left(\frac{\omega}{\omega_d}\right) \ , \tag{C2}$$

where $k_d = \omega_d/c_d$ is the wavenumber at the resonant frequency, and γ is a dimensionless constant. In the example we used the numerical value $\gamma = 0.025$.

APPENDIX D: DERIVATION OF THE ROTATION VECTOR RELATIVE TO THE DIRECTION OF WAVE PROPAGATION ON A SPHERE

In this appendix, we use the coordinate system where the z-axis is aligned with the rotation vector, hence

$$\Omega = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \ . \tag{D1}$$

We determine the component of this rotation vector relative to a path passing a point on the sphere with colatitude θ, longitude φ and an azimuth ψ defined eastward from north, as shown in Fig. 4. As shown in that figure we define a unit vector $\hat{\mathbf{t}}$ along the path on the surface, a unit vector $\hat{\mathbf{n}}$ perpendicular to the path, and a unit vector $\hat{\mathbf{d}}$ that points downward. We also define the unit vectors $\hat{\theta}$, $\hat{\varphi}$ and $\hat{\mathbf{r}}$ that point southward, eastwards, and radially, respectively. Using the geometry of Fig. 4 it follows that these sets of unit vectors are related by

$$\begin{aligned}\hat{\mathbf{t}} &= -\cos\psi\ \hat{\theta} + \sin\psi\ \hat{\varphi} \ , \\ \hat{\mathbf{n}} &= \sin\psi\ \hat{\theta} + \cos\psi\ \hat{\varphi} \ , \\ \hat{\mathbf{d}} &= -\hat{\mathbf{r}} \ . \end{aligned} \tag{D2}$$

We seek to find the components of the rotation vector Ω along the unit vectors $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{d}}$. According to expression (D1) these components are given by

$$\begin{aligned}\Omega_t &\equiv (\Omega \cdot \hat{\mathbf{t}}) = \Omega\ t_z \ , \\ \Omega_n &\equiv (\Omega \cdot \hat{\mathbf{n}}) = \Omega\ n_z \ , \\ \Omega_d &\equiv (\Omega \cdot \hat{\mathbf{d}}) = \Omega\ d_z \ . \end{aligned} \tag{D3}$$

We thus only need the z-components of the unit vectors $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{d}}$. Using expression (D2) and expressions (4.7) of Snieder & van Wijk (2015) for $\hat{\theta}$, $\hat{\varphi}$ and $\hat{\mathbf{r}}$ gives

$$\begin{aligned} \Omega_t &= \Omega \cos\psi \sin\theta \ , \\ \Omega_n &= -\Omega \sin\psi \sin\theta \ , \\ \Omega_d &= -\Omega \cos\theta \ . \end{aligned} \tag{D4}$$

The component Ω_t is the projection of the rotation vector along the path of propagation, this is the x-direction in the main text. Similarly, Ω_n corresponds to the component of Ω in the y-direction of the main text, and Ω_d corresponds to the vertical component Ω_z of the main text. Eq. (D4) thus corresponds to expression (76).