ON THE ADDITION OF INDUCTION VECTORS

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Induction vectors were originally introduced by Parkinson (1962) and, independently, by Wiese (1962), and have been interpreted in terms of transfer functions in the frequency domain by Schmucker (1970). In the horizontal plane an induction vector is defined by the ordered pair (A, B) given by the linear relation

$$Z = AX + BY \tag{1}$$

between the vertical magnetic field at some particular site, and the components of the horizontal magnetic field $\mathbf{H} = (X, Y)$ at the same site. Except in equatorial or polar regions, the vertical component Z is an anomalous field associated with conductivity gradients in the earth whereas X and Y are usually dominated by the 'normal' or 'regional' field.

The quantities A and B are generally frequency dependent so that it is convenient to express the field components in (1) in the frequency domain, i.e. they are given by the (complex) Fourier transforms of the time dependent magnetic variations. It follows that

$$A = A_r + iA_i, \quad B = B_r + iB_i \tag{2}$$

are complex numbers, and it is usual to plot the real (A_r, B_r) and imaginary (A_i, B_i) induction vectors separately. They are obtained by minimizing $(Z - AX - BY)(Z^* - A^*X^* - B^*Y^*)$ summed over an ensemble of events (Everett and Hyndman, 1967). The real (Parkinson) vector $(-A_r, -B_r)$ points towards good conductors and its length is indicative of the magnitude of the anomalous Z variation.

Many authors avoid the word 'vector' and prefer to use 'arrow' instead because vectorial addition is not meaningful in the physical sense. It seems entirely appropriate, however, to call the ordered pairs

$$\mathbf{P}_r = (A_r, B_r), \quad \mathbf{P}_i = (A_i, B_i) \tag{3}$$

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vectors in the horizontal plane because they satisfy the well-known transformation definition of a vector under a rotation of the coordinate system about the z-axis. For it follows from (1), (2) and (3) that for any horizontal magnetic field $\mathbf{H}$,

$$Z_r = \mathbf{P}_r \cdot \mathbf{H}_r - \mathbf{P}_i \cdot \mathbf{H}_i, \quad Z_i = \mathbf{P}_r \cdot \mathbf{H}_i + \mathbf{P}_i \cdot \mathbf{H}_r, \tag{4}$$

and these equations show that the scalar products of the quantities $\mathbf{P}_r$ and $\mathbf{P}_i$ with the two-dimensional magnetic vectors $\mathbf{H}_r$ and $\mathbf{H}_i$ yield Z_r and Z_i which are known to be scalars i.e. quantities that are invariant under a rotation of the coordinates. Hence $\mathbf{P}_r$ and $\mathbf{P}_i$ must be two-dimensional vectors by the well-known quotient theorem for tensors (see e.g. Arfken, 1966, p. 118; Butkov, 1968, p. 682). If $\mathbf{P} = \mathbf{P}_r + i\mathbf{P}_i$, then the formal sum $\mathbf{P}^{(1)} + \mathbf{P}^{(2)}$ of two such vectors has *mathematical* meaning according to the usual definition of vectorial addition; the fact that it does not have an obviously useful *physical* interpretation is no reason to disqualify $\mathbf{P}$ as a vector.

Notwithstanding this cautionary linguistic approach, there are examples in the literature (e.g. Hebert *et al.*, 1983) where induction vectors have indeed been combined additively and interpreted physically with plausible results (see also the contribution from B. Siemon in this Protokoll). A typical example is the removal of the 'coast effect' from field measurements made in coastal regions by subtracting Parkinson vectors associated with the coast effect alone from those actually observed, the former being calculated theoretically or measured in the laboratory with the aid of an analogue model. The resulting (real) 'difference vectors' are then assumed to point towards other conductive anomalies in the region, usually geological features. Such an interpretation is questionable, however, because the ocean and the geological anomaly do not generate separate anomalous fields independently; they are coupled electromagnetically by mutual induction (Wolf, 1983) and by the redistribution of the charge distributions on their boundaries. Indeed, it is this very coupling that renders the physical interpretation of the resultant of two induction vectors so obscure. Only if the two bodies are effectively isolated from each other (and the anomalous horizontal magnetic fields associated with them are negligible compared with the regional magnetic field), can we possibly claim that $Z^{(1)} + Z^{(2)} \equiv (\mathbf{P}^{(1)} + \mathbf{P}^{(2)}) \cdot \mathbf{H}$ represents the total vertical field when both anomalous conductors are present.

This effect is examined in more detail with the aid of a computer program for induction in thin sheets developed by Dawson and Weaver (1979) and McKirdy, Weaver and Dawson (1985). Consider a non-uniform surface layer comprising an ocean of uniform conductance 10^4 S separated by a straight line from a surface layer of crustal rocks whose conductance is 25 S. With the conductivity of seawater equal to 4 S/m, the ocean is 2.5 km deep. The underlying half-space is assumed to have a conductivity of 5×10^{-3} S/m down to a depth of 30 km, and 0.1 S/m below 30 km. In the numerical model the surface layer is represented by a thin sheet divided into 29×29 square cells whose sides are of length 60 km and to which different

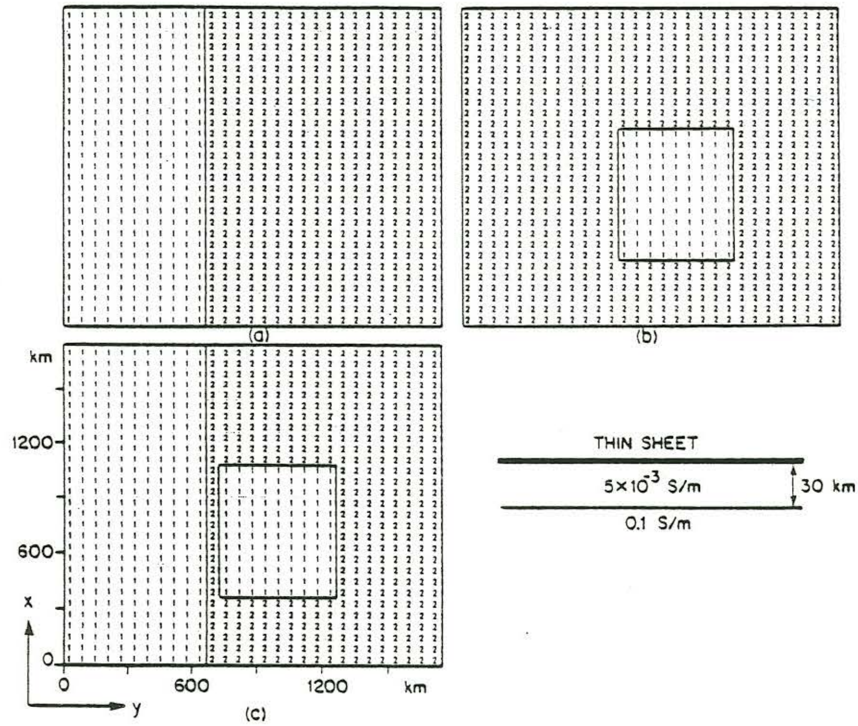

Figure 1: Numerical thin sheet models: (a) coastline (b) surface anomaly (c) coastline and anomaly. The numbers 1 and 2 refer to conductances of 10^4 S and 25 S respectively. The layered structure beneath the thin sheet is also shown.

conductances are assigned according to the grid shown in Fig. 1a. For geomagnetic variations of period 1 hr the cell size is 0.14 skin depths in the layer beneath the thin sheet which is small enough for the discretization of the problem to be valid. The total surface area covered by the model is 1740 km $\times$1740 km.

We shall study the effect of a 540 km $\times$ 720 km rectangular surface anomaly introduced on the landward side of the boundary. For simplicity we take the conductance of this anomaly to be the same as that of the ocean. In Fig. 1b we show the anomaly alone with the ocean removed, while in Fig. 1c both anomaly and ocean are present, with the anomaly displaced 60 km inland from the coastline. The slightly asymmetrical placement of the rectangle relative to the top and bottom of the grid has not affected the symmetry of the calculated fields which is confirmation that the overall size of the grid is large enough to eliminate any boundary effects.

Induction vectors have been computed from the magnetic fields, (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) respectively, for two perpendicular polarizations of the regional magnetic field (i) parallel to (x-direction) and (ii) perpendicular to (y-direction) the coastline. It follows from (1) that

$$A = \frac{Z_1 Y_2 - Z_2 Y_1}{X_1 Y_2 - X_2 Y_1}, \quad B = \frac{Z_2 X_1 - Z_1 X_2}{X_1 Y_2 - X_2 Y_1}. \tag{5}$$

For the coastline alone (Fig. 1a) the problem is strictly two-dimensional so that $X_1 = X_0$ (const), $X_2 = Y_1 = Z_1 = 0$, and (5) degenerates to

$$A = 0, \quad B = Z_2/Y_2. \tag{6}$$

In Figs. 2a, 2b and 2c we have plotted the Parkinson vectors $-\mathbf{P}_r$ (arrow heads) and $\mathbf{P}_i$ (flat heads) corresponding to the three models in Figs. 1a, 1b and 1c. The coast effect vectors $\mathbf{P}^{(c)}$ in Fig. 2a are perpendicular to the coastline with the real vectors pointing towards the ocean in typical coast effect fashion. In Fig. 2b the anomaly vectors $\mathbf{P}^{(a)}$ also orient themselves in directions normal to the conductivity boundaries with some distortions near the corners of the rectangle. Finally, the vectors $\mathbf{P}$ in Fig. 2c display a combination of these two effects. The coastal vectors are much reduced in length near the inland anomaly, and the real vectors actually reverse direction inland towards the anomaly.

Now if we subtract the vectors in Fig. 2a from those in Fig. 2c, how close will we be to recovering the vectors in Fig. 2b? The result of these subtractions is shown in Fig. 3 where the (negative) real and imaginary 'difference vectors' $\mathbf{P} - \mathbf{P}^{(c)}$ are displayed. Even though the pattern of directions viewed as a whole does give a reasonably good qualitative indication of where the remaining anomaly is, this is deceptive because in a field experiment data will be available at only a few sites and it is clear that certain individual difference vectors are not at all the same as the corresponding induction vectors in Fig. 2b. For example, if field data were available only from the stations with circular bases in Fig. 3 near the left-hand corners of the anomaly, then it might be concluded that the anomaly was located to the left of the stations rather than to the right. On the other hand the dramatic reversal of the difference vectors at the coastal sites shown with crosses in Fig. 3 reveals in no uncertain fashion the correct position of the geological anomaly. It is concluded, at least for this particular model, that the electromagnetic coupling between the two conductive bodies is strong enough for difference vectors to give very misleading interpretations at some sites even though their use would be perfectly justified at others.

In Fig. 4 we have plotted the (negative) real and imaginary parts of the vector $\boldsymbol{\Delta} = \mathbf{P} - \mathbf{P}^{(a)} - \mathbf{P}^{(c)}$. If the difference vectors $\mathbf{P} - \mathbf{P}^{(c)}$ truly represented the induction vector $\mathbf{P}^{(a)}$ associated with the anomalous rectangle, then we would expect $\boldsymbol{\Delta}$ to vanish. It is immediately apparent from Fig. 4 that this is not the case, a further indication of the effect of electromagnetic coupling.

The importance of this coupling will depend, of course, on the shape, size, conductivity and relative positions of the two bodies as well as on the period of the magnetic variations. In a more detailed investigation (Weaver and Agarwal, 1990) we have studied the effect of two of these variables by moving the anomalous rectangle progressively inland and calculating the response for a period of 20 min as well. A complete understanding of the behaviour of difference vectors would, however, require a numerical investigation of a fully three-dimensional model which includes

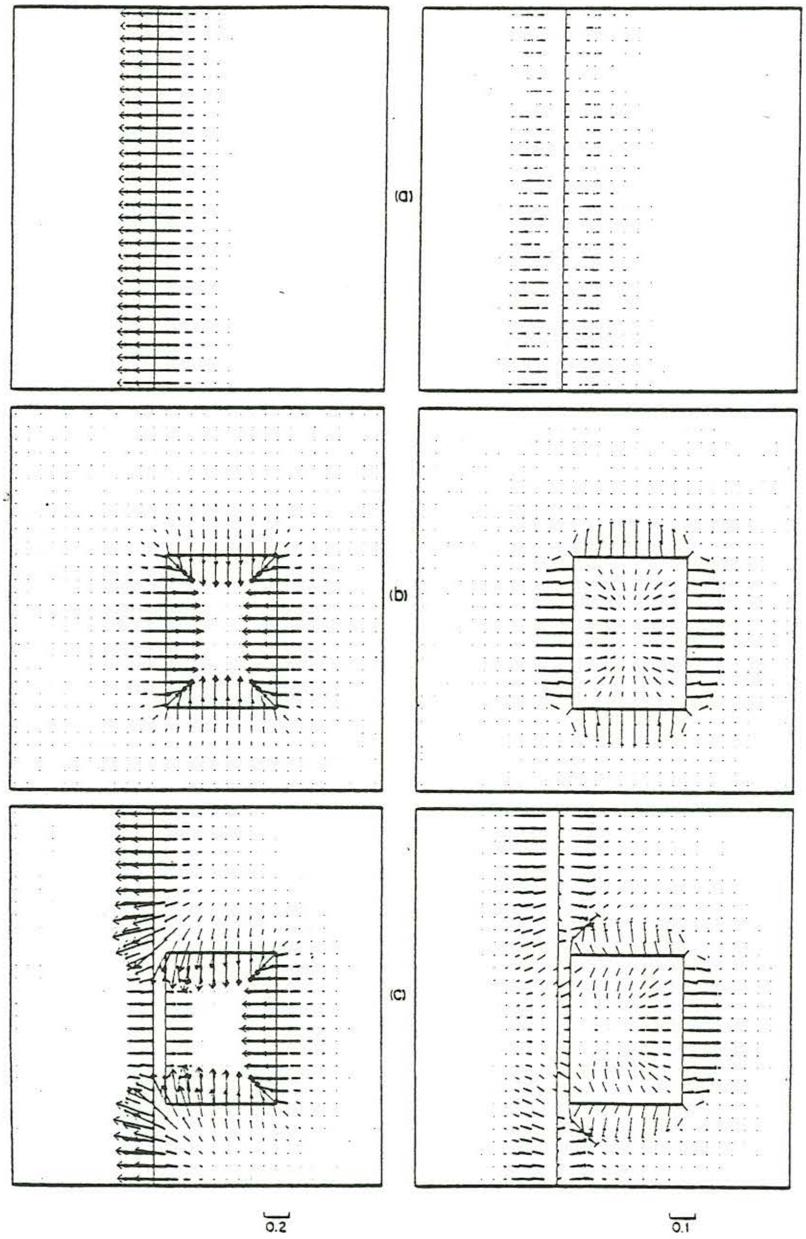

Figure 2: Real (left hand diagrams with arrow heads) and imaginary (right hand diagrams with flat heads) induction vectors (a) $\mathbf{P}^{(c)}$ (b) $\mathbf{P}^{(a)}$ and (c) $\mathbf{P}$ for the three models in Fig. 1. The signs (directions) of the real vectors have been reversed. The period of the field is 1 hr.

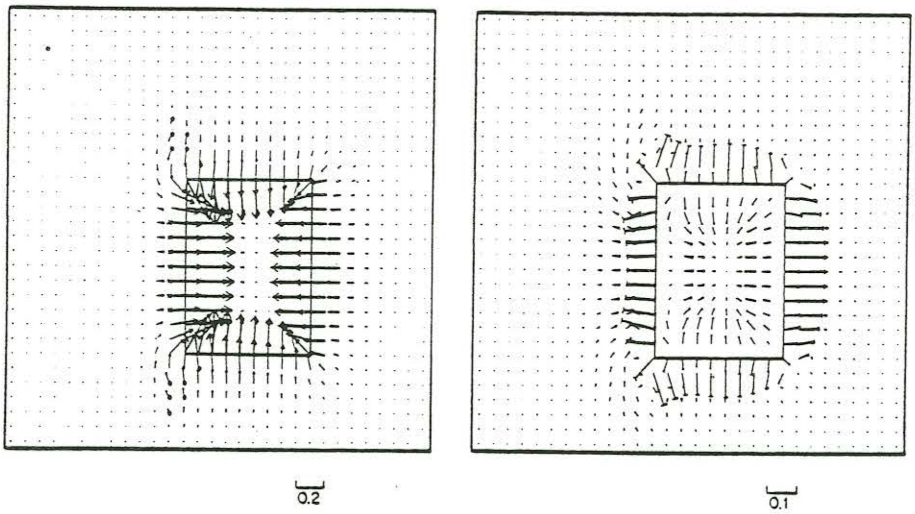

Figure 3: The 'difference vectors' $\mathbf{P} - \mathbf{P}^{(c)}$ obtained from Fig. 2. The signs of the real vectors have been reversed.

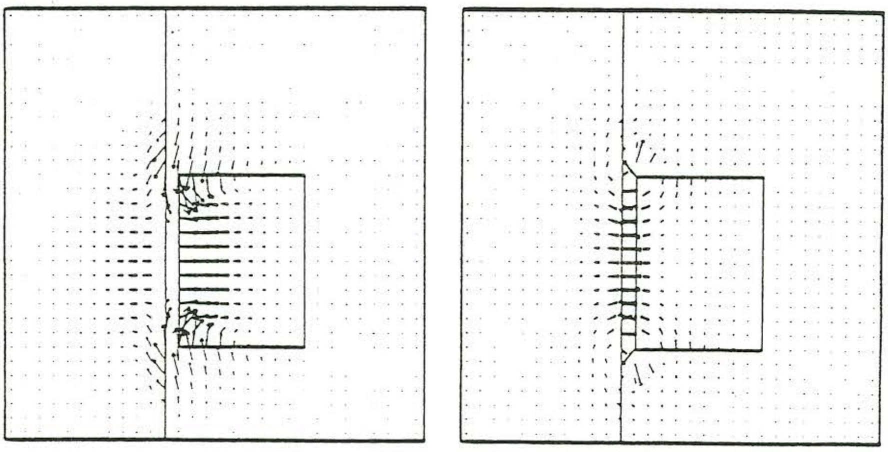

Figure 4: Vectors Δ for the three positions of the anomaly shown in Fig. 4.

anomalous bodies of different sizes and conductivities located at various depths. Wolf's (1983) two-dimensional study of the inductive coupling between anomalous bodies of perfect conductivity suggests that the coupling would be stronger for a conductive anomaly at depth and extending beneath the ocean than for one displaced laterally from the ocean as in the model we have considered here.

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