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Inversion of Three-Dimensional Electromagnetic Induction Through Inverting Scattering Current Distribution With Tikhonov Regularisation

SUMMARY

The nonlinear 3D EM conductivity inverse problem is treated iteratively as inversion of an ill-posed linear Fredholm integral equation of the first kind relating the scattering current and the data (EM fields) using Tikhonov regularisation. This equation is regularized with the integral equation for the forward modeling which describes the relation between inhomogeneous conductivity, scattering current and total electrical field within the inhomogeneity. Beginning with an initial conductivity for the inhomogeneity, the regularized minimal problem is solved for the scattering current from which the conductivity is again calculated, which initiates the iteration process. The solution converges in few iterations. Numerical results show the efficiency of this algorithm. The inversion is completed with a little more computation time than forward modeling.

Key words: 3D EM inversion, scattering current, Tikhonov regularisation

1 INTRODUCTION

While the forward modeling in the study of electromagnetic (EM) induction has made great progress, the theory for the inverse problems, especially for multidimensional inverse problems, is only at a experimental stage, although they have attracted many geophysicists. The grounds lie mainly in the inherency of the nonlinearity of EM induction.

In most reports on 2D and 3D EM inversion, the nonlinear inverse problems are approached with linearization methods which linearize the data-model-relation, or the direct relation between observed data and model parameters (Chave & Booker, 1987). Those methods suffer in general a burden of tremendous computation time.

Our new approach is based on the fact that surface observations of electromagnetic and telluric fields are correlated linearly to the subsurface scattering current distributions through a Fredholm integral equation of the first kind whose inverse is ill-posed and whose solution requires some kind of stabilization. Once the scattering current is obtained, the anomalous conductivity of the inhomogeneous structure can be calculated using the relation between the scattering current and the total electric field within the inhomogeneity which in turn is derived from the scattering current itself.

The concept to infer anomalous internal currents from surface observations is not new. In the simplest form equivalent line or sheet currents are deduced from the anomalous magnetic surface field, using potential theory or, with further assumptions, Biot-Savart law (Banks 1978). Jones (1983) has reviewed the studies on current channelling, discussing the distortions of electric currents by conductive heterogeneities. Those are qualitative interpretations of the induction problem. Here we try to approach the inverse problem of EM induction in a quantitative way.

2 INTEGRAL EQUATIONS FOR THE FORWARD PROBLEM

For a given 3D inhomogeneity Σ with conductivity σ_a in a homogeneous medium of conductivity σ_n there exists an integral equation governing the scattering current $\mathbf{J}_s$ in Σ (Hohmann, 1983; Xiong et al, 1986):

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_i(\mathbf{r}) + \int_\Sigma \underline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\mathbf{J}_s(\mathbf{r}')dv', \qquad (1)$$

where $\mathbf{E}$ is the total electric field, $\mathbf{E}_i$ the incident electric field, and $\underline{\mathbf{G}}$ the tensor Green's function. The second term in the right side of eq. (1) represents the secondary, or the induced electric field in Σ. $\mathbf{J}_s$ is defined by

$$\mathbf{J}_s(\mathbf{r}) = \Delta\sigma(\mathbf{r})\mathbf{E}(\mathbf{r}), \qquad (2)$$

Where $\Delta\sigma(\mathbf{r}) = \sigma_a(\mathbf{r}) - \sigma_n$. Thus we can write a Fredholm integral equation that $\mathbf{J}_s$ must satisfy:

$$\mathbf{J}_s(\mathbf{r}) = \Delta\sigma(\mathbf{r})\mathbf{E}_i(\mathbf{r}) + \Delta\sigma(\mathbf{r})\int_\Sigma \underline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\mathbf{J}_s(\mathbf{r}')dv'. \qquad (3)$$

In the forward calculation, equation (3) is solved for $\mathbf{J}_s$ and the secondary electric field $\mathbf{E}_s$ at the earth's surface is calculated as follows:

$$\mathbf{E}_s(\mathbf{r}) = \int_\Sigma \underline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\mathbf{J}_s(\mathbf{r}')dv' \text{ , } for\ z = 0 \text{ .} \qquad (4)$$

The secondary magnetic field is calculated in the same way with appropriate Green's function.

3 INVERSION

The inversion of EM conduction problems is nonlinear. Here we try to approach this nonlinear problem indirectly by means of an ill-posed linear inverse problem which defines a third parameter linking the data and the model parameters, and then to reconstruct the conductivity structure. We find that the scattering current distribution may be the proper parameter that is directly related to the observable data ($\mathbf{E}_s$ and secondary magnetic field $\mathbf{H}_s$ at the surface of the Earth) through eq. (4) and to the anomal conductivity through eq. (2).

In fact, any anomalous induced current distribution $\mathbf{J}_s$ that reproduces the data is to some extend already a solution to the inverse problem, for the conductive structure can be qualitatively outlined by $\mathbf{J}_s$.

But the inversion of eq. (4) is not a simple matter. Eq. (4) is a Fredholm's integral equation of the first kind whose inversion is always ill-posed (Baker, 1977; Miller, 1974; Tikhonov & Arsenin, 1977). The solution to the inverse problem (4) might not exist, or it might not be unique, even for exact data. Even if it exists the solution does not depend continuously on the data. Small perturbations in the data result in large errors in the current distribution. Remedies for this problem are to find a quasi-solution, to solve for $\mathbf{J}_s$ subject to certain conditions, or to approach the problem with some other kind of regularisation.

Generalized Inversion

A possible solution of (4) is the least squares solution of minimal norm (Baker, 1974), which finds $\mathbf{J}_s$ of minimal L^2 norm from the least squares solution of eq. (4), i.e.:

$$\begin{cases} \| \mathbf{E}_s - \int_\Sigma \underline{\mathbf{G}}\ \mathbf{J}_s dv \|^2 = min \\ \| \mathbf{J}_s \|^2 = min \end{cases} \quad . \tag{5}$$

This is achieved by the *generalized inversion* (Ben-Israel and Greville, 1974; Peters and Wilkinson, 1970), which will be refered later again.

The minimal norm solution is not stable. For data with noise the minimal norm solution could be of no physical meaning. That is to say, it could be likely a pure mathematical solution to (5) rather than a physical scattering current. For example, thus inverted conductivity could be negative or complex. Numerical calculations have proved this. We obtained the correct conductivity of an inhomogeneity only with exactly the same parameterisation in the inversion as in the forward computation.

Tikhonov Regularisation

Physical constrainsts must be considered in the inversion. We find that one possibble constraint might be the definition of the scattering current $\mathbf{J}_s$, or eq. (3), which is in fact the Ohm's law. From the viewpoint of functional analysis $\mathbf{J}_s$ can be considered as a functional mapping $\Delta\sigma$ onto $\mathbf{J}_s$ by eq. (3). The solution of eq. (4) must be an element from the function space of all the $\mathbf{J}_s$. From a physical viewpoint eq. (3) takes into account that $\mathbf{J}_s$ is produced by the induction of $\mathbf{E}_i$.

Another powerful tool for treating ill-posed problems that merges constraints for the solution into the minimum problem is *Tikhonov regularisation* (Groetsch, 1984; Miller, 1974; Tikhonov & Glasko, 1964; Tikhonov & Arsenin, 1977). For an ill-posed problem

$$A\,x = b, \tag{6}$$

in which A is a compact operator, one can approach it approximately by a well-posed least squares problem

$$\|\,A\,x - b\,\|^2 + \alpha^2\|\,L\,x\,\|^2 = min, \tag{7}$$

where α is a small number, and L a well-posed linear operator. L can be any physical constraint for x which is well-posed.

So far, we can construct an inverse algorithm for the 3D EM problem using a kind of Tikhonov regularisation as follows:

$$\|\,\mathbf{E}_s - \int_\Sigma \underline{\mathbf{G}}\;\mathbf{J}_s dv\,\|^2 + \alpha^2\|\,\Delta\sigma\mathbf{E}_i - (\mathbf{J}_s - \Delta\sigma\int_\Sigma \underline{\mathbf{G}}\;\mathbf{J}_s dv)\,\|^2 = min. \tag{8}$$

The minimazation is done with respect to $\mathbf{J}_s$ and $\Delta\sigma$. The first term represents the ill-posed least squares problem of eq. (4). The second term is the regularization operator which is the physical constraint for $\mathbf{J}_s$ that confines it in the family of scattering currents produced by EM induction.

Problem (8) connects the unknown scatterring current $\mathbf{J}_s$ and the unknown anomalous conductivity $\Delta\sigma$ with the known scatterring field $\mathbf{E}_s$ at the earth's surface and the known source field $\mathbf{E}_i$ within the inhomogeneity Σ. We can add in problem (8) the secondary magnetic field analogously to $\mathbf{E}_s$.

Here we solve the two unknowns in problem (8), $\mathbf{J}_s$ and $\Delta\sigma$, *iteratively*: given an initial guess for $\Delta\sigma$, a $\mathbf{J}_s$ can be found from (8) by minimazing $\mathbf{J}_s$ only. A new $\Delta\sigma$ is then calculated according to (3) with some physical constraints for $\Delta\sigma$. This begins the iteration.

4 NUMERICAL TREATMENTS

Problem (8) reduces numerically to a matrix form of

$$\| E_s - K\ J_s \|^2 + \alpha^2 \| B - \Gamma\ J_s \|^2 = min. \tag{9}$$

The dimensions of the two terms in the left side of problem (9) are not equal, for the second term contains a factor $\Delta\sigma$ comparing with the first term. In order to obtain a dimensionless α we rewrite problem (9) as

$$\| E_s - K\ J_s \|^2 + \alpha^2 \| \beta(B - \Gamma\ J_s) \|^2 = min, \tag{10}$$

with

$$\beta = \begin{cases} 1/\mid \Delta\sigma \mid_m, & \mid \Delta\sigma \mid_m \neq 0 \\ 1, & \mid \Delta\sigma \mid_m = 0 \end{cases}, \tag{11}$$

in which $\mid \Delta\sigma \mid_m$ means the arithmetical mean value of $\mid \Delta\sigma(\mathbf{r}) \mid$. The solution of problem (10) is obtained by solving its normal equation,

$$(K^*K + \alpha^2\beta^2\Gamma^*\Gamma) J_s = K^*E_s + \alpha^2\beta^2\Gamma^*B, \tag{12}$$

where the index '*' means transpose conjugate. Eq. (12) is solved by Cholesky decomposition, because the coefficient matrix is symmetrical.

As α goes to zero, eq. (12) can be arbitrarily bad conditioned, because $K^*K + \alpha^2\beta^2\Gamma^*\Gamma$ differs numerically not much from the ill-conditioned K^*K, so does the right side of (12) also, and the numerical results tend to inaccurate.

An alternative possibility to solve problem (10) is (Lawson & Hanson, 1974)

$$\left\| \begin{pmatrix} K \\ \alpha\beta\Gamma \end{pmatrix} J_s - \begin{pmatrix} E_s \\ \alpha\beta B \end{pmatrix} \right\| = min, \tag{13}$$

which can be interpreted as a weighting approximation to eq. (4). It is also called ridge regression or damped least squares. Problem (13) can again be solved by generalized inversion using Householder transformations or using any other method.

Here we shall introduce briefly an algorithm for generalized inverse with Householder transformations (Lawson & Hanson, 1974).

Suppose a $m \times n$ real matrix A with rank r has an orthogonal decomposition

$$A = URV^T, \tag{14}$$

where U is an $m \times m$ orthogonal matrix, V an $n \times n$ orthogonal matrix, R an $m \times n$ matrix of the form

$$R = \begin{pmatrix} R_{11} & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$

with R_{11} being a $r \times r$ matrix of rank r. Then, the unique generalized inverse matrix $A^\dagger$ of A is

$$A^\dagger = V \begin{pmatrix} {R_{11}}^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T \tag{16}$$

and the minimal norm solution of $Ax = b$ is given by $x^* = A^\dagger b$. A FORTRAN program HFTI is supplied by Lawson and Hanson (1974) for solving generalized inverse problems with Househoulder transformations. Solving eq. (13) using Householder transformations require more computation time and more computer storage than solving eq. (12) with Cholesky decompostion, because the complex coefficient matrix of eq. (13) must be rewritten as real matrix and thus the storage requirement and the dimension of the linear system double. But the former is much more accurate than the later, especially when α is very small.

In our applications the solution of (13) on the eight-digit IBM-3090 computer of the computer center of Universität Göttingen converges in general for an α ranging from 10^{-8} to 10, but for problem (12) α should not be smaller than 10^{-2}. α depends on the number of data and data error. When the data contain random noises, as in practice, α should not be too small, for it acts as a smoothing factor also. Empirically we have $0.1 \leq \alpha \leq 10$. With this value for α we can solve the normal equation (12) fairly well.

The reconstruction of the conductivity from $\mathbf{J}_s$ can be performed as a minimum problem too. The total electric field $\mathbf{E}$ is readily calculated from $\mathbf{J}_s$ using eq. (4). At every point $\Delta\sigma$ can be calculated such that

$$\| \mathbf{J}_s - \Delta\sigma \mathbf{E} \|^2 = min, \ \Delta\sigma > -\sigma_n, \ real. \tag{17}$$

However, for inconsistent $\mathbf{J}_s$ and $\mathbf{E}$ it is very likely that $\Delta\sigma$ is complex and $| \Delta\sigma | \leq \sigma_n$. Therefor, we have simply $\Delta\sigma = | J_s/E |$ for better convergence.

The conductivity of the body can be taken as a constant or piecewise constant, as suggested by Weidelt (private communication). For this we need only to establish an overdetermined system in a similar way as problem (17).

Problem (8) is constructed for one frequency. In case of multi-frequency inversion one must evaluate the scatterring current separately for each frequency and solve an overdetermined system for the frequency-independent $\Delta\sigma$.

5 NUMERICAL RESULTS

Fig. 1 shows a cube of 1 km with conductivity 1 Ωm embedded at a depth of 0.25 km in a half-space of conductivity 100 Ωm. Only a quarter of the body was considered in the calculations because of symmetry. In the forward calculation the whole cube was divided into $6 \times 6 \times 6$ cells. The computed secondary fields as data for inversion were E_x, E_y, H_x, H_y and H_z at four sites in the first quadrant of the surface (XOY) plane, (0,0), (0,2), (2,0), and (2,2). The incident field is a plane wave of frequency 1 Hz propagating in z-direction.

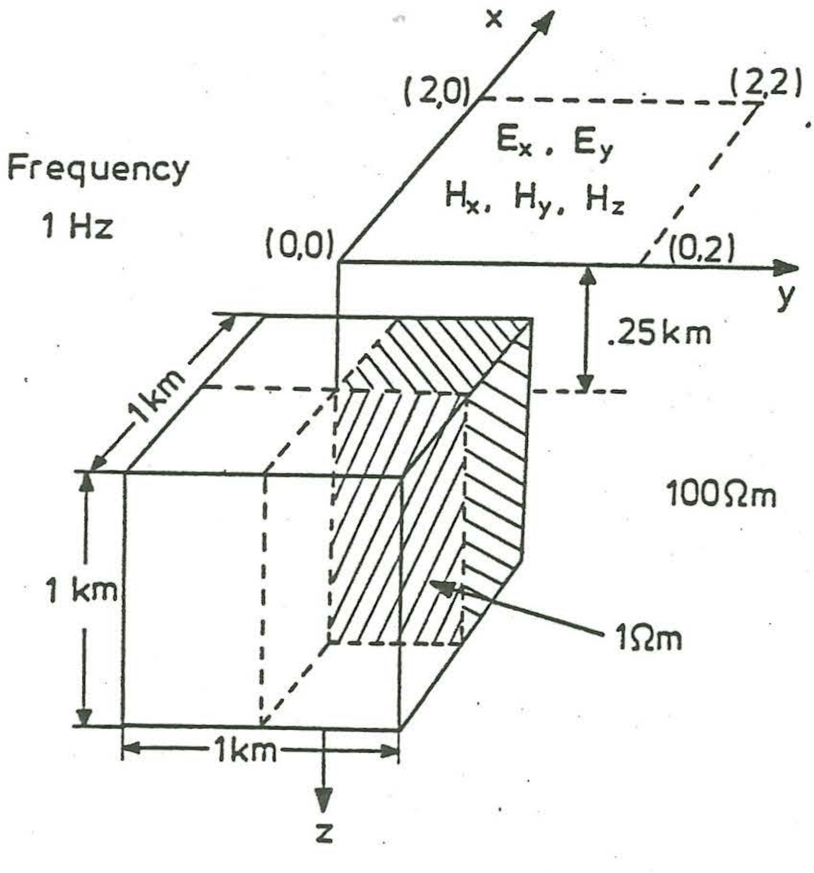

FIG. 1: A cube in a half-space. Only a quarter in shadow was considered in the calculation because of symmetry. Responses at the first quadrant of the surface was calculated and was used as input data for inversion.

In the inversion all the parameters except the conductivity of the body were assumed known. As initial guess for σ_a we had 0.1 Ωm. Solving eq. (13) with $\alpha = 0.001$ we needed only one iteration to get a solution of 1.0 Ωm. When we added 20 percent artificial random noises to the data, the solution converged in 7 iterations for $\alpha = 1$ to 0.96Ωm which reproduced data with 4 percent error; we got after one iteration a solution of 0.62 Ωm reproducing data with 14 percent error which was within the limit of the random noises, thought σ_a varied a little in the later iterations; for comparison, the solution of eq. (12) converged for this case to $\sigma_a = 1.8$ Ωm after 2 iterations which reproduced data with 18 percent error.

The computation time on the IBM 3090 is about 31 sec. for the forward calulation, and 63 sec. for solving eq. (13) with one iteration of which the execution of HFTI costed 32 sec.. For further iterations we needed mainly the computation time for HFTI. The inversion of eq. (12) needed about 42 sec. for one iteration, and 10 sec. for one further iteration. We see that for slightly more computation time the inverse problem is solved by this algorithm.

With three frequencies, 0.01, 0.1, and 1 Hz we got almost the same results.

6 DISCUSSION

The efficiency of this inverse scheme is proved by the above numerical results. We have used data at 4 measurement points only which are commonly available in practice. The algorithm withstands error disturbences. The computation time for inversion is in the order of that for the forward calculation. Therefore, we hope that our method can be applied in data interpretations for practical measurements.

It remains the problem to determine the geometry of a conductivity structure. Theoretically our algorithm should be applicable to such problems. When one inverts a structure larger than the real one, one should get for the conductivity outside the inhomogeneity a value of that of the host medium. Unfortunately we have no good numerical examples so far. We could only see that those regions were more resistive than the real conductor in some model examples. For any geometry we may obtain a conductivity reproducing data with minimal error to the original data. This is again a problem of non-uniqueness. It may be possible to study the non-uniqueness of conductivity inversion using the above-mentioned scheme.

It is a disadvantage of this algorithm that for every frequency the different scatterring currents must be evaluated and the multi-frequency data influence one another only at the step of solving eq. (17) instead of the more important eq. (12) or (13). We shall suggest to use one frequency only.

Anyhow, this novel approach to 3D EM inversion has been proved successful.

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