DISTORTION ANALYSIS OF MAGNETOTELLURIC DATA CONSIDERING THE ERRORS OF THE MEASURED IMPEDANCE TENSOR.

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SUMMARY

With an a priori distribution of the tensor element confidence limits the distribution of any impedance based parameter can be determined by using the Jacobi-matrix transformation of random variables. Independently, it is well known that the linear error propagation of the parameter - function of the tensor elements - will be true if these are independent variables with small relative errors.

In the present work, I derived the probability distribution of the regional skew parameter of Bahr (1991), as it is a continuos and differentiable function of the impedance tensor elements.

A Gaussian distribution with a 68% confidence limit is assumed for the spectral analysis of MT data measured in the Bolivian Altiplano after processing with Egbert's (1998) code with a remote reference station.

For MT sites located in the Bolivian Altiplano, the skew parameter with its confidence limit is compared with its respective linear propagated errors. Also, the regional strike angles of Bahr with their respective linear propagated errors are compared with those from Chave's (1993) decomposition code, giving the strike of the possible two-dimensional structure.

I. INTRODUCTION.

- When determining a parameter function of the impedance tensor, its respective error is normally not taken into account.
- One specific example treated here is the regional skew parameter defined by Bahr (1991) which estimate the 2-D impedance phase deviation.
- I derive the confidence limit of the regional skew by expressing its distribution function in terms of the tensor element $\{Z_{ij}\}$ density functions using the Jacobi-matrix transformation of random variables.
- The results are tested with measured data from the Ancorp-profile of the Bolivian Altiplano assuming that the tensor elements have uncorrelated errors and that they are Gaussian distributed.
- The hypothesis will fail with significant correlation of the tensor element distributions and/or with few number of sample data recorded at the respective frequency range. It is actually possible to obtain the probability distribution of the skew parameter when the covariance matrix of the impedance tensor is known. The function will be expressed as a complicated system of integral complex equations, but it will be possible to resolve them numerically.

II. DERIVATION OF THE PROBABILITY DISTRIBUTION OF THE REGIONAL SKEW PARAMETER

Regional skew (η) is a continuous function of the tensor elements and continuous differentiable. It has the following expression:

$$\eta = \frac{\sqrt{2|x_1 x_7 - x_4 x_6 + x_2 x_8 - x_3 x_5|}}{\sqrt{(x_2 - x_3)^2 + (x_6 - x_7)^2}}$$

where,

$$x_1 = \mathrm{Re}\{Z_{xx}\} \quad x_2 = \mathrm{Re}\{Z_{xy}\} \quad x_3 = \mathrm{Re}\{Z_{yx}\} \quad x_4 = \mathrm{Re}\{Z_{yy}\}$$
$$x_5 = \mathrm{Im}\{Z_{xx}\} \quad x_6 = \mathrm{Im}\{Z_{xy}\} \quad x_7 = \mathrm{Im}\{Z_{yx}\} \quad x_8 = \mathrm{Im}\{Z_{yy}\}$$

and the variables correspond to the real and imaginary part of the impedance tensor elements defined in magnetollurics:

$$Z = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}$$

η is rotationally invariant and in case of two dimensionality it is reduced to its minimum value 0, which means that each pair Z_{xx},Z_{yx} and Z_{xy},Z_{yy} has equally phases, respectively.

Assuming a known density function distribution f for the tensor elements –defined as random variables-, we can derive the function distribution of η in terms of Z using the Jacobi-matrix. The transformation is valid as η is continuously differentiable in Z.

The density function g of η has thus the form:

$$g(\eta) = |\det[J(\vec{x}/\eta)]| \cdot f(\vec{x}) \qquad \vec{x} = (x_1, .., x_8)$$

The space transformation will be determined by inverting one of the $\vec{x}$'s element into the space of η:

$$X_p(x) = \begin{pmatrix} x_1 \\ \vdots \\ x_{p-1} \\ x_p(t) \\ \vdots \end{pmatrix} \xrightarrow{T} t = \begin{pmatrix} x_1 \\ \vdots \\ x_{p-1} \\ \eta \\ \vdots \end{pmatrix}$$

in this way the Jacobi-matrix will be expressed as:

$$[J] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 0 \\ & & & 1 & \vdots \\ \frac{\partial x_p}{\partial x_1} & \frac{\partial x_p}{\partial x_2} & \cdots & \frac{\partial x_p}{\partial x_{p-1}} & \frac{\partial x_p}{\partial \eta} \end{bmatrix}$$

Because it is a diagonal matrix on the first (p-1) rows, its determinant reduces to a simple form:

$$\det[J] = \left|\frac{\partial x_p}{\partial \eta}\right|$$

It follows to determine the proper x_p variable for a valid transformation of spaces in order to assert the equivalence of density function integration between the $\overrightarrow{x_p}$ and the $\vec{t}$ vector spaces. Thus the variable function x_p should accomplish the following conditions:

- $\frac{\partial x_p}{\partial \eta}$ is always positive which means that x_p is strictly monotonous with respect to η, i.e., $\eta :\rightarrow x_p(\eta_a) < x_p(\eta_b)$ with $\eta_a < \eta_b$.

- $\frac{\partial x_p}{\partial \eta} = \left(\frac{\partial \eta}{\partial x_p}\right)^{-1}$

$\Rightarrow$ This is true for x_1, x_4, x_5 and x_8 having partial derivatives of the form:

$$\left|\frac{\partial x_p}{\partial \eta}\right| = \left|\frac{\eta \cdot \left[(x_2 - x_3)^2 - (x_6 - x_7)^2\right]}{x_i}\right| \qquad (p,i) = (1,7),(4,6),(5,3),(8,2)$$

In consequence, choosing x_p will be arbitrary in the transformation because of the symmetry observed in $\frac{\partial x_p}{\partial \eta}$. Note that these variables are the diagonal impedance elements Z_{xx}, Z_{yy} (see Fig. 1), meanwhile η encounters a minimal & maximal local extreme in the anti-diagonal elements Z_{xy}, Z_{yx} (Fig. 2).

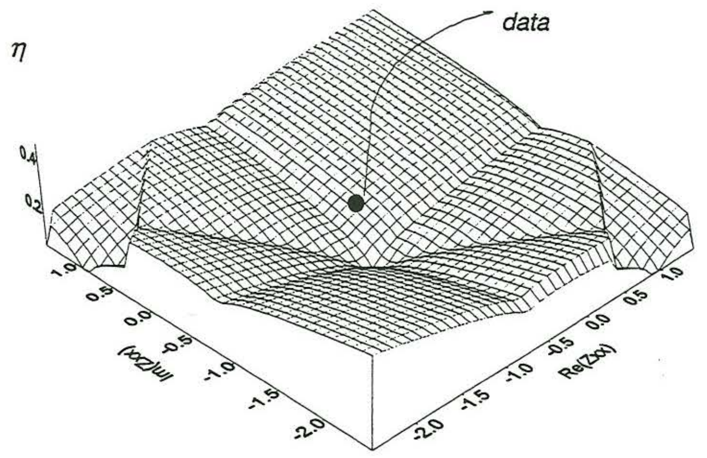

Fig. 1.- Regional skew parameter (η) as function of the real and imaginary part of the tensor element Z_{xx}, while the other tensor elements were kept fixed. The parameter has no extreme minimum neither maximum. This is also true for the other diagonal tensor element Z_{yy}.

The symmetrical shape of the curve indicates the arbitrarly Jacobi-transformation by using any of these 4 elements.

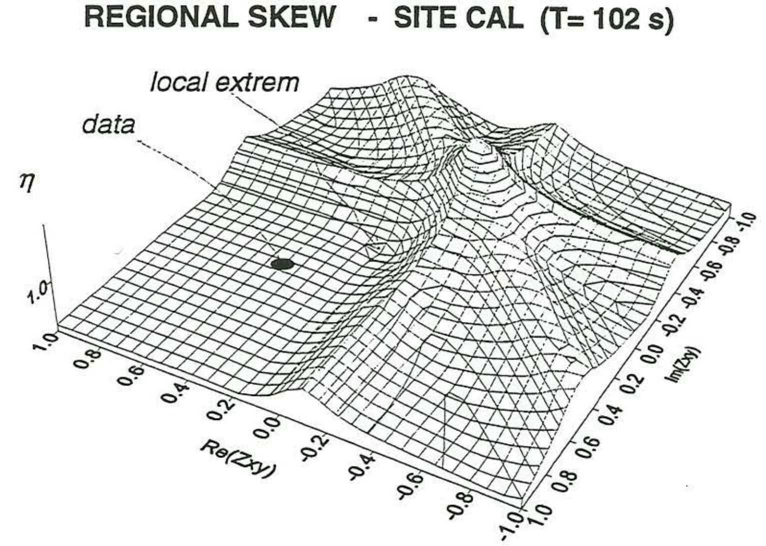

Fig. 2.- Regional skew parameter (η) as function of the real and imaginary part of the tensor element Z_{xy}, while the other tensor elements were kept fixed. The parameter has local positive & negative extreme. This is also true for the other anti-diagonal tensor element Z_{yx}. These 4 elements cannot be used in the Jacobi-transformation (see text).

After the previous analysis, we can express the probability distribution of η as function of f -the probability density function of $[Z]$- as following:

$$G(\eta_o) = \int_{-\infty}^{\eta_o} g(\eta) d\eta = \int_{-\infty}^{\eta_o} \cdot \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \frac{\partial x_p(\eta, x_{p-1},..., x_1)}{\partial \eta} \right| \cdot f(x_p, x_{p-1},..., x_1) dx_{p-1} dx_{p-2} \cdots dx_1 \right\} d\eta$$

$$= \int_{-\infty}^{x_p(\eta)} \underbrace{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_p, x_{p-1},..., x_1) dx_{p-1} dx_{p-2} \cdots dx_1 \right)}_{(a)} dx_p = \int_{-\infty}^{x_p(\eta)} g(\eta(x_p)) \cdot dx_p \qquad (*)$$

This means, the function $G(\eta_o) = P(\eta < \eta_o)$ can be expressed directly in terms of x_p, and thus the system is considerably simplified. But it is conditioned to the other tensor elements, which means to have a priori information of their expected values.

III. NORMAL DISTRIBUTION OF UNCORRELATED DATA

In case of normal distribution for the random variables $[Z]$ and assuming the simplest case of independence within the tensor elements, then the statistical density function f will be simplified to a factorisation of 8 Gaussian density functions:

$$f(\vec{x}) = \phi(x_1, u_1, \sigma_1) \cdot \phi(x_2, u_2, \sigma_2) \cdots \phi(x_8, u_8, \sigma_8)$$

where $u_1, \ldots, u_8$ are the respective expected values of $[Z]$ and $\sigma_1^2, \ldots, \sigma_8^2$ the variances. In our case, the expected values are the measured Z_{ij} data with known σ^2 variances.

Then the integration part (a) written on equation (*) will be reduced to:

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_p, x_{p-1},\ldots,x_1)dx_{p-1}dx_{p-2}\cdots dx_1\right) = \int_{-\infty}^{\infty}\phi(x_1)dx_1\int_{-\infty}^{\infty}\phi(x_2)dx_2\cdots\int_{-\infty}^{\infty}\phi(x_{p-1})dx_{p-1} = 1$$

Resulting in:

$$G(\eta_o) = P(\eta < \eta_o) = \int_{-\infty}^{x_p(\eta_o)}\phi(x_p)dx_p = \psi_o\left(\frac{x_p(\eta_o) - u_p}{\sigma_p}\right) \qquad (**)$$

where ψ_o is the well known standard distribution of variance 1 and mean 0.

The confidence limit $C.L.$ of η laying between a certain range $[\eta_{a,},\eta_b]$ is defined:

$$P(\eta_a < \eta_o < \eta_b) = \psi_o\left(\frac{x_p(\eta_b) - u_p}{\sigma_p}\right) - \psi_o\left(\frac{x_p(\eta_a) - u_p}{\sigma_p}\right) = C.L. = 2\psi_o\left(\frac{x_p(\eta_b) - u_p}{\sigma_p}\right) - 1$$

because ψ_o is symmetrical, the desired confidence limit $C.L.$ will accomplish the relationship:

$$\frac{x_p(\eta_b) - u_p}{\sigma_p} = \frac{u_p - x_p(\eta_a)}{\sigma_p} \qquad \Rightarrow \qquad x_p(\eta_a) + x_p(\eta_b) = 2 \cdot u_p \ ,$$

and both limits are found for the desired confidence limit $C.L.$

$$x_p(\eta_b) = \psi_o^{-1}(\frac{C.L.+1}{2}.) \cdot \sigma_p + u_p \ \rightarrow \ x_p(\eta_a) = -\psi_o^{-1}(\frac{C.L.+1}{2}) \cdot \sigma_p + u_p$$

a) 95% CONFIDENCE LIMIT

It is chosen x_p as $x_1 = \mathrm{Re}\{Z_{xx}\}$. Using Eq.(**) the variable $x_1(\eta)$ takes the following form:

$$x_p(\eta, x_{p-1},\ldots) = x_1(\eta, \hat{x}_2,\ldots,\hat{x}_8) = \frac{\eta^2}{2\hat{x}_7}\cdot\left[(\hat{x}_2 - \hat{x}_3)^2 + (\hat{x}_6 - \hat{x}_7)^2\right] + \frac{(\hat{x}_4\hat{x}_6 - \hat{x}_2\hat{x}_8 + \hat{x}_3\hat{x}_5)}{\hat{x}_7}$$

where each $\hat{x}_i$ is the measured data considered to be the respective expected value.

We want to estimate the 95% confidence limit of η. It will be resolved by subtracting the standard distributions at the values of 0.975 and 0.025 respectively:

$\psi_o|_{0.975} - \psi_o|_{0.025}$

which can be obtained from any statistical table

$$\psi_o\left(\frac{x_i(\eta_b)-u_{xx}}{\sigma_{xx}}\right)=0.975 \Rightarrow \qquad \left(\frac{x_1(\eta_b)-u_{xx}}{\sigma_{xx}}\right)=1.96$$

where $u_{xx}=\mathrm{Re}\{\hat{Z}_{xx}\}$,i.e., the measured data and σ_{xx} its corresponding error.

IV. APPLICATION OF THE THEORY ON DATA MEASURED IN THE BOLIVIAN ALTIPLANO

The data were processed with the robust code of Egbert (1998) which includes a remote reference station. The spectral analysis is realised assuming Gaussian distribution with a 68% confidence limit. The code computes a covariance matrix for the MT tensor which is expected to be asymptotically Gaussian. Thus we assume a normal distribution on $[Z]$ and take the simplest case of independence within the tensor elements.

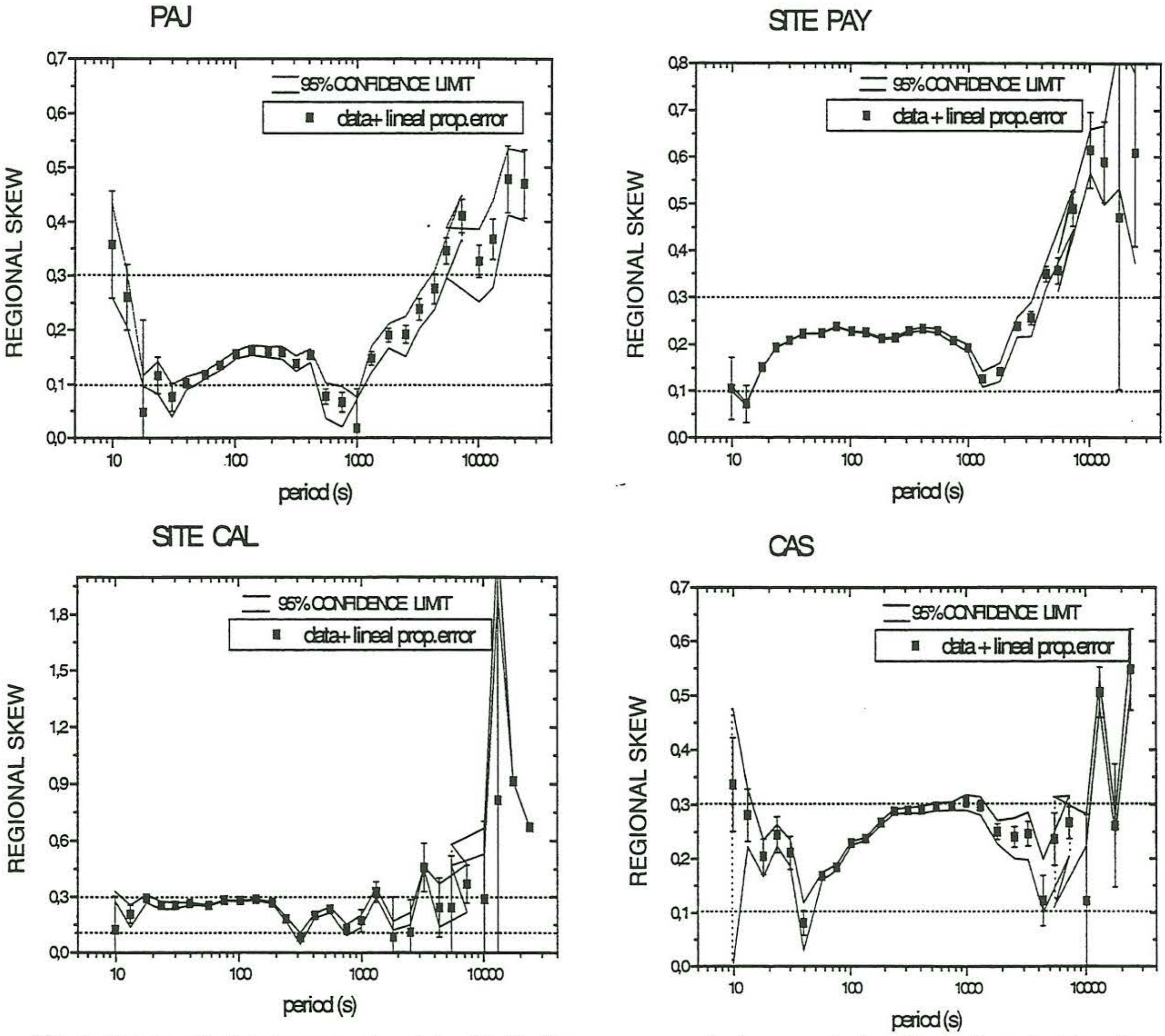

Fig.3- Regional skew parameter (η) with its linear propagated errors in function of period for 4 sites of the Ancorp-profile. Shown is the 95% confidence limit (see text). Observe the general agreement between the linear propagated errors and the 95% confidence limit, mainly at mid period bands.

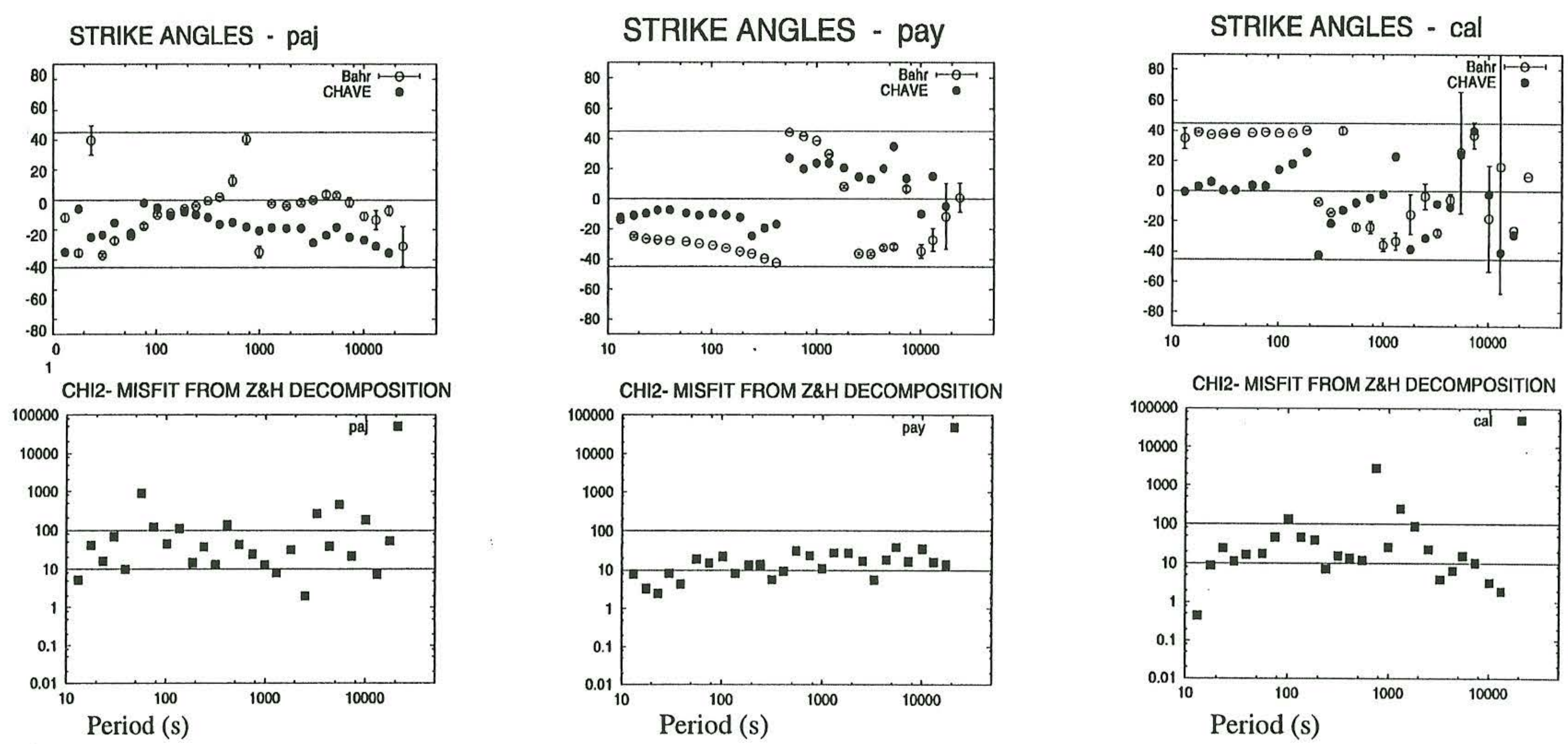

Fig. 4 .- Regional strike angle of Bahr (with its linear error) and the strike angle after Chave decomposition as function of the period for 3 sites of the Ancorp-profile. Below is shown the corresponding X^2-misfit of the Chave decomposition, which included also the magnetic transfer function.

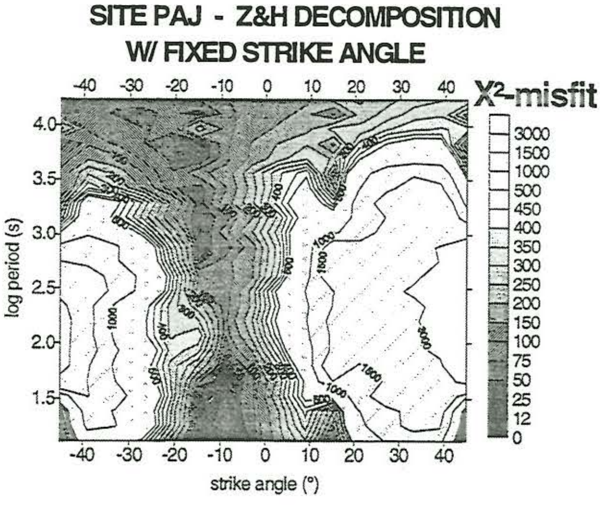

Fig.5.a)- X^2-misfit of Chave MT &magnetic tensor decomposition realised with fixed strike angles.

The minimum misfits are found for angles between -20° and 0°.

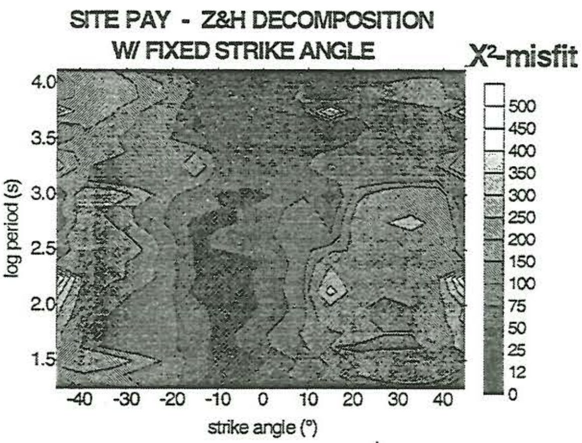

Fig.5.b).- X^2-misfit of Chave MT &magnetic tensor decomposition realised with fixed strike angles.

The minimum misfits along the period band are found at angles between -15°and 0° .

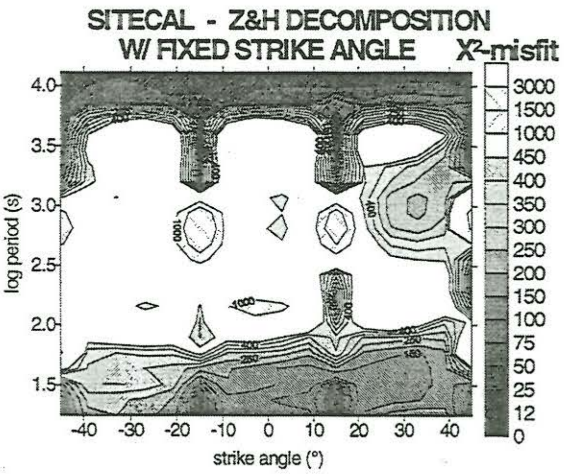

Fig. 5.c)- X^2-misfit of Chave MT &magnetic tensor decomposition realised with fixed strike angles.

No tendency of minimal misfit is observed along the period band for a certain angle range .

Real Induction Arrows - ANCORP profile

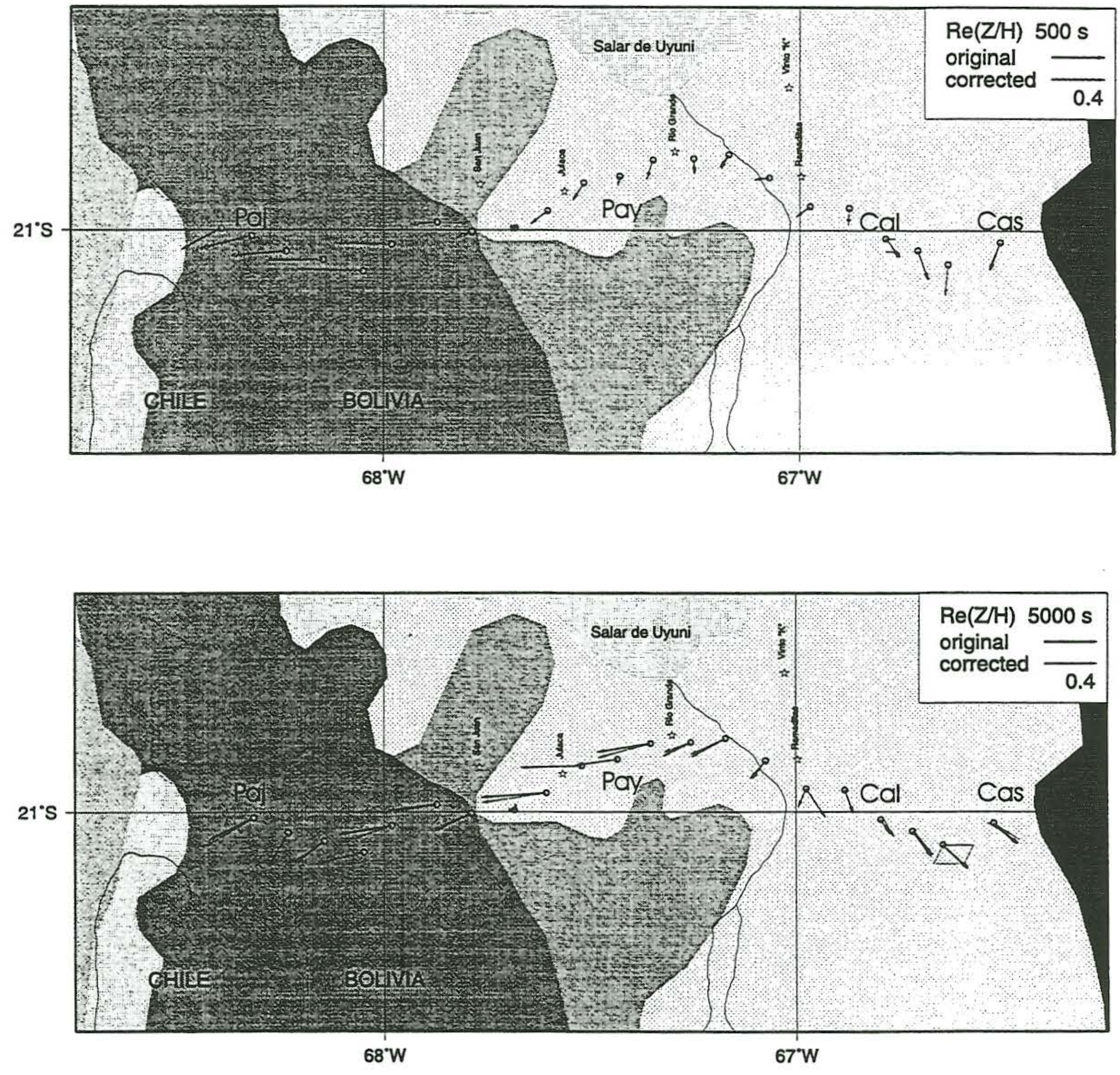

Fig.6.- Real Induction arrows of measured data (original) and the ones corrected by Chave decomposition code which included both impedance & magnetic transfer functions, realised with all parameters free. Data of period 500 s and 5000 s are shown above and below, respectively. There is almost no difference between corrected and original data for shorter period bands, whereas only data of longer periods seem to be corrected. The corresponding X^2-misfits are shown for 3 sites in Fig.4.

Fig.3 shows an application of the previous confidence limit hypothesis for real data measured in the Bolivian Altiplano.

CONCLUSIONS AND PERSPECTIVES

By taking into account the measured data errors, regional 2-D dimensionality may be assumed with a higher certainty when the confidence limits of the parameters involved in the analysis are estimated. This can be done by Jacobi-matrix transformation of random variables, which is applicable for the parameter whose function is continuous and differentiable on the tensor elements, provided a known statistical distribution of the impedance tensor.

The hypothesis of Gaussian distribution of the impedance elements seems to be valid at periods where the number of sample data was higher enough to deal with the assumption of asymptotically Gaussian distribution.

Assuming uncorrelated tensor elements might be a good approximation when the parameter confidence limit is within its linear propagated error (see Fig. 3). Anyhow, it is still possible to calculate the probability distribution of the based tensor element parameter by considering correlated error, if the covariance matrix of the measured data is known.

The Jacobi-transformation for the regional skew parameter is also applicable to a Fisher distribution of the tensor elements, provided that the number of sample data is known.

Tensor decomposition applied in measured data of the Bolivian Altiplano (Figs. 4 and 5) indicates 2-D dimensionality with a strong tendency of approx. N-S regional strike (i.e. ~0°) for sites located on the western and central profile (see Fig.6).

The western sites of the profile have regional skew values below 0.2-0.3 with at least 95% probability for the period band 20-4000 s (Fig. 3), thus indicating a 2-D regional structure with high certainty.

References:

Bahr, K., 1991. Geological noise in magnetotelluric data: a classification of distortion types, *Phys. Earth Planet. Int.*, **66**, 24-38.

Brandt S., Datenanalyse, *Wissenschaftsverlag*, Mannheim, 651 pp, 1992.

Chave A.D. & Smith T.J., 1994. On electric and magnetic galvanic distortion tensor decompositions, *J. Geophys. Res.*, **99**, B3: 4669-4682.

Dudewicz E.J.& Mishira S.N., Modern Mathematical Statistics, *John Wiley & Sons*, New York, 838 pp, 1987.

Egbert G.D. & Booker J.R., 1986. Robust estimation of geomagnetic transfer function, *Geophys. J. R. astr. Soc.*, **87**, 173-194.

Egbert G.D., 1998. EMTF: Programs for Robust Single Station and Remote Reference Analysis of MT Data & Error Bars for Transfer Function Elements in Z_files, College of Oceanic and Athmospheric Sciences, Oregon State University, Corvaills.

Fisz M., Wahrscheinlichkeitsrechnung und mathematische Statistik, *VEB Deutscher Verlag der Wissenschaften*, Berlin, 718 pp, 1976.

Heuser H., Lehrbuch der Analysis, Teil 2, *B.G. Teubner Stuttgart,*736 pp, 1988.