TECHNISCHE UNIVERSITÄT
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2D & 3D INVERSION OF MT DATA CONSIDERING TOPOGRAPHY

A case study based on the Chemnitz University of Technology and TU Bergakademie Freiberg FE-Toolbox

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MOTIVATION

As part of the GEOSAX project (2017-2021) the TU Chemnitz together with the TU Bergakademie Freiberg developed an FE-Toolbox that provides blueprints for implementing forward operators and inversion routines for arbitrary geophysical EM problems. The presented work uses this toolbox to invert MT data from the Tarawera Volcanic Complex, New Zealand, that was acquired by GNS Science (2012-2017).

MESHING AND TOPOGRAPHY INTEGRATION

Free 3D finite element mesh generator gmsh[a]

DEM

2D mesh: local refinement around observation points (z-coordinate neglected)

3D surface mesh: interpolation of DEM using inverse distance weighting → raise mesh points

3D simulation domain: intersect 3D surface with cuboid volume

Inversion domain: local refinement in a cuboid (side length depends on approximated average skin depth)

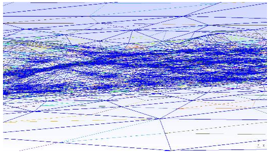

Figure 1a: section of Tarawera surface mesh

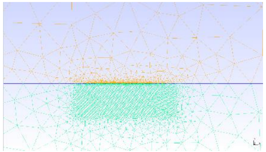

Figure 1b: section of Coprod_2S1 mesh

FORWARD PROBLEM

- MT basic equations in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$:

$$\left(\nabla \times \frac{1}{\mu} \nabla \times + \imath\omega\sigma - \varepsilon\omega^2\right) \boldsymbol{E} = 0 \qquad \text{in } \Omega$$

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{g}^{TE|TM} \times \boldsymbol{n} \qquad \text{on } \delta\Omega$$

$$\boldsymbol{H} = \frac{1}{\imath\omega\mu_0} \nabla \times \boldsymbol{E} \qquad \boldsymbol{n} \text{ ... outer normal vector}$$

- $\boldsymbol{g}$: $\boldsymbol{E}$ excited by plane $\boldsymbol{H} = 1\frac{A}{m}$ in the air
- $\boldsymbol{E}$-field formulation: Prevention of ill-conditioned matrices caused by large σ contrasts

3D:

$$a_{\sigma,\omega_j}^{TE|TM}(\boldsymbol{\varphi},\boldsymbol{\psi}) = \int_\Omega \nabla \times \boldsymbol{\varphi} \cdot \nabla \times \overline{\boldsymbol{\psi}} + (\imath\omega_j\mu\sigma - \varepsilon\omega_j^2\mu)\boldsymbol{\varphi} \cdot \overline{\boldsymbol{\psi}}$$

$\forall \boldsymbol{\varphi},\boldsymbol{\psi} \in H(curl,\Omega)$ **Nédélec elements**

2D:

$$a_{\sigma,\omega_j}^{TE}(\boldsymbol{\varphi},\boldsymbol{\psi}) = \int_\Omega \nabla \times \boldsymbol{\varphi} \cdot \nabla \times \overline{\boldsymbol{\psi}} + (\imath\omega_j\mu\sigma - \varepsilon\omega_j^2\mu)\boldsymbol{\varphi} \cdot \overline{\boldsymbol{\psi}}$$

$\forall \boldsymbol{\varphi},\boldsymbol{\psi} \in H(curl,\Omega)$ **Nédélec elements**

$$a_{\sigma,\omega_j}^{TM}(\varphi,\psi) = \int_\Omega \nabla \varphi \cdot \nabla \overline{\psi} + (\imath\omega_j\mu\sigma - \varepsilon\omega_j^2\mu)\varphi \cdot \overline{\psi}$$

$\forall \varphi,\psi \in H^1(\Omega)$ **Lagrange elements**

$$\boldsymbol{A}'\boldsymbol{u}' = 0$$

Inclusion of inhomogeneous Dirichlet BC

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$$

- Simulated quantity $(\boldsymbol{E}, \boldsymbol{H})$: $d_{ijk} \coloneqq \boldsymbol{Q}_{jk}\, \boldsymbol{u}_{ij}(\sigma)$ (polarization index i, frequency index j, observation index k)
- Interpolation operator $\boldsymbol{Q}_{jk}$: interpolation of FE-solution $\boldsymbol{u}$ to $\boldsymbol{E}$ and $\frac{1}{\imath\omega\mu_0} \nabla \times \boldsymbol{E}$
- Equation solver: distributed multifrontal solver MUMPS[b]

JACOBIAN MATRIX

- Primal solution $\boldsymbol{u}$: solution of $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$
- Dual solution $\boldsymbol{q}$: solution of $\boldsymbol{A}\boldsymbol{q} = \boldsymbol{Q}$
- Jacobian: linear Gateaux derivative of $d_{ijk}(\sigma)$

$$\boldsymbol{J}_{ijk}(\sigma)\,\delta\sigma = -\imath\omega_j\mu \int_\Omega \delta\sigma\, \boldsymbol{u}_{ij} \cdot \overline{\boldsymbol{q}}_{jk}$$

$\overline{\boldsymbol{q}}$ complex conjugate of $\boldsymbol{q}$

- Note: Additional transformation of $d_{ijk} \rightarrow g^k$ and $\boldsymbol{J}_{ijk}$ (chain rule) to obtain Z, T, ρ_a, ϕ and its derivatives

INVERSE PROBLEM

- Regularized Gauss-Newton method[c] with $\boldsymbol{m} = f(\boldsymbol{\sigma})$ (e.g. $\boldsymbol{m} = log(\boldsymbol{\sigma})$):

$$\Phi = \min_{\Delta\boldsymbol{m}} \left(\sum_k \left| \boldsymbol{W}(\boldsymbol{J}^k(\boldsymbol{m})\Delta\boldsymbol{m} - (b^k - g^k(\boldsymbol{m}))) \right|^2 + \beta \int_\Omega \left| \nabla \left(\Delta\boldsymbol{m} - (\boldsymbol{m}^{ref} - \boldsymbol{m}) \right) \right|^2 \right)$$

$$\boldsymbol{m} = \boldsymbol{m}_0 + \Delta\boldsymbol{m}$$

- Solution[d] of each Gauss-Newton step:

$$\nabla\Phi \,!= 0$$

Euler-Lagrange & mixed weak formulation (Raviart-Thomas and discontinuous Lagrange elements)

$$\begin{bmatrix} -\boldsymbol{M} & \boldsymbol{D}^T \\ \boldsymbol{D} & \frac{1}{\beta}\boldsymbol{J}^T\boldsymbol{W}^T\boldsymbol{W}\boldsymbol{J} \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta} \\ \Delta\boldsymbol{m} \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}^T(\boldsymbol{m}^{ref} - \boldsymbol{m}) \\ \frac{1}{\beta}\boldsymbol{J}^T\boldsymbol{W}^T\boldsymbol{W}(\boldsymbol{b} - \boldsymbol{g}) \end{bmatrix}$$

Solve using the direct method (see below) or iteratively

- $\mathbf{U} = \frac{1}{\sqrt{\beta}}[\boldsymbol{0} \quad \boldsymbol{W}\boldsymbol{J}]$ & $\boldsymbol{C}^{-1}$ by sparse $\mathbf{LDL}^\mathrm{T}$ decomposition of $\mathbf{C} = \begin{bmatrix} -\boldsymbol{M} & \boldsymbol{D}^T \\ \boldsymbol{D} & \boldsymbol{0} \end{bmatrix}$
- Application of Woodbury formula[e]:

$$(\boldsymbol{C} + \boldsymbol{U}^T\boldsymbol{U})^{-1} = \boldsymbol{C}^{-1} - \boldsymbol{C}^{-1}\boldsymbol{U}^T(\boldsymbol{I} + \boldsymbol{U}\boldsymbol{C}^{-1}\boldsymbol{U}^T)^{-1}\boldsymbol{U}\boldsymbol{C}^{-1}$$

- Solve via backward substitution

$\boldsymbol{J}$ ... Jacobian matrix
β ... regularization parameter,
$\boldsymbol{D}$ ... divergence on $H(div,\Omega)$
$\boldsymbol{M}$ ... mass matrix on $H(div,\Omega)$
$\boldsymbol{\zeta} = -\nabla\,(\Delta\boldsymbol{m})$ ... flux
$\Delta\boldsymbol{m}$ ... change in parameter and
$\boldsymbol{m}^{ref}$ ... reference model
$\boldsymbol{b}$ ... observations

2D MODEL: COPROD_2S1

- 2D synthetic model[f] (Figure 2b)
- Observations: Impedance $\boldsymbol{Z}$, Tipper $\boldsymbol{T}$
- 61 sites; 11 periods: [2, 10000] s
- Data weighting: mean value over all points for one frequency and for one component
- Additional balancing between $\boldsymbol{T}$ and $\boldsymbol{Z}$ in $\boldsymbol{g}$

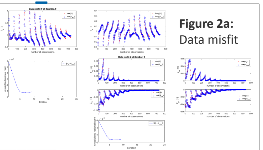

Figure 2a: Data misfit

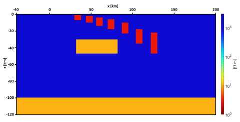

Figure 2b: Coprod_2S1 synthetic model

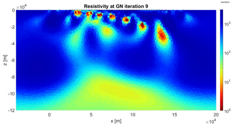

Figure 2c: Inversion result

- Seven well resolved conductive bodies
- Block in 40 km depth and conductive substratum recognizable
- Background: 1000 to 2000 Ωm
- Residual norm < $1 \cdot 10^{-3}$, but fit for long periods is worse than for shorter periods

TARAWERA

- 3D model of Tarawera Volcanic Complex
- Observations: Impedance $\boldsymbol{Z}$
- 68 sites; 16 periods [0.012, 341] s
- Subsequent inversion of frequency bands (using previous result as starting model)
- Data weighting: same as for Coprod_2S1

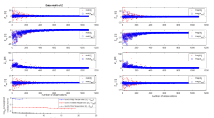

Figure 3a: Data misfit

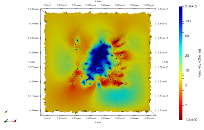

Figure 3b: Inversion result, top view

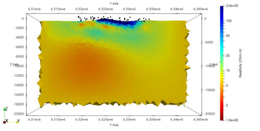

Figure 3c: Inversion result, section parallel to y

- Resistive young rhyolite at surface (blue) and greywacke (green) well resolved
- Older conductive ignimbrite (2 -3 km depth) and conductive body in 10 km depth partially resolved
- Effects at stations and bad fit of low periods due to coarse grid and over-smoothing of model at depth

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