GFZ

Helmholtz-Zentrum
POTSDAM

HELMHOLTZ-ZENTRUM POTSDAM
DEUTSCHES
GEOFORSCHUNGSZENTRUM

Jan M. Hagedoorn, Hans Greiner-Mai, Ludwig Ballani

Core-Mantle Coupling

Part III:
Gravitational coupling torques

Scientific Technical Report STR12/01

www.gfz-potsdam.de

HELMHOLTZ
GEMEINSCHAFT

Imprint

HELMHOLTZ CENTRE POTSDAM
GFZ GERMAN RESEARCH CENTRE
FOR GEOSCIENCES

Telegrafenberg
D-14473 Potsdam

Printed in Potsdam, Germany
March 2012

ISSN 1610-0956

This work is published in the GFZ series
Scientific Technical Report (STR)
and electronically available at GFZ website
www.gfz-potsdam.de > News > GFZ Publications

Jan M. Hagedoorn, Hans Greiner-Mai, Ludwig Ballani

Core-Mantle Coupling

Part III:
Gravitational coupling torques

Scientific Technical Report STR12-01

Jan M. Hagedoorn
Hans Greiner-Mai
Ludwig Ballani

Core-Mantle Coupling
Part III:
Gravitational coupling torques

Scientific Technical Report STR12/01

Table of Contents

Table of Contents i

1 Introduction 1
1.1 Introduction to this Scientific Technical Report . . . 1
1.2 Basic concept . . . 1

2 Gravitational coupling torques 3
2.1 Basic equations . . . 3
2.2 Surface integral of the gravitational coupling torque . . . 3
2.3 Spherical harmonic representation . . . 4
2.3.1 SH representation of different field quantities . . . 4
2.3.2 Rotation of the SH representation of the ICB . . . 5
2.4 Cartesian components of the gravitational coupling torque . . . 6
2.4.1 Axial coupling torque . . . 6
2.4.2 Non-axial coupling torques . . . 6
2.4.3 Uniform coupling coefficient . . . 7

3 Internal gravitational potential 9
3.1 Geometrical settings for the determination of the gravitational potential at the inner-core boundary . . . 9
3.2 Internal gravitational potential for layers of constant density . . . 9
3.2.1 Contribution from below the ICB . . . 10
3.2.2 Contribution from above the ICB . . . 11
3.2.3 Combined SH coefficients of the gravitational potential . . . 13
3.3 Determination of the p-th power of the radial distance . . . 14
3.4 Internal gravity potential . . . 15

4 Examples of gravitational coupling torques 17
4.1 Simplified example for misaligned ellipsoid . . . 17
4.2 Consideration of CMB topography models . . . 19
4.2.1 Gravitational potential at the ICB . . . 19
4.2.2 Gravitational coupling torques for elliptical ICB . . . 21

References 25

A Spherical harmonics and related derivations 27
A.1 Definition of scalar spherical harmonics . . . 27
A.2 Vector spherical harmonics . . . 28

B Additional derivation for the calculation of the torques 29
B.1 Surface integral with respect to the unit sphere . . . 29
B.2 Derivation of the combined coefficient . . . 29
B.3 Derivation of the Cartesian components . . . 30
B.4 Derivation of equatorial coupling torque . . . 30

C Solution for the j-depending integral in eq. (3.24) 33

D Concept of derivation of the pth power of the radial distance 37

List of symbols 39

1 Introduction

1.1 Introduction to this Scientific Technical Report (STR)

In recent years a whole series of reports (STR) was published, which present a consistent theoretical description of the individual core-mantle coupling torques. In addition to a brief summary of the derivation of the theoretical description, the concept for the numerical determination was presented. The published STR treated the electromagnetic (Hagedoorn & Greiner-Mai, 2008) and the topographic core-mantle coupling (Greiner-Mai & Hagedoorn, 2008). Like in the previous STR, our aim is to derive a theoretical description of the gravitational coupling torques acting between the solid inner core and the Earth's mantle in a semi-analytical form, based on spherical harmonic representation.

The presented approach is based on certain assumptions, which are explained in the following paragraph. In Sec. 2, we present the derivation of the analytical expression for the gravitational coupling torques and the related Cartesian components. Moreover, we illustrate the determination of the orientated surface element for an arbitrary inner-core boundary (ICB) topography. For the calculation of the gravitational coupling torque, we need also to determine the gravitational potential of the mass distribution in the Earth's interior. This problem is discussed in detail in Sec. 3. In Sec. 4 are presented a simplified theoretical description and its application under additional assumptions as well as the estimation of the gravitational coupling torques for a set of certain core-mantle boundary topographies which deviate from elliptical shape.

1.2 Basic concept for the determination of the gravitational coupling torque

The density structure of the Earth's mantle deviates from a simple radially dependent stratification, which causes an inhomogeneous gravitational potential. The geometry of the solid inner core is not a sphere and, therefore, the inhomogeneous gravitational potential of the Earth's mantle causes a gravitational torque onto the inner core (IC).

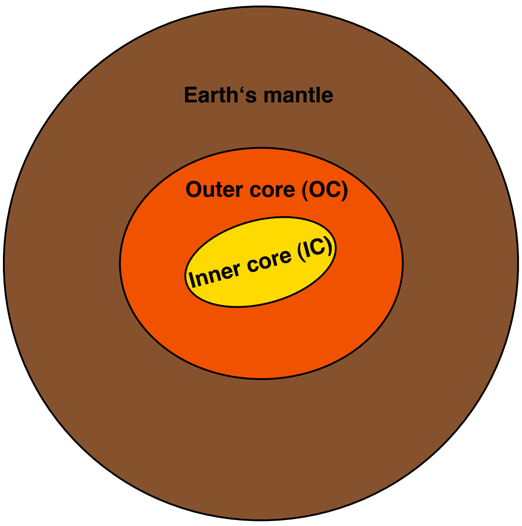


Figure 1.1: Sketch of the basic geometrical set-up for the prescribed model structure of the Earth. Details are given in the text.

The basic geometrical set-up for the description of the gravitational coupling torque is shown in Fig. 1.1 . Beside the inhomogeneous density stratification in the Earth's mantle, constant densities for the fluid outer core (OC), ρ^{OC}, and the solid inner core (IC), ρ^{IC}, are assumed. The core-mantle boundary (CMB) between the Earth's mantle and the (outer) core, and the inner-core boundary (ICB) between fluid and solid cores are in principle non-regular surfaces, which deviate from spherical or ellipsoidal surfaces. To describe this derivation, we choose a mantle-fixed coordinate system. Any relative rotation of the IC with respect to the Earth's mantle yields to a time-dependent ICB, $S_{\text{ICB}}(\Omega, t)$. In general, the inner core should take a position in which the gravitational torque vanishes.

Any displacement of the IC out of this position yields to a torque driving the IC back to this position, which is counter balanced by a similar torque acting on the Earth's mantle. The widely discussed oscillation of the inner core (see e.g. Buffett, 1996; Dumberry, 2010) is also based on this basic physical principle. In addition to those geometrical settings, we assume for the derivations in this report that any change of the gravity potential do not lead to a redistribution of the masses within the particular components considered (e.g. inner core), i.e. we ignore any dynamic effects caused by the self-gravitation. For a more realistic model, any change of the gravity potential leads also to a deformation of a (visco-)elastic body. If the actual density distribution and the shape of the related interfaces are known, the presented formulae for the calculation of the gravitational coupling torques could be applied to compute the instantaneous acting gravitational torque.

Derivation of the gravitational coupling torques | 2

2.1 Basic equation for the gravitational coupling torque

In this study, we do not consider external gravitational forcing by the sun, moon or other planetary bodies. The gravitational coupling torque then describes the torque which is caused by the attraction of the mass of the inner core by the gravitational potential due to the mass of the Earth's mantel and outer core. The general integral form of the gravitational torque on the inner core is given by (e.g. Smylie et al., 1984)

$$\boldsymbol{L} = -\int\limits_{V_{\text{IC}}(t)} \left[\boldsymbol{r} \times \rho^{\text{IC}} \nabla\phi(\boldsymbol{r})\right] \mathrm{d}V_{\text{IC}}(t). \tag{2.1}$$

In addition, any displacement of the inner core (which deviates in general from a rotational ellipsoid) relative to the mantle creates a pressure torque on the ICB due to the hydrostatic fluid pressure of the outer core. The consideration of this fluid pressure torque leads for a rigid inner core to (e.g. Dumberry, 2008)

$$\boldsymbol{L} = -\int\limits_{V_{\text{IC}}(t)} \left[\boldsymbol{r} \times \left(\rho^{\text{IC}} - \rho^{\text{OC}}\right) \nabla\phi(\boldsymbol{r})\right] \mathrm{d}V_{\text{IC}}(t). \tag{2.2}$$

We introduce the density difference, $\rho^{\Delta} = \rho^{\text{IC}} - \rho^{\text{OC}}$, because we assume a homogeneous inner-core density ρ^{IC} and a fluid layer with homogeneous density ρ^{OC}, which surrounds the solid inner core. Here, the integration volume, $V_{\text{IC}}(t)$, is time-dependent, expressed by a relative rotation of the inner core in the chosen mantle-fixed coordinate system (see Sec. 2.3.2).

2.2 Derivativon of the surface integral of the gravitational coupling torque

In eq. (2.2), we give the basic volumen integral expression for the gravitational coupling torque, which reads for the described assumption of a difference density, ρ^{Δ}, as follows

$$\boldsymbol{L} = -\rho^{\Delta} \int\limits_{V_{\text{IC}}(t)} \boldsymbol{r} \times \nabla\phi(\boldsymbol{r})\, \mathrm{d}V_{\text{IC}}(t)\,,$$

which can be transformed by Gauss integral theorem (e.g. Smirnow, 1964) into the surface integral

$$\boldsymbol{L} = -\rho^{\Delta} \int\limits_{\boldsymbol{S}_{\text{ICB}}(t)} \boldsymbol{r} \times \phi(\boldsymbol{r})\, \mathrm{d}\boldsymbol{S}_{\text{ICB}}(t)\,,$$

where $\mathrm{d}\boldsymbol{S}_{\text{ICB}}(t)$ is the orientated and time dependent surface element of S_{ICB}, which is the enclosing surface of the volume $V_{\text{ICB}}(t)$.

For the determination of this surface integral, we follow Heuser (1993, Sec. 208) to substitute the integration domain of the orientated surface $\boldsymbol{S}_{\text{ICB}}$ by the unit sphere, Ω. Therefore, we have to consider the surface normal $\boldsymbol{N}$, defined by

$$\boldsymbol{N}(\Omega, t) = \frac{\partial \boldsymbol{S}_{\text{ICB}}(\Omega, t)}{\partial\vartheta} \times \frac{\partial \boldsymbol{S}_{\text{ICB}}(\Omega, t)}{\partial\varphi}\,, \tag{2.3}$$

where the surface of the inner core is given by

$$\boldsymbol{S}_{\text{ICB}}(\Omega,t) = \boldsymbol{e}_r a_{\text{ICB}} E(\Omega,t)\,. \tag{2.4}$$

Here, $\boldsymbol{e}_r$ is the unit basis vector in radial direction in spherical coordinates and the scalar function $E(\Omega,t)$ describes the departure of the inner core from a sphere with radius a_{ICB}. For a simplified notation, we suppress the variable t in the following derivation. In the surface integral of the gravitational coupling torque, we can now substitute the integration domain $\boldsymbol{S}_{\text{ICB}}$ by its defining space Ω by considering eq. (2.3)

$$\boldsymbol{L} = -\rho^{\Delta} \int\limits_{\Omega} \phi(\boldsymbol{r}) \left[\boldsymbol{r}_{\text{ICB}}(\Omega) \times \boldsymbol{N}(\Omega)\right] \mathrm{d}\vartheta \mathrm{d}\varphi\,. \tag{2.5}$$

Applying the angular partial derivatives on the orientated surface $\boldsymbol{S}_{\text{ICB}}$, which are summarized in Appendix B.1, leads to the following expression for the gravitational coupling torque

$$\boldsymbol{L} = \rho^{\Delta} {a_{\text{ICB}}}^3 \int\limits_{\Omega} E(\Omega)\, E(\Omega)\, \phi(\Omega) \left[\boldsymbol{e}_r \times \nabla_{\Omega}\, E(\Omega)\right] \mathrm{d}\Omega\,. \tag{2.6}$$

This way, we have reduced the problem of solving an integral over an arbitrary surface to the integration of a product of scalar function and a vector differential operator over the unit sphere. For the further derivation, we express all scalar field quantities by spherical harmonics and use their orthogonality to solve the integral. This is shown in the following Sec. 2.3.

2.3 Spherical harmonic representation

2.3.1 SH representation of different field quantities

In eq. (2.6), the gravitational coupling torque is given by a surface integral over the time-dependent ICB and the gravitational potential. The scalar function $E(\Omega)$, defined in eq. (2.4), and the gravitational potential $\phi(\Omega)$ at the ICB are represented by spherical harmonics (SH)

$$E(\Omega) = \sum_{jm} E_{jm}(t)\, Y_{jm}(\Omega)\,, \tag{2.7}$$

$$\phi(\Omega) = \sum_{jm} \Phi_{jm}(t)\, Y_{jm}(\Omega)\,. \tag{2.8}$$

Considering the definition of the vector spherical harmonic $\boldsymbol{S}^{(0)}_{jm}(\Omega)$ given in eq. (A.18) the gravitational coupling torque can be expressed by

$$\boldsymbol{L} = \rho^{\Delta} a_{\text{ICB}}^3 \int\limits_{\Omega} \Big(\sum_{j_1 m_1} E_{j_1 m_1} Y_{j_1 m_1}(\Omega)\Big)\Big(\sum_{j_2 m_2} E_{j_2 m_2} Y_{j_2 m_2}(\Omega)\Big)\Big(\sum_{j_3 m_3} \Phi_{j_3 m_3} Y_{j_3 m_3}(\Omega)\Big)\Big(\sum_{jm} E_{jm} \boldsymbol{S}^{(0)}_{jm}(\Omega)\Big) \mathrm{d}\Omega\,, \tag{2.9}$$

Moreover, it is possible to combine the product of the first three coefficients and related spherical harmonic basis functions and define a new combined coefficient, whose derivation is summarized in Appendix B.2:

$$\Theta_{j'm'} = \sum_{\substack{j_1 m_1 j_2 m_2 \\ j_3 m_3 j_4 m_4}} E_{j_1 m_1} E_{j_2 m_2} \Phi_{j_3 m_3} \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{(4\pi)^2 (2j'+1)}} \cdot \mathbf{C}^{j_4 0}_{j_1 0\, j_2 0}\, \mathbf{C}^{j' 0}_{j_4 0\, j_3 0}\, \mathbf{C}^{j_4 m_4}_{j_1 m_1\, j_2 m_2}\, \mathbf{C}^{j' m'}_{j_4 m_4\, j_3 m_3}\,. \tag{2.10}$$

Here, **C** denotes the Clebsch-Gordan coefficients as introduced in Varshalovich et al. (1989, Sec. 8.1). The resulting expression for the gravitational coupling torque is reduced to the integral of the product of a scalar and vector spherical harmonic basis function:

$$\boldsymbol{L} = \rho^{\Delta} a_{\text{ICB}}^3 \sum_{j'm'\, jm} \Theta_{j'm'}\, E_{jm} \int\limits_{\Omega} Y_{j'm'}(\Omega)\, \boldsymbol{S}^{(0)}_{jm}(\Omega)\, \mathrm{d}\Omega\,. \tag{2.11}$$

It is possible to solve this remaining integral analytically. For our investigation of the gravitational coupling torque, it is necessary to transform this expression from the spherical coordinates into Cartesian components of the coupling torque, which is prescribed in Sec. 2.4. Prior to this derivation, we have to provide a relation between SH coefficient E_{jm} of the ICB topography in the reference state and in another rotated state for an arbitrary new orientation.

2.3.2 Rotation of the SH representation of the ICB for arbitrary Euler angles

In eqs. (2.10)–(2.11) the time-dependent topography of the ICB is represented by SH coefficients, where the time-dependence is realized by a solely rotation of the ICB topography with respect to the mantle. This rotation is prescribed by the three Euler angles, $\alpha(t)$,$\beta(t)$ and $\gamma(t)$, which describe the rotation of the mantle-fixed coordinate system, $C_{\mathsf{M}}\{x,y,z\}$, into the inner-core-fixed coordinate system, $C_{\mathsf{I}}\{X,Y,Z\}$, as illustrated in Fig. 2.1. In $C_{\mathsf{I}}\{X,Y,Z\}$ the ICB is represented by

$$S_{\mathsf{ICB}} = a_{\mathsf{S}} \sum_{jm} E_{\mathsf{I}jm}\, Y_{jm}(\Omega_{\mathsf{I}})\,. \qquad (2.12)$$

According to (Varshalovich et al., 1989, Sec. 5.5.1 Eq. (1)), we can express the SH in $C_{\mathsf{I}}\{X,Y,Z\}$ by

$$Y_{jm}(\Omega_{\mathsf{I}}) = \sum_{n} \mathbf{D}^{j}_{nm}(\alpha(t),\beta(t),\gamma(t))\, Y_{jn}(\Omega_{\mathsf{M}})\,, \qquad (2.13)$$

where $Y_{jn}(\Omega_{\mathsf{M}})$ are the scalar SH defined in the mantle-fixed coordinate system $C_{\mathsf{M}}\{x,y,z\}$. Moreover, we use here the Wigner D-function, defined by (Varshalovich et al., 1989, Sec. 4.3 eq. (1))

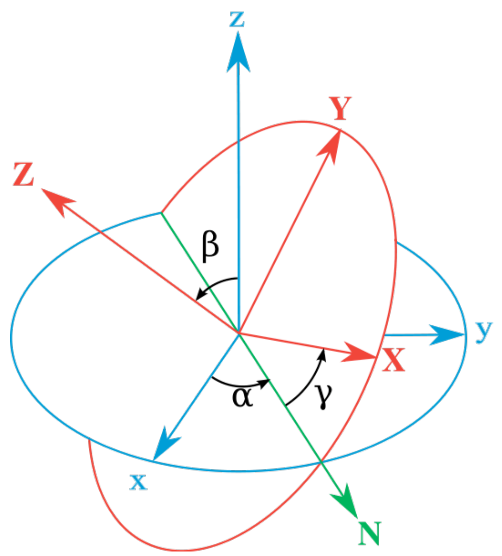


Figure 2.1: Illustration of the Euler angles α, β and γ, which rotate the mantle-fixed coordinate system $C_{\mathsf{M}}\{x,y,z\}$ into the inner-core-fixed coordinate system $C_{\mathsf{I}}\{X,Y,Z\}$.

$$\mathbf{D}^{j}_{nm}(\alpha(t),\beta(t),\gamma(t)) = e^{-i\,n\alpha(t)}\, \mathbf{d}^{j}_{nm}(\beta(t))\, e^{-i\,m\gamma(t)}. \qquad (2.14)$$

The explicit form for the function $\mathbf{d}^{j}_{nm}$ is given by (Varshalovich et al., 1989, Sec. 4.3 Eq. (2))

$$\mathbf{d}^{j}_{nm}(\beta(t)) = (-1)^{(j-m)} \sqrt{(j+n)!\,(j-n)!\,(j+m)!\,(j-m)!} \cdot \sum_{k} (-1)^{k} \frac{\left(\cos\frac{\beta(t)}{2}\right)^{(n+m+2k)} \left(\sin\frac{\beta(t)}{2}\right)^{(2j-n-m-2k)}}{k!\,(j-n-k)!\,(j-m-k)!\,(n+m+k)!}. \qquad (2.15)$$

The index k in the eq. (2.15) runs over all integer values for which the arguments of the four factorials in the nominator are non-negative. For each index combination of j,m and n, there exist $(N+1)$ terms, where $N = \min\{(j+m),(j-m),(j+n),(j-n)\}$.

We can now express the ICB in eq. (2.12) in terms of SH defined in $C_{\mathsf{M}}\{x,y,z\}$, which leads to

$$S_{\mathsf{ICB}} = a_{\mathsf{S}} \sum_{jm} E_{\mathsf{I}jm} \sum_{n} \mathbf{D}^{j}_{nm}(\alpha(t),\beta(t),\gamma(t)) Y_{jn}(\Omega_{\mathsf{M}})\,,$$
$$S_{\mathsf{ICB}} = a_{\mathsf{S}} \sum_{jn} E_{\mathsf{M}}(t)_{jn}\, Y_{jn}(\Omega_{\mathsf{M}})\,. \qquad (2.16)$$

The time-dependent SH coefficients of the ICB in the mantle-fixed coordinate system $C_{\mathsf{M}}\{x,y,z\}$ are

$$E_{\mathsf{M}}(t)_{jn} = \sum_{m} E_{\mathsf{I}jm}\, \mathbf{D}^{j}_{nm}(\alpha(t),\beta(t),\gamma(t))\,,$$
$$E_{\mathsf{M}}(t)_{jn} = \sum_{m} E_{\mathsf{I}jm}\, e^{-i\,n\alpha(t)}\, \mathbf{d}^{j}_{nm}(\beta(t))\, e^{-i\,m\gamma(t)}. \tag{2.17}$$

The relation in eq. (2.17) allows us to represent the time dependency of the ICB topography in the mantle fixed coordinate system by the Euler angles and to calculate the necessary SH coefficients $E_{\mathsf{M}}(t)_{jn}$, required in eqs. (2.10)–(2.11).

2.4 Cartesian components of the gravitational coupling torque

Based on the definition of the vector spherical harmonic basis function $\boldsymbol{S}^{(0)}_{jm}(\Omega)$, given in eq. (A.18), and the transformation between spherical and Cartesian coordinates, given by

$$\boldsymbol{e}_\vartheta = \boldsymbol{e}_x \cos\vartheta\cos\varphi + \boldsymbol{e}_y \cos\vartheta\sin\varphi - \boldsymbol{e}_z \sin\vartheta, \tag{2.18}$$
$$\boldsymbol{e}_\varphi = -\boldsymbol{e}_x \sin\varphi + \boldsymbol{e}_y \cos\varphi, \tag{2.19}$$

we can derive the Cartesian components of the gravitational coupling torque according to eq. (2.11):

$$L_x = \rho^{\Delta} a^3_{\mathsf{ICB}} \sum_{j'm'\,jm} \Theta_{j'm'}\, E_{jm} \int\limits_{\Omega} Y_{j'm'}(\Omega)\left[-\sin\varphi\,\frac{\partial}{\partial\vartheta}Y_{jm}(\Omega) - \cos\varphi\,\cot\vartheta\,\frac{\partial}{\partial\varphi}Y_{jm}(\Omega)\right]\mathrm{d}\Omega\,, \tag{2.20}$$
$$L_y = \rho^{\Delta} a^3_{\mathsf{ICB}} \sum_{j'm'\,jm} \Theta_{j'm'}\, E_{jm} \int\limits_{\Omega} Y_{j'm'}(\Omega)\left[\cos\varphi\,\frac{\partial}{\partial\vartheta}Y_{jm}(\Omega) - \sin\varphi\,\cot\vartheta\,\frac{\partial}{\partial\varphi}Y_{jm}(\Omega)\right]\mathrm{d}\Omega\,, \tag{2.21}$$
$$L_z = \rho^{\Delta} a^3_{\mathsf{ICB}} \sum_{j'm'\,jm} \Theta_{j'm'}\, E_{jm} \int\limits_{\Omega} Y_{j'm'}(\Omega)\,\frac{\partial}{\partial\varphi}Y_{jm}(\Omega)\,\mathrm{d}\Omega\,. \tag{2.22}$$

More details of the derivation are given in Appendix B.3. The further derivation is split into the axial (z) and the equatorial components, summarized in the following sub-sections.

2.4.1 Axial component of the gravitational coupling torque

Considering the partial derivative with respect of φ in eq. (2.22), which is given in eq. (A.13), leads to the expression

$$L_z = i\,\rho^{\Delta} {a_{\mathsf{ICB}}}^3 \sum_{j'm'\,jm} m\,\Theta_{j'm'}\, E_{jm} \int\limits_{\Omega} Y_{j'm'}(\Omega)\, Y_{jm}(\Omega)\,\mathrm{d}\Omega\,.$$

We apply now the relation between spherical harmonics and its complex conjugate, as given in eq. (A.7), and the relation for the coefficients, which allow us to solve the surface integral by the orthonormality condition (A.6). We find the analytical expression for the axial component of the gravitational coupling torque

$$L_z = -i\,\rho^{\Delta} {a_{\mathsf{ICB}}}^3 \sum_{jm} m\,\Theta_{jm}\, E^*_{jm}\,. \tag{2.23}$$

The superscript $*$ denotes the complex conjugate of the related quantity (see eq. (A.7) in Appendix A.1).

2.4.2 Equatorial components of the gravitational coupling torque

In the theoretical description of the variation of the Earth's rotation, it is common to combine the non-axial components, the Cartesian x- and y-component by the following complex quantity:

$$L = L_x + i\,L_y\,. \tag{2.24}$$

According to this rule, we combine eqs. (2.20) and (2.21) to

$$L = -\rho^{\Delta} {a_{\text{ICB}}}^3 \sum_{j'm'\,jm} \Theta_{j'm'}\, E_{jm} \int_{\Omega} Y_{j'm'}(\Omega) \Big[\big(\sin\varphi - i\,\cos\varphi\big) \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) + \big(\cos\varphi + i\,\sin\varphi\big) \cot\vartheta \frac{\partial}{\partial\varphi} Y_{jm}(\Omega) \Big] \,\mathrm{d}\Omega$$

For further simplification of this expression, we transform the complex combination of trigonometric functions into exponential functions according to

$$e^{i\,\varphi} = \cos\varphi + i\,\sin\varphi\,, \tag{2.25}$$

$$-i\,e^{i\,\varphi} = \sin\varphi - i\,\cos\varphi\,, \tag{2.26}$$

which leads to

$$L = -\rho^{\Delta} {a_{\text{ICB}}}^3 \sum_{j'm'\,jm} \Theta_{j'm'}\, E_{jm} \int_{\Omega} Y_{j'm'}(\Omega) \Big[-i\,e^{i\,\varphi} \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) + e^{i\,\varphi} \cot\vartheta \frac{\partial}{\partial\varphi} Y_{jm}(\Omega) \Big] \,\mathrm{d}\Omega\,. \tag{2.27}$$

Applying the relations for partial derivatives of spherical harmonics in eqs. (A.13)–(A.14) and the combination with $\cot\vartheta$ in eq. (A.15) leads after some algebraic transformations, which are summarized in Appendix B.4, to the expression

$$L = i\,\rho^{\Delta} {a_{\text{ICB}}}^3 \sum_{j'm'\,jm} \sqrt{j(j+1) - m(m+1)}\, \Theta_{j'm'}\, E_{jm} \int_{\Omega} Y_{j'm'}(\Omega)\, Y_{j(m+1)}(\Omega) \,\mathrm{d}\Omega\,. \tag{2.28}$$

The surface integral can again solved analytically with help of the orthonormality condition (A.6). In addition, the relation for complex conjugate spherical harmonics in eq. (A.7) is considered and the final expression reads:

$$L = -i\,\rho^{\Delta} {a_{\text{ICB}}}^3 \sum_{jm} \sqrt{j(j+1) - m(m+1)}\, \Theta_{jm}\, E^{*}_{j(m+1)}\,. \tag{2.29}$$

2.4.3 Uniform coupling coefficient

The expressions for the axial and equatorial coupling torques given by eqs. (2.23) and (2.29) combine the product coefficients Θ_{jm} (see eq. (2.10) and Sec. **??**) with coefficients of the ICB E_{jm}. The idea for the following derivation is based on a simplification for the implementation by the construction of a uniform coupling coefficient for both coupling torque expressions. For this purpose, we split up the coefficient Θ_{jm} again according to eq. (2.10) and define the coupling coefficient:

$$\mathbf{\Gamma}^{jm}_{j_1m_1j_2m_2j_3m_3} = \sum_{j'=0}^{j'_{\text{max}}} \sum_{m'=-j'}^{j'} \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{(4\pi)^2(2j+1)}}\, \mathbf{C}^{j'0}_{j_10\,j_20}\, \mathbf{C}^{j0}_{j'0\,j_30}\, \mathbf{C}^{j'm'}_{j_1m_1\,j_2m_2}\, \mathbf{C}^{jm}_{j'm'\,j_3m_3}\,. \tag{2.30}$$

With this coupling coefficient, we can reformulate the eqs. (2.29) and (2.23) into

$$L = -i\,\rho^{\Delta} {a_{\text{ICB}}}^3 \sum_{\substack{jm\,\forall j\neq m \\ j_1m_1j_2m_2j_3m_3}} \sqrt{j(j+1) - m(m+1)}\, E_{j_1m_1}\, E_{j_2m_2}\, \Phi_{j_3m_3}\, E^{*}_{j(m+1)}\, \mathbf{\Gamma}^{jm}_{j_1m_1j_2m_2j_3m_3}\,, \tag{2.31}$$

$$L_z = -i\,\rho^{\Delta} {a_{\text{ICB}}}^3 \sum_{\substack{jm\,\forall m\neq 0 \\ j_1m_1j_2m_2j_3m_3}} m\, E_{j_1m_1}\, E_{j_2m_2}\, \Phi_{j_3m_3}\, E^{*}_{jm}\, \mathbf{\Gamma}^{jm}_{j_1m_1j_2m_2j_3m_3}\,. \tag{2.32}$$

This expressions enable us to implement the gravitational coupling torque for the same coupling coefficients in the equatorial and axial components. Moreover, if the inner core rotates relative to the mantle the coefficients of its surface, E_{jm}, and the gravitational potential, represented by Φ_{jm}, are time dependent, but not the uniform coupling coefficients $\mathbf{\Gamma}$. Therefore, the uniform coupling coefficients can be calculated by eq. (2.30) once and can be used for the calculation of the whole time series.

3 Internal gravitational potential

3.1 Geometrical settings for the determination of the gravitational potential at the inner-core boundary

A layered density structure is assumed to derive the internal gravitational potential. Between certain boundaries, the layer B_n has a constant volume mass density, $\rho_n = \mathrm{const.}$ These surfaces reflect the depth-dependent hydrostatical flattening, i.e. the degree-two SH coefficient is determined by the flattening, but all other SH coefficients are arbitrary. There exist two special boundary surfaces, the core-mantle boundary (CMB) and the inner-core boundary (ICB), which are physical boundaries in the Earth. The largest density jump in the Earth is located at the CMB of about $4400\,\mathrm{kg\,m^{-3}}$. Moreover, the outer core is fluid, whereas the Earth's mantle is assumed here to be solid. The related density jump from the liquid outer core to the solid inner core at the ICB is much smaller (about $600\,\mathrm{kg\,m^{-3}}$). In Fig. 3.1, this geometrical settings are illustrated by introducing such boundary surfaces ∂B_n between layers of constant density. For the determination of the internal gravitational potential, we use a similar SH representation of these boundaries as given for the ICB in eq. (2.12)

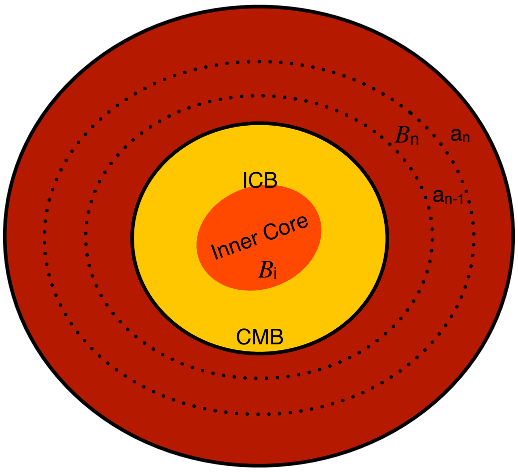


Figure 3.1: Geometrical settings for the determination of the internal gravitational potential at the ICB. For further details, we refer to the text.

$$\partial B_n(r,\Omega) = a_n \sum_{jm} B_{jm}(a_n)\, Y_{jm}(\Omega)\,, \tag{3.1}$$

where a_n is the reference radius of the related boundary. The related SH coefficients of this representation are denoted by $B_{jm}(a_n)$.

3.2 Internal gravitational potential for layers of constant density

The gravitational potential $\phi(P)$ at the point P is given by

$$\phi(P) = -G \int_B K(P,P')\,\rho(P')\,\mathrm{d}V'\,, \tag{3.2}$$

where G denotes the gravitational constant and $\rho(P')$ is the density at the integration point P'. The integration kernel, $K(P,P')$, which is the reciprocal of the distance between P and P', is expressed by SH base functions differently for $r > r'$ and $r < r'$. For the further derivation, we split the integration volume B into two domains, where B_i is the volume of the inner core and B_o is the remaining volume without the dividing ICB, which reads

$$B = B_\mathrm{i} \cup \partial B_\mathrm{ICB} \cup B_\mathrm{o}\,. \tag{3.3}$$

We choose such a splitting of the integration volume, because we want to determine the gravitational potential at the ICB and for any point P of the ICB we find

$$P \in \partial B_{\text{ICB}} \quad \Rightarrow \quad P \notin B_{\text{i}} \quad \wedge \quad P \notin B_{\text{o}} \,. \tag{3.4}$$

Moreover, for the integration point P' we find the relations

$$P' \in B_{\text{i}} \quad \Rightarrow \quad r' < r \,, \tag{3.5}$$

$$P' \in B_{\text{o}} \quad \Rightarrow \quad r' > r \,. \tag{3.6}$$

For the kernel, we find for each of the domains a different representation by SH base functions (e.g. Arfken, 1985, eqs. 12.207 ff):

$$K(P,P') = \frac{1}{r} \sum_{j=0}^{\infty} \left(\frac{r'}{r}\right)^j \frac{4\pi}{2j+1} \sum_{m=-j}^{j} Y^*_{jm}(\Omega') Y_{jm}(\Omega) \qquad \forall\, P' \in B_{\text{i}} \,, \tag{3.7}$$

$$K(P,P') = \frac{1}{r'} \sum_{j=0}^{\infty} \left(\frac{r'}{r}\right)^j \frac{4\pi}{2j+1} \sum_{m=-j}^{j} Y^*_{jm}(\Omega') Y_{jm}(\Omega) \qquad \forall\, P' \in B_{\text{o}} \,. \tag{3.8}$$

For each of the domains, we determine the contribution to the gravitational potential

$$\phi(P) = \phi^{\text{i}}(P) + \phi^{\text{o}}(P) \,, \tag{3.9}$$

and represent this contributions by SH basis functions and related coefficients according to

$$\phi^{\text{i}}(P) = -\frac{G}{r} \sum_{jm} \left(\frac{a_{\text{ICB}}}{r}\right)^j A^{\text{i}}_{jm} Y_{jm}(\Omega) \qquad \forall\, P' \in B_{\text{i}} \,, \tag{3.10}$$

$$\phi^{\text{o}}(P) = -G \sum_{jm} \left(\frac{r}{a_{\text{ICB}}}\right)^j A^{\text{o}}_{jm} Y_{jm}(\Omega) \qquad \forall\, P' \in B_{\text{o}} \,. \tag{3.11}$$

In the following subsections, we present the derivation for the determination of the coefficients A^{i}_{jm} and A^{o}_{jm}.

3.2.1 Contribution from below the ICB

The contribution from the mass distribution in B_{i}, i.e. below the ICB, to the gravitational potential is prescribed by eq. (3.10), which defines also the SH coefficient A^{i}_{jm} by

$$A^{\text{i}}_{jm} = \frac{4\pi}{2j+1} a_{\text{ICB}}^{-j} \int\limits_{B_{\text{i}}} r'^j \rho(P')\, Y^*_{jm}(\Omega')\, \mathrm{d}V' \,. \tag{3.12}$$

We reformulate this expression in spherical coordinates and introduce the following substitution for the angular-depending radial distance r',

$$r' = r_{\text{ICB}}(\Omega')\, x \quad , \quad \mathrm{d}r' = r_{\text{ICB}}(\Omega') \mathrm{d}x \,, \tag{3.13}$$

which leads to

$$A^{\text{i}}_{jm} = \frac{4\pi}{2j+1} a_{\text{ICB}}^{-j} \int\limits_{\Omega} r_{\text{ICB}}^{j+3}(\Omega') Y^*_{jm}(\Omega') \int\limits_{x=0}^{1} \rho(x,\Omega')\, x^{j+2}\, \mathrm{d}x\, \mathrm{d}\Omega' \,. \tag{3.14}$$

Furthermore, we assume a constant density for the whole inner core, i.e. $\rho(x,\Omega') = \rho_{\text{ICB}} = \text{const}$. Hence, the integral over x can be solved analytically:

$$A^{\text{i}}_{jm} = \frac{4\pi}{2j+1} a_{\text{ICB}}^{-j} \int\limits_{\Omega} r_{\text{ICB}}^{j+3}(\Omega') Y^*_{jm}(\Omega')\, \rho_{\text{ICB}} \left[\frac{1}{j+3}\, x^{j+3}\right]_0^1 \mathrm{d}\Omega' \,. \tag{3.15}$$

This equation shows that we need an expression for a power of r_{ICB} for the further evaluation. Similarly to eq. (3.1), the ICB is represented by

$$r_{\text{ICB}}(\Omega) = a_{\text{ICB}} \sum_{jm} E_{jm}\, Y_{jm}(\Omega)\,. \tag{3.16}$$

Based on this representation, we derive an approximation of the p th power of the radial distance by

$$r^p_{\text{ICB}}(\Omega) \approx a^p_{\text{ICB}} \sum_{jm} E^{\langle p\rangle}_{jm}\, Y_{jm}(\Omega)\,, \tag{3.17}$$

where the related SH coefficients $E^{\langle p\rangle}_{jm}$ are defined in Sec. 3.3. For the integral in eq. (3.15), we find by applying the condition of orthonormal SH basis functions (A.6)

$$A^{\text{i}}_{jm} = \frac{4\pi}{(2j+1)(j+3)}\, a^3_{\text{ICB}}\, \rho_{\text{ICB}}\, E^{\langle j+3\rangle}_{jm}\,. \tag{3.18}$$

The gravitational potential ϕ has to be expressed by SH for the determination of the gravitational coupling torques in eqs. (2.31) and (2.32). Therefore, we have to express also the contribution ϕ^{i} by SH coefficients and to take into account that the potential must be determined on the ICB, i.e. for all $P \in \partial B_{\text{icb}}$. According to eqs. (3.10) and (3.18), we find

$$\phi^{\text{i}}(r_{\text{ICB}},\Omega) = -G \sum_{jm} a^j_{\text{ICB}} \left(a_{\text{ICB}} \sum_{j'm'} E_{j'm'}\, Y_{j'm'}(\Omega) \right)^{-(j+1)} A^{\text{i}}_{jm}\, Y_{jm}(\Omega)\,,$$

which transforms with eq. (3.17) to

$$\phi^{\text{i}}(r_{\text{ICB}},\Omega) = -\frac{G}{a_{\text{ICB}}} \sum_{j_1 m_1 j_2 m_2} A^{\text{i}}_{j_1 m_1}\, E^{\langle -(j_1+1)\rangle}_{j_2 m_2} Y_{j_1 m_1}(\Omega) Y_{j_2 m_2}(\Omega)\,. \tag{3.19}$$

The related SH coefficients of the potential ϕ^{i} at the ICB are determined by

$$\Phi^{\text{i}}_{jm} = -\frac{G}{a_{\text{ICB}}} \sum_{j_1 m_1 j_2 m_2} A^{\text{i}}_{j_1 m_1}\, E^{\langle -(j_1+1)\rangle}_{j_2 m_2} \int_{\Omega} Y_{j_1 m_1}(\Omega)\, Y_{j_2 m_2}(\Omega)\, Y^*_{jm}(\Omega)\, \mathrm{d}\Omega\,.$$

The surface integral has an analytical solution in terms of Clebsh-Gordan coefficients, as given in Varshalovich et al. (1989, see 5.9.1) and applying those leads to

$$\Phi^{\text{i}}_{jm} = -\frac{G}{a_{\text{ICB}}} \sum_{j_1 m_1 j_2 m_2} A^{\text{i}}_{j_1 m_1}\, E^{\langle -(j_1+1)\rangle}_{j_2 m_2} \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j+1)}}\, \mathbf{C}^{j0}_{j_1 0 j_2 0}\, \mathbf{C}^{jm}_{j_1 m_1 j_2 m_2}\,. \tag{3.20}$$

3.2.2 Contribution from above the ICB

The contribution from the mass distribution in B_{o}, i.e. above the ICB, to the gravitational potential is given in eq. (3.11), which defines also the SH coefficients A^{o}_{jm} by

$$A^{\text{o}}_{jm} = \frac{4\pi}{2j+1}\, a^j_{\text{ICB}} \int_{B_{\text{o}}} r'^{-(j+1)}\, \rho(P')\, Y^*_{jm}(\Omega')\mathrm{d}V'\,. \tag{3.21}$$

For the further derivation, we split the volume B_{o} into N sub-domains with constant densities ρ_n in each of the concentric layered domains, as illustrated in Fig. 3.1. The expression for the coefficients A^{o}_{jm} reads then

$$A^{\text{o}}_{jm} = \frac{4\pi}{2j+1}\, a^j_{\text{ICB}} \sum_{n=1}^{N} \rho_n \int_{\Omega} \int_{r_{n-1}(\Omega')}^{r_n(\Omega')} r'^{-(j-1)}(\Omega')\, Y^*_{jm}(\Omega')\, \mathrm{d}r'\mathrm{d}\Omega'\,.$$

To solve the integral in the equation above, we introduce the following substitution for the angular depending radial distance r'

$$r'(\Omega') = (r_n - r_{n-1})x + r_{n-1} \quad , \quad \mathrm{d}r' = (r_n - r_{n-1})\mathrm{d}x \, . \tag{3.22}$$

Applying this substitution for the radial distance leads to

$$A^{\circ}_{jm} = \frac{4\pi}{2j+1}\, a^{j}_{\text{ICB}} \sum_{n=1}^{N} \rho_n \int\limits_{\Omega} \int\limits_{0}^{1} \left[(r_n - r_{n-1})x + r_{n-1}\right]^{-(j-1)} (r_n - r_{n-1})\,\mathrm{d}x\, Y^{*}_{jm}(\Omega')\,\mathrm{d}\Omega' \, . \tag{3.23}$$

The different solutions of the x-depending integral in eq. (3.23) are j-dependent. Therefore, we define

$$A^{\circ}_{jm} = \frac{4\pi}{2j+1}\, a^{j}_{\text{ICB}} \sum_{n=1}^{N} \rho_n\, I_j \, ,$$

with

$$I_j = \int\limits_{\Omega} \int\limits_{0}^{1} \left[(r_n - r_{n-1})x + r_{n-1}\right]^{-(j-1)} (r_n - r_{n-1})\,\mathrm{d}x\, Y^{*}_{jm}(\Omega')\,\mathrm{d}\Omega' \, . \tag{3.24}$$

For $j = 0$ we find the solution (details are given in Appendix C)

$$I_0 = \frac{1}{2} \int\limits_{\Omega} \left[r_n^2 - r_{n-1}^2\right] Y^{*}_{00}(\Omega')\,\mathrm{d}\Omega' \, .$$

The radial distances r_n are angular depending and belong to the boundaries between constant densities $r_n \in \partial B_n$. Its SH representation reads according to eq. (3.1)

$$r_n(\Omega) = a_n \sum_{jm} B_{jm}(a_n) Y_{jm}(\Omega) \, . \tag{3.25}$$

Analogous to eq. (3.17), we approximate the p th power of r_n by

$$r_n^p(\Omega) = a_n^p \sum_{jm} B^{\langle p \rangle}_{jm}(a_n) Y_{jm}(\Omega) \, , \tag{3.26}$$

where the related SH coefficients $B^{\langle p \rangle}_{jm}$ are defined in Sec. 3.3. For I_0, we find by applying the condition of orthonormal SH base functions (A.6)

$$I_0 = \frac{1}{2}\left[a_n^2 B^{\langle 2 \rangle}_{00}(a_n) - a_{n-1}^2 B^{\langle 2 \rangle}_{00}(a_{n-1})\right] .$$

With the last expression for I_0, the zero-degree coefficient reads

$$A^{\circ}_{00} = 2\pi \sum_{n=1}^{N} \rho_n \left[a_n^2 B^{\langle 2 \rangle}_{00}(a_n) - a_{n-1}^2 B^{\langle 2 \rangle}_{00}(a_{n-1})\right] . \tag{3.27}$$

For $j = 1$, the integral I_1 has the analytical solution

$$I_1 = \left[a_n B_{1m}(a_n) - a_{n-1} B_{1m}(a_{n-1})\right] ,$$

which yields

$$A^{\circ}_{1m} = \frac{4\pi}{3}\, a_{\text{ICB}} \sum_{n=1}^{N} \rho_n \left[a_n B_{1m}(a_n) - a_{n-1} B_{1m}(a_{n-1})\right] . \tag{3.28}$$

For the case $j = 2$, an analytical solution for I_2 can not be derived, but an approximate solution, accurate upon the fourth-order of the flattening of the Earth. The related derivation of the approximate solution is summarized in Appendix C. We find for I_2 (see eq. (C.9))

$$I_2 \approx \sum_{p=1}^{4} \frac{(-1)^{p+1}}{p} \left[\left(\frac{1}{B_{00}(a_n) Y_{00}} \right)^p B_{2m}^{\langle p \rangle}(a_n) - \left(\frac{1}{B_{00}(a_{n-1}) Y_{00}} \right)^p B_{2m}^{\langle p \rangle}(a_{n-1}) \right].$$

The degree-two coefficient of the gravitational potential due to the mass distribution in $B_\circ$ reads then

$$A_{2m}^{\circ} = \frac{4\pi}{5}\, a_{\mathsf{ICB}}^2 \sum_{n=1}^{N} \rho_n \sum_{p=1}^{4} \frac{(-1)^{p+1}}{p} \left[\left(\frac{1}{B_{00}(a_n) Y_{00}} \right)^p B_{2m}^{\langle p \rangle}(a_n) - \left(\frac{1}{B_{00}(a_{n-1}) Y_{00}} \right)^p B_{2m}^{\langle p \rangle}(a_{n-1}) \right]. \quad (3.29)$$

For all $j \geq 3$, we can find an analytical solution again for I_j. In Appendix C the derivation is summarized, which leads to eq. (C.10), which reads

$$I_j = -\frac{1}{j-2} \left[a_n^{-(j-2)} B_{jm}^{\langle -(j-2) \rangle}(a_n) - a_{n-1}^{-(j-2)} B_{jm}^{\langle -(j-2) \rangle}(a_{n-1}) \right].$$

Using this analytical expression of I_j for $j \geq 3$, the related SH coefficients reads

$$A_{jm}^{\circ} = -\frac{4\pi}{(2j+1)(j-2)}\, a_{\mathsf{ICB}}^j \sum_{n=1}^{N} \rho_n \left[a_n^{-(j-2)} B_{jm}^{\langle -(j-2) \rangle}(a_n) - a_{n-1}^{-(j-2)} B_{jm}^{\langle -(j-2) \rangle}(a_{n-1}) \right]. \quad (3.30)$$

Based on eq. (3.11), we can calculate the gravitational potential, ϕ°, due to the mass distribution in $B_\circ$ on the ICB for the prescribed geometrical settings. Analogously to eq. (3.20), we derive now the SH coefficient of ϕ° considering the SH representation of the ICB, which is given in eq. (3.16)

$$\phi^\circ(r_{\mathsf{icb}}, \Omega) = -G \sum_{jm} a_{\mathsf{ICB}}^{-j} \left(a_{\mathsf{ICB}} \sum_{j'm'} E_{j'm'}\, Y_{j'm'}(\Omega) \right)^j A_{jm}^{\circ} Y_{jm}(\Omega)\,,$$

and the p-th power of the radial distance, given in eq. (3.17), which leads to

$$\phi^\circ(r_{\mathsf{icb}}, \Omega) = -G \sum_{j_1 m_1 j_2 m_2} A_{j_1 m_1}^{\circ}\, E_{j_2 m_2}^{\langle j_1 \rangle} Y_{j_1 m_1}(\Omega)\, Y_{j_2 m_2}(\Omega)\,. \quad (3.31)$$

The SH coefficients of ϕ° on the ICB are determined by

$$\Phi_{jm}^{\circ} = -G \sum_{j_1 m_1 j_2 m_2} A_{j_1 m_1}^{\circ}\, E_{j_2 m_2}^{\langle j_1 \rangle} \int_{\Omega} Y_{j_1 m_1}(\Omega)\, Y_{j_2 m_2}(\Omega)\, Y_{jm}^*(\Omega)\, \mathrm{d}\Omega\,.$$

We consider here again Varshalovich et al. (1989, eq. (4) §5.9.1), and find a solution in terms of Clebsh-Gordan coefficients

$$\Phi_{jm}^{\circ} = -G \sum_{j_1 m_1 j_2 m_2} A_{j_1 m_1}^{\circ}\, E_{j_2 m_2}^{\langle j_1 \rangle} \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j+1)}}\, \mathbf{C}_{j_1 0 j_2 0}^{j0}\, \mathbf{C}_{j_1 m_1 j_2 m_2}^{jm}\,. \quad (3.32)$$

3.2.3 Combined SH coefficients of the gravitational potential

The summation of the contributions of the gravitational potential from mass distributions below and above the ICB, as it is expressed in eq. (3.9), leads to the combined SH coefficient based on A_{jm}^{i} in eq. (3.18), A_{jm}° in eqs. (3.27)–(3.30), and $E_{jm}^{\langle p \rangle}$ in eq. (3.17),

$$\Phi_{jm} = -G \sum_{j_1 m_1 j_2 m_2} \left[\frac{1}{a_{\mathsf{ICB}}} A_{j_1 m_1}^{\mathsf{i}}\, E_{j_2 m_2}^{\langle -(j_1+1) \rangle} + A_{j_1 m_1}^{\circ}\, E_{j_2 m_2}^{\langle j_1 \rangle} \right] \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j+1)}}\, \mathbf{C}_{j_1 0 j_2 0}^{j0}\, \mathbf{C}_{j_1 m_1 j_2 m_2}^{jm}\,. \quad (3.33)$$

By eq. (3.33), the SH coefficients of the gravitational potential on the ICB are determined, which are required in eqs. (2.31)–(2.32) for the calculation of the gravitational coupling torque components.

3.3 Determination of the p-th power of the radial distance

To determine the p-th power of the radial distance by the coefficients $B_{jm}^{\langle p\rangle}$ given in eq. (3.25) and repeated here

$$r_n^p(\Omega) = a_n^p \sum_{jm} B_{jm}^{\langle p\rangle}(a_n) Y_{jm}(\Omega)\,,$$

we follow here the idea of Pěč & Martinec (1984). Therein, the approximation of the coefficients $B_{jm}^{\langle p\rangle}$ up to the fourth order of the flattening is given in detail. We have summarized the ideas and basic steps of the derivation in the Appendix D. An application for the gravitational potential of an irregular shaped body is presented by Pěč & Martinec (1988). We follow the notation therein and introduce

$$\mathbf{Q}_{j_1m_1j_2m_2}^{jm} = \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j+1)}}\,\mathbf{C}_{j_10j_20}^{jm}\,\mathbf{C}_{j_1m_1j_2m_2}^{jm}\,, \tag{3.34}$$

which realises the expansion of the product of two scalar spherical harmonics

$$Y_{j_1m_1}(\Omega)\,Y_{j_2m_2}(\Omega) = \sum_{jm} \mathbf{Q}_{j_1m_1j_2m_2}^{jm} Y_{jm}(\Omega)\,, \tag{3.35}$$

where **C** denotes the Clebsh-Gordan coefficient as defined in Varshalovich et al. (1989, Sec. 8.1). Based on such combination of Clebsh-Gordan coefficients, $B_{jm}^{\langle p\rangle}$ can be approximated in terms of B_{jm} retaining the terms of the order of magnitude α^4, where α is the flattening at the radial distance. The approximation then reads

$$B_{jm}^{\langle p\rangle}(a_n) \approx \left(\frac{B_{00}}{\sqrt{4\pi}}\right)^p \left[\mathcal{B}^0 + \binom{p}{1}\mathcal{B}^1 + \binom{p}{2}\mathcal{B}^2 + \binom{p}{3}\mathcal{B}^3 + \binom{p}{4}\mathcal{B}^4\right], \tag{3.36}$$

The terms $\mathcal{B}^i$ are defined by

$$\mathcal{B}^0 = \begin{cases} \sqrt{4\pi} & \text{for} \quad j=0 \\ 0 & \text{for} \quad j\neq 0 \end{cases}, \tag{3.37}$$

$$\mathcal{B}^1 = \begin{cases} 0 & \text{for} \quad j=0 \\ \dfrac{\sqrt{4\pi}}{B_{00}}B_{jm} & \text{for} \quad j\neq 0 \end{cases}, \tag{3.38}$$

$$\mathcal{B}^2 = \left(\frac{\sqrt{4\pi}}{B_{00}}\right)^2 \sum_{\substack{j_1m_1j_2m_2\\ j_1,j_2\geq 1}} B_{j_1m_1}\,B_{j_2m_2}\,\mathbf{Q}_{j_1m_1j_2m_2}^{jm} \tag{3.39}$$

$$\mathcal{B}^3 = \left(\frac{\sqrt{4\pi}}{B_{00}}\right)^3 \left[B_{20}^3 \sum_{j_1\geq 1} \mathbf{Q}_{2020}^{j_10}\mathbf{Q}_{j_1020}^{jm} + 3\,B_{20}^2 \sum_{\substack{j_1j_2m_2\\ j_1,j_2\geq 1\\ j_2\neq 2}} B_{j_2m_2}\,\mathbf{Q}_{2020}^{j_10}\mathbf{Q}_{j_10j_2m_2}^{jm}\right], \tag{3.40}$$

$$\mathcal{B}^4 = \left(\frac{\sqrt{4\pi}}{B_{00}}\right)^4 B_{20}^4 \sum_{j_1j_2} \mathbf{Q}_{2020}^{j_10}\mathbf{Q}_{j_1020}^{j_20}\mathbf{Q}_{j_2020}^{jm}\,. \tag{3.41}$$

In eq. (3.36) the parentheses in front of the $\mathcal{B}$-coefficients denote binomial coefficients. The eqs. (3.36)–(3.41) can be implemented into numerical code to approximate the p-th power of the radial distance for any integer p.

3.4 Internal gravity potential

In the gravitational coupling torque the acting potential ϕ is the gravity potential. Therefore, we have to consider beside the gravitational potential also the centrifugal potential at the ICB, where this torque is determined. We chose a similar SH representation as for the gravitational potential in eq. (2.8), given by

$$\psi(\boldsymbol{r}) = \sum_{jm} \Psi_{jm}(r) Y_{jm}(\Omega) \,. \tag{3.42}$$

We neglect the time dependency of the centrifugal potential, because its time depending increment is at least six orders of magnitude smaller as the first order contribution. The centrifugal potential is given by (e.g. Kertz, 1995, eq. (6.11)),

$$\psi(\boldsymbol{r}) = -\frac{1}{2}\omega_0^2\, r^2\, \sin^2\vartheta \,, \tag{3.43}$$

which can be expressed by SH basis functions, as defined in eq. (A.1):

$$\psi(\boldsymbol{r}) = -\omega_0^2\, r^2 \sqrt{\frac{4\pi}{9}} \left(Y_{00}(\Omega) - \frac{1}{\sqrt{5}} Y_{20}(\Omega) \right) . \tag{3.44}$$

A comparison of both expressions for the centrifugal potential leads to the following SH coefficients, according to eq. (3.42)

$$\Psi_{00}(r) = -\frac{2}{3}\sqrt{\pi}\omega_0^2 r^2 \,, \tag{3.45}$$

$$\Psi_{20}(r) = \frac{2}{3}\sqrt{\frac{\pi}{5}}\omega_0^2 r^2 \,. \tag{3.46}$$

These two SH coefficients are the only non-zero ones, which have to be added to the related degree and orders SH coefficients of the gravitational potential to achieve the gravity potential.

In Martinec & Hagedoorn (2005, eqs. (94)–(95)) the Eulerian increment of the centrifugal potential is given in dependency of the vector of the variation of the Earth's rotation, $\boldsymbol{m}$. The degree zero and two related SH coefficients read

$$\Psi^{\mathsf{E}}_{00}(r,t) = -\frac{4}{3}\sqrt{\pi}\omega_0^2\, r^2\, m_3(t) \,, \tag{3.47}$$

$$\Psi^{\mathsf{E}}_{20}(r,t) = \frac{4}{3}\sqrt{\frac{\pi}{5}}\omega_0^2\, r^2\, m_3(t) \,. \tag{3.48}$$

The Eulerian increments differ from the related SH coefficients of the centrifugal potential by the factor $2m_3(t)$, where $\mathcal{O}(m_3(t)) = 10^{-7}$, which supports neglecting these contributions, due to the difference to the related coefficients of at least six orders of magnitude.

Examples of gravitational coupling torques | 4

4.1 Simplified example for misaligned ellipsoid

In this section, we derive an analytical expression for the gravitational coupling torque for simplified geometrical settings. The basic assumption for the geometrical setting is the ellipsoidal shape of all boundaries within the Earth with the same origin and a depth-dependent flattening. The density is also depth-dependent and the surfaces of same density are given by the introduced ellipsoids.

The basic idea is to use an approximation for the expression of the normal vector on the elliptical ICB (similar to Sec. 2.1 Greiner-Mai & Hagedoorn, 2008), where the ICB can be described by the following expression

$$\boldsymbol{r}_{\mathsf{ICB}} = \boldsymbol{e}_r \left[a_{\mathsf{ICB}} + \widetilde{E}(\Omega) \right] . \tag{4.1}$$

Herein, $\widetilde{E}$ denotes the deviation from a sphere with the radius a_{ICB}. The normal on this surface can by approximated by (see Greiner-Mai & Hagedoorn, 2008, eqs. (2.4)–(2.10))

$$\boldsymbol{n}_{\mathsf{ICB}} \approx \boldsymbol{e}_r - \frac{1}{a_{\mathsf{ICB}}} \nabla_\Omega \widetilde{E}(\Omega) . \tag{4.2}$$

These approximations of the ICB surface and the normal vector are given with respect to the unit sphere Ω. Thus, the gravitational coupling torque can be prescribed by a surface integral over the sphere Ω, which is derived from eq. (2.5):

$$\boldsymbol{L} \cong -\rho^{\Delta} \int\limits_{\Omega} \boldsymbol{r}_{\mathsf{ICB}} \times \boldsymbol{n} \, \phi(\Omega) \, {a_{\mathsf{ICB}}}^2 \, \mathrm{d}\Omega . \tag{4.3}$$

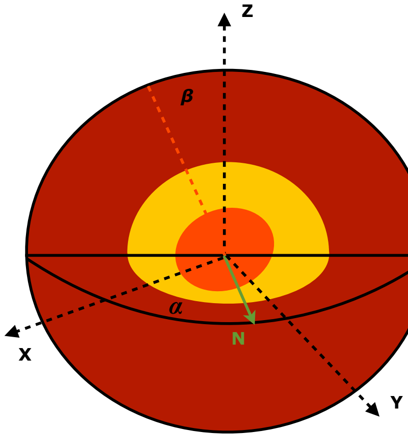


Figure 4.1: Simplified geometrical settings for the approximative description of the gravitational coupling torque in dependency of the angles of misalignment α and β as indicated in the sketch.

Performing the vector product in eq. (4.3) and considering eqs. (4.1) and (4.2) yields

$$\boldsymbol{L} \cong \rho^{\Delta} {a_{\mathsf{ICB}}}^2 \left[\int\limits_{\Omega} \left(\boldsymbol{e}_r \times \nabla_\Omega \widetilde{E}(\Omega) \right) \phi(\Omega) \, \mathrm{d}\Omega + \frac{1}{a_{\mathsf{ICB}}} \int\limits_{\Omega} \widetilde{E}(\Omega) \left(\boldsymbol{e}_r \times \nabla_\Omega \widetilde{E}(\Omega) \right) \phi(\Omega) \, \mathrm{d}\Omega \right] . \tag{4.4}$$

The second term is at least two orders of magnitude smaller ($\sim \widetilde{E}/a_{\mathsf{ICB}}$) then the fist one, considering the approximation in eq. (4.2). Therefore, we derive here an estimation for the first term and neglect the second term. Moreover, we introduce the SH of $\widetilde{E}$ and ϕ by

$$\widetilde{E}(\Omega) = \sum_{jm} \widetilde{E}_{jm} \, Y_{jm}(\Omega) , \tag{4.5}$$

$$\phi(\Omega) = \sum_{jm} \Phi_{jm} \, Y_{jm}(\Omega) . \tag{4.6}$$

We introduce now the Cartesian components of $\boldsymbol{L}$ according to eqs. (2.18) and (2.19), which are given by

$$L_x \cong \rho^{\Delta}\, {a_{\mathsf{ICB}}}^2 \int\limits_{\Omega} \sum_{jm\,kl} \widetilde{E}_{jm}\Phi_{kl} \left[-\sin\varphi \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) - \frac{1}{\sin\vartheta}\cos\vartheta\cos\varphi Y_{jm}(\Omega) \right] Y_{kl}(\Omega)\mathrm{d}\Omega\,, \tag{4.7}$$

$$L_y \cong \rho^{\Delta}\, {a_{\mathsf{ICB}}}^2 \int\limits_{\Omega} \sum_{jm\,kl} \widetilde{E}_{jm}\Phi_{kl} \left[\cos\varphi \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) - \frac{1}{\sin\vartheta}\cos\vartheta\sin\varphi \frac{\partial}{\partial\varphi} Y_{jm}(\Omega) \right] Y_{kl}(\Omega)\mathrm{d}\Omega\,, \tag{4.8}$$

$$L_z \cong \rho^{\Delta}\, {a_{\mathsf{ICB}}}^2 \int\limits_{\Omega} \sum_{jm\,kl} \widetilde{E}_{jm}\Phi_{kl}\, \sin\vartheta \frac{1}{\sin\vartheta} \frac{\partial}{\partial\varphi} Y_{jm}(\Omega) Y_{kl}(\Omega)\,\mathrm{d}\Omega\,. \tag{4.9}$$

Analogous to the derivation in Sec. 2.4.2, we use the complex combination of the equatorial components, L (see eq. (2.24)), of the coupling torque, expressed by

$$L \cong \rho^{\Delta}\, {a_{\mathsf{ICB}}}^2 \sum_{jm\,kl} \widetilde{E}_{jm}\Phi_{kl} \int\limits_{\Omega} e^{i\,\varphi} \left[i\, \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) - \cot\vartheta \frac{\partial}{\partial\varphi} Y_{jm}(\Omega) \right] Y_{kl}(\Omega)\,\mathrm{d}\Omega\,. \tag{4.10}$$

We apply now the partial derivatives on the SH basis functions according to eqs. (A.13)–(A.14) and consider the expression (A.15) for $\cot\vartheta\, Y_{jm}(\Omega)$. Furthermore, the orthogonality of the SH basis functions in eq. (A.6) and the relation for the complex conjugate of SH coefficients are taken into account. We obtain

$$L \cong i\,\rho^{\Delta}\, {a_{\mathsf{ICB}}}^2 \sum_{jm} \sqrt{j(j+1) - m(m+1)}\, \widetilde{E}_{jm}\Phi^*_{j(m+1)}\,. \tag{4.11}$$

Applying the orthogonality condition (A.6) on eq. (4.9) leads to the following form for the z-component of the coupling torque:

$$L_z \cong i\,\rho^{\Delta}\, {a_{\mathsf{ICB}}}^2 \sum_{jm} m\, \widetilde{E}_{jm}\Phi^*_{jm}\,. \tag{4.12}$$

To derive simplified expressions for the gravitational coupling torques, we have now to determine the SH coefficients of the gravitational potential at the ICB and the ICB itself in a mantle-fixed coordinate system. For the assumed geometrical case and ellipsoidal surfaces of constant density, their exist only two non-zero SH coefficients Φ_{00} and Φ_{20}, which are computed numerically according to eq. (3.33). In the reference state of an aligned inner core to the mantle equatorial plane (labeled by $^{\mathsf{REF}}$), the ICB is described by the following two coefficients

$$\widetilde{E}^{\mathsf{REF}}_{00} = \sqrt{\frac{4\pi}{9}}\, a_{\mathsf{ICB}}\,\epsilon^2 \quad \text{and} \quad \widetilde{E}^{\mathsf{REF}}_{20} = -\sqrt{\frac{4\pi}{45}}\, a_{\mathsf{ICB}}\,\epsilon^2\,. \tag{4.13}$$

A detailed derivation of these expression is given in Greiner-Mai & Hagedoorn (2008, Sec. 3). Therein is also defined $\epsilon^2 = (b^2 - {a_{\mathsf{ICB}}}^2)/b^2$, where $b = (1+f)a_{\mathsf{ICB}}$ is valid for a given flattening f.

Taking into account that only Φ_{00} and Φ_{20} differ from zero, we find that $\widetilde{E}_{2-1}\Phi^*_{20}$ is the only combination which contributes to the equatorial coupling torque given in eq. (4.11). Therefore, we need to express $\widetilde{E}_{2-1}(\alpha,\beta)$ by the related coefficients of the reference state. According to eq. (2.17), we find

$$\widetilde{E}_{21}(\alpha,\beta) = e^{-i\,\alpha}\mathbf{d}^2_{10}(\beta(t))\,\widetilde{E}^{\mathsf{REF}}_{20}\,,$$

which leads with eq. (2.15) to

$$\widetilde{E}_{21}(\alpha,\beta) = e^{-i\,\alpha}\sqrt{6}\left[\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)\left(\sin^2\left(\frac{\beta}{2}\right) - \cos^2\left(\frac{\beta}{2}\right)\right)\right]\widetilde{E}^{\mathsf{REF}}_{20}\,,$$

$$\widetilde{E}_{21}(\alpha,\beta) = -e^{-i\,\alpha}\sqrt{\frac{3}{2}}\left[2\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)\left(\cos^2\left(\frac{\beta}{2}\right) - \sin^2\left(\frac{\beta}{2}\right)\right)\right]\widetilde{E}^{\mathsf{REF}}_{20}\,,$$

$$\widetilde{E}_{21}(\alpha,\beta) = -e^{-i\,\alpha}\sqrt{\frac{3}{2}}\,\sin(\beta)\,\cos(\beta)\,\widetilde{E}^{\mathsf{REF}}_{20}\,, \tag{4.14}$$

where α and $beta$ are Euler angles as illustrated in Fig. 4.1. Herein, we have considered the following relations (e.g. Bronstein et al., 1997, eqs. (2.97) & (2.99))

$$\sin(2\beta) = 2\sin(\beta)\,\cos(\beta)\,, \tag{4.15}$$

$$\cos(2\beta) = \cos^2(\beta) - \sin^2(\beta)\,. \tag{4.16}$$

Moreover, we have to consider that $\widetilde{E}_{2-1} = -\widetilde{E}^*_{21}$, which leads to the equatorial coupling torque

$$L(\alpha,\beta) \cong 3\,\rho^\Delta\,{a_{\text{ICB}}}^2\big(i\,\cos(\alpha) - \sin(\alpha)\big)\,\sin(\beta)\,\cos(\beta)\,\widetilde{E}^{\text{REF}}_{20}\,\Phi_{20}\,. \tag{4.17}$$

The individual Cartesian components read

$$L_x(\alpha,\beta) \cong -\Gamma\,\sin(\alpha)\,\sin(\beta)\,\cos(\beta)\,, \tag{4.18}$$

$$L_y(\alpha,\beta) \cong \quad \Gamma\,\cos(\alpha)\,\sin(\beta)\,\cos(\beta)\,, \tag{4.19}$$

where Γ considering eq. (4.13) is given by

$$\begin{aligned}\Gamma &= 3\,\rho^\Delta\,{a_{\text{ICB}}}^2\,\widetilde{E}^{\text{REF}}_{20}\,\Phi_{20}\,,\\ \Gamma &= -2\,\rho^\Delta\,{a_{\text{ICB}}}^3\sqrt{\frac{\pi}{5}}\,\epsilon^2\Phi_{20}\,.\end{aligned} \tag{4.20}$$

The determination of the gravitational coupling torques for the simplified geometrical settings allow us also to derive inverse expressions. For the further assumption of small misalignment angle $\beta \ll 1$ it follows that $\sin\beta \approx \beta$, which leads to

$$\alpha \approx \arctan\left(-\frac{L_x}{L_y}\right), \tag{4.21}$$

$$\beta \approx \arcsin\left(-\frac{L_x}{\Gamma}\,\frac{1}{\sin\alpha}\right). \tag{4.22}$$

For given equatorial coupling torque components, eqs. (4.21) and (4.22) give estimations for the necessary misalignment of an elliptical inner core out of the equatorial plan of the mantle, to produce such torques. For PREM density profiles (Dziewonski & Anderson, 1981) and the related values of the hydrostatic flattening the numerical value of the coupling coefficient is approximately

$$\Gamma \cong -1.09892\,10^{23}\ \text{Nm}\,. \tag{4.23}$$

4.2 Consideration of CMB topography models

The theoretical description in the Sec. 3 allows us to determine the gravity potential on the ICB considering an arbitrary CMB topography. To illustrate the effect of a CMB topography which deviate from a hydrostatic ellipsoid, we determine for two published CMB topography models the gravity potential on the ICB. Moreover, we compare the equatorial gravitational coupling torques for an elliptical structure with those for the prescribed CMB topographies.

4.2.1 Gravitational potential at the ICB

The both chosen examples of CMB topography models published in the last decades are based on seismic wave travel-time differences between different core phases. The model of Morelli & Dziewonski (1987) is denoted with **M** and the model of Tanaka (2010) is labeled with **T**. For further details of the estimation of the individual models of the CMB topography we refer to the cited publications. The SH representation of the CMB models are truncated at degree and order four and are reduced by the degree one term (no shift of the whole CMB with respect to the mass center of the Earth is allowed). The resulting spatial distribution of the CMB topography with respect to the hydrostatic ellipsoid are

(a) Morelli & Dziewonski (1987) CMB topography **M** (b) Tanaka (2010) CMB topography **T**

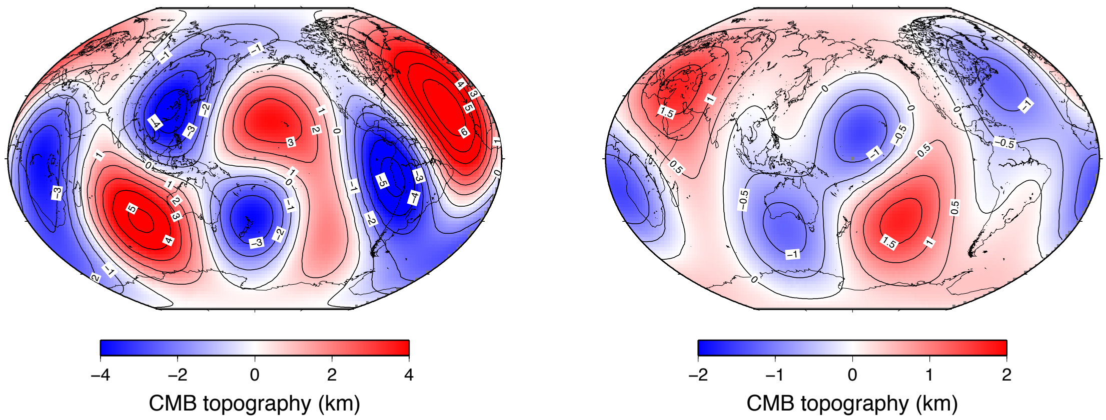


Figure 4.2: For comparison is shown the deviation from the hydrostatic ellipsoid of the CMB topography. **(a)** model **M** (Morelli & Dziewonski, 1987), **(b)** model **T** (Tanaka, 2010).

(a) $\Delta\Phi$ based on CMB model **M** (b) $\Delta\Phi$ based on CMB model **T**

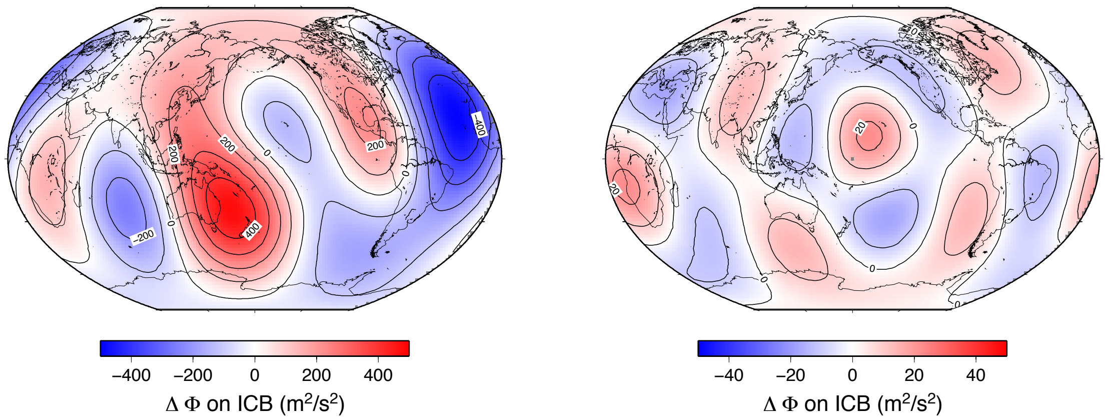


Figure 4.3: For comparison is shown the deviation of the gravity potential on an elliptical ICB for a model with ellipsoidal CMB from **(a)** a model with CMB topography **M** and **(b)** from a model with CMB topography **T**, which are presented in Fig. 4.2

compared in Fig. 4.2. The model **M** shows peak-to-peak variations of up to 10 km, whereas the model **T** varies between -1 km and 1.5 km. A possible explanation of such different CMB topographies are the different data basis for the individual computations, due to the enlarged number of seismic events throughout the years, the pre-processing and/or improved data selection.

The dominant spatial pattern of the gravity potential on the ICB is produced by the depth-depending flattening of the surfaces of equal density and the centrifugal potential. To highlight the influence of the chosen CMB topography model, we compute the gravity potential on the elliptical ICB according to eq. (3.33) for the individual CMB models and subtract in the spatial domain the related gravity potential of an Earth model with purely elliptical surfaces of equal density. Fig. 4.3 shows those differences in the gravity potential on the ICB considering the CMB topography models **M** and **T**, respectively. The presented differences in the gravity potential for the CMB model **M** are one order of magnitude larger than the differences for the CMB model **T**. Also the spatial pattern show no similarities, even both models are truncated at degree and order four. For the further discussion we have to recall that the absolute value of the gravity potential on the ICB is in the order of 10^8 m^{2}s^{-2}, which leads us to the conclusion, that the CMB topography has only an very minor influence on the gravity potential on the ICB.

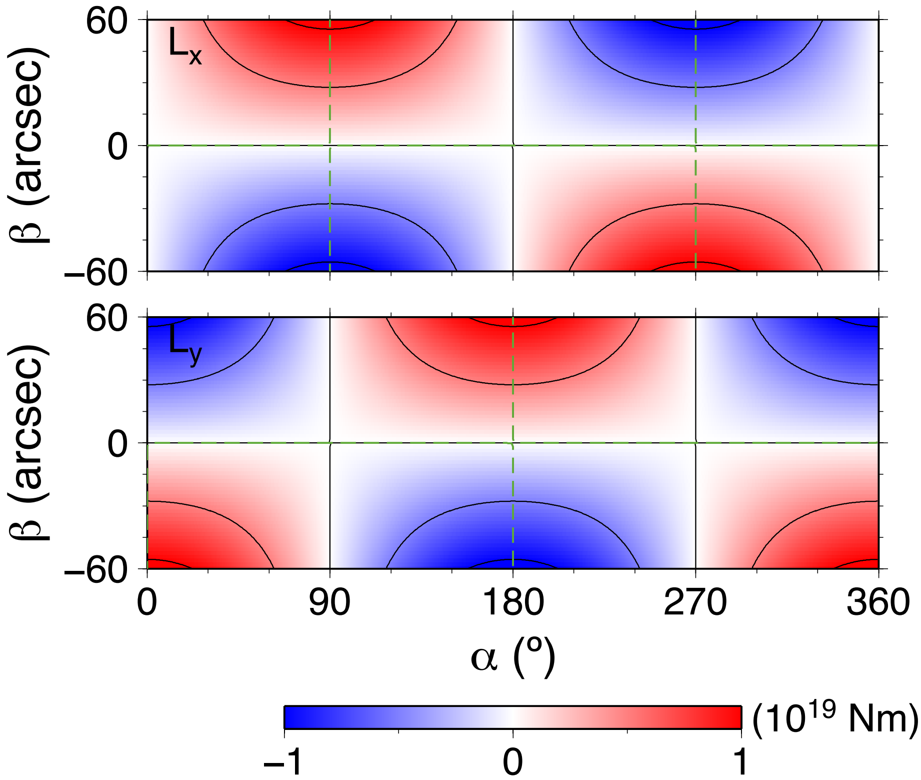


Figure 4.4: Equatorial coupling torques for a model with elliptical equal-density surfaces. Contour lines are plotted for every $5 \cdot 10^{18}$ Nm.

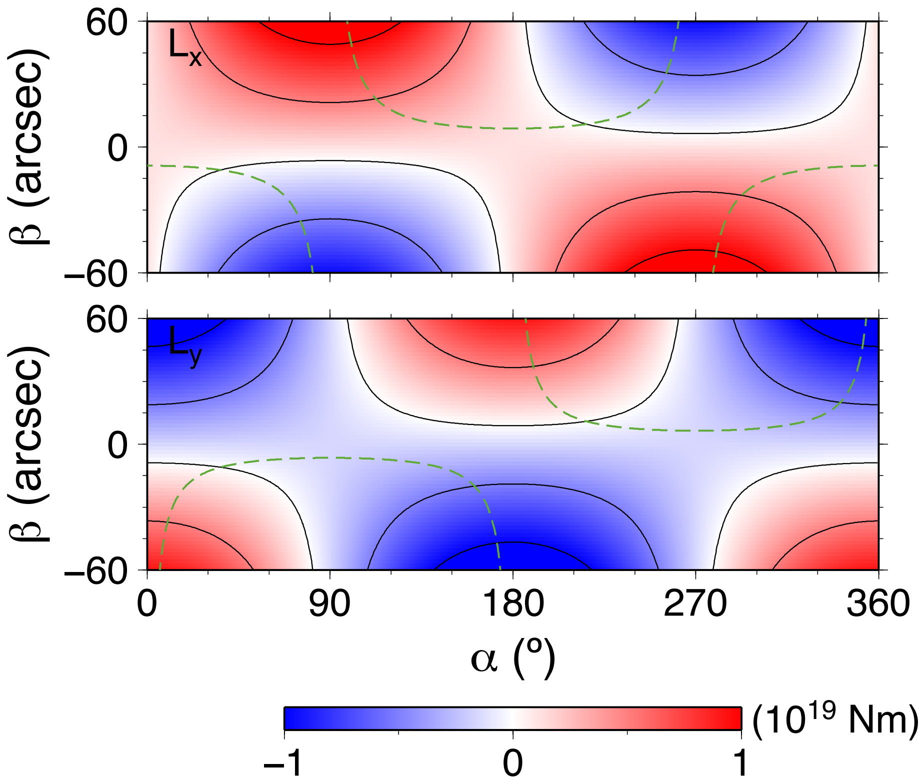
4.2.2 Gravitational coupling torques for elliptical ICB

The equatorial gravitational coupling torques are calculated according to eq. (2.31) for $\alpha \in [0°; 360°]$ and $\beta \in [-60\,\text{arcsec}; +60\,\text{arcsec}]$ ('arcsec' denotes arc-seconds), whereas the ICB topography coefficients are computed using eq. (2.17) and the related gravity potential at the ICB is determined by eqs. (3.33) and (3.45)–(3.46). Fig. 4.4 shows the equatorial coupling torque components L_x and L_y for the different inner core misalignment, whereas β prescribes the tilt of the ellipsoidal inner core with respect to the equatorial plane and α prescribes the direction of the tilt projected onto the equatorial plane with respect to the zero-meridian (see Fig. 4.1). Both equatorial torque components are in the order of 10^{19} Nm. For the first calculation, it is assumed that all boundaries between layers of equal-density have elliptical shape. Due to this assumption there is no γ-dependency (see eq. (2.17)). The zero-contour line of the other equatorial torque component is drawn in green to highlight the positions (α, β), for which both components vanish. For $\beta = 0$ arcsec both equatorial components vanish for any α, i.e. it does not exist a specific torque-free (α, β) combination. In Fig. 4.4, the basic pattern of the components of the gravitational torque reflect the combination of trigonometric functions, as illustrated in the approximate solution of the equatorial components in eqs. (4.18)–(4.19).

We perform additional calculations of the equatorial torque components considering the CMB models **M** and **T** introduced in Sec. 4.2.1. Fig. 4.5 compares the resulting L_x and L_y for the same interval of misalignment angles as in Fig. 4.4. The amplitudes of the equatorial coupling torques are all in the same order of magnitude for the different CMB models. The overlay (green dashed lines) of the zero-contour of the other component highlights specific (α, β) combinations, for which $L_x = L_y = 0$ Nm. Those differ from each other for model **M** and **T**. In contrast to the elliptical CMB model both components do not vanish for any α and $\beta = 0$ arcsec. This is caused by the asymmetric gravity potential (Fig. 4.3) due to the asymmetric CMB topography models **M** and **T**. As a consequence, the inner core would rotate into such a position, which would not cause any torque, if no other forces act on it. Therefore, a specific reference position exist for each CMB topography even for an elliptical inner core. This has to be considered in any joint core-mantle coupling model, because any coupling torque acting on the CMB will change the actual rotation of the mantle but not necessarily the inner core rotation. Such a relative rotation between mantle and inner core would cause gravitational coupling torques, which is described in Sec. 2.

Moreover, the comparison of Figs. 4.4 and 4.5 shows the pattern dominated of the elliptical Earth model, which is altered by the influence of the CMB topography models **M** and **T**. In contrast to the results for the pure elliptical model in Fig. 4.4, the zero-contour lines in the results for model **M** and **T** are curved. The x-component for model **M** is slightly shifted by around $1.1 \cdot 10^{18}$ Nm with respect to the results of the elliptical model, whereas the y-component is reduced by approximate $1.6 \cdot 10^{18}$ Nm (see Fig. 4.5 **(a)**). The influence of the CMB topography model **T** results in an slightly positive offset of about $0.75 \cdot 10^{18}$ Nm in both equatorial components (see Fig. 4.5 **(b)**). This comparison leads to the conclusion that the influence of the chosen CMB topography model is about one order of magnitude smaller than the dominant contribution of an elliptical Earth model, which means still an variation by about $7 - 10$ % due to the considered CMB topography models.

(a) Equatorial torques for CMB model **M**



(b) Equatorial torques for CMB model **T**

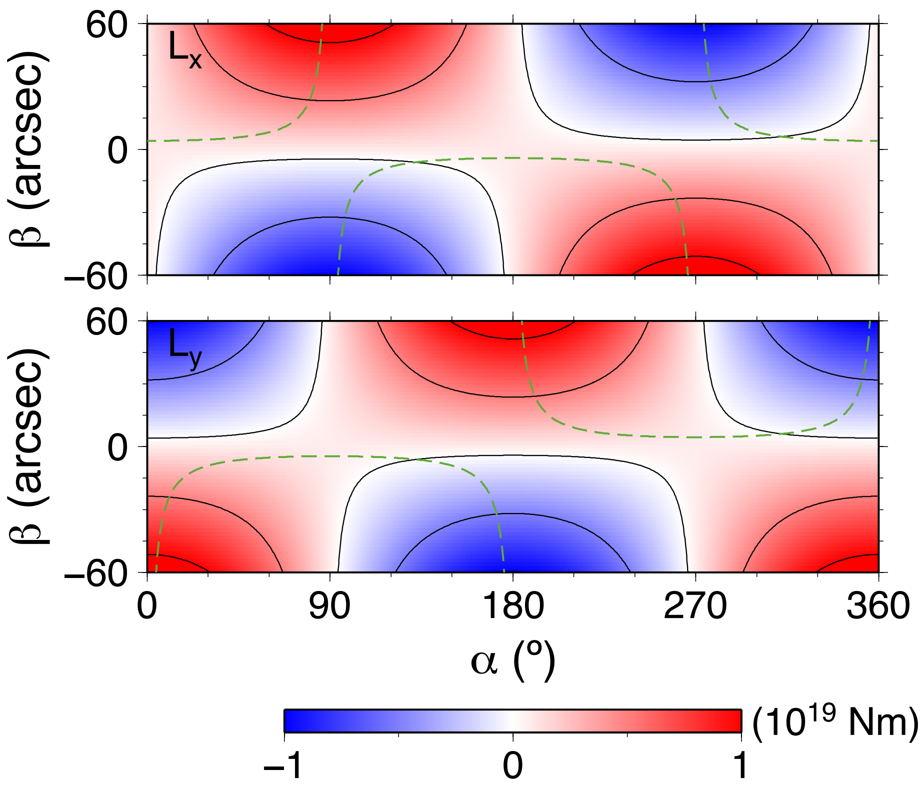


Figure 4.5: Equatorial coupling torques considering an elliptical ICB and **(a)** CMB model **M** and **(b)** CMB model **T**, respectively.

References | R

Arfken, G., 1985. Mathematical Methods for Physicists, 3. Edition. Academic Press, San Diego.

Bronstein, I., Semendjajew, K., Musiol, G., Mühlig, H., 1997. Taschenbuch der Mathematik, 3. überarbeitete und erweiterte Auflage. Verlag Harri Deutsch, Thun, Franfurt am Main.

Buffett, B., 1996. Gravitational oscillations in the length of day. Geophys. Res. Lett. 23, 2279–2282.

Dumberry, M., 2008. Gravitational torque on the inner core and decadal polar mation. Geophys. J. Int. 172, 903–920.

Dumberry, M., 2010. Gravitationally driven inner core differential rotation. Earth Planet. Sci. Lett. 297, 387–394.

Dziewonski, A., Anderson, D., 1981. Preliminary reference earth model. Phys. Earth Planet. Inter. 25, 297–356.

Greiner-Mai, H., Hagedoorn, J., 2008. Core-Mantle Coupling - Part II: Topographic coupling torques. Sci. Techn. Rep. GFZ Potsdam, STR 08/11 http://bib.gfz-potsdam.de/pub/str0811/0811.pdf.

Hagedoorn, J., Greiner-Mai, H., 2008. Core-Mantle Coupling - Part I: Electromagnetic coupling torques. Sci. Techn. Rep. GFZ Potsdam, STR 08/06 http://bib.gfz-potsdam.de/pub/str0806/0806.pdf.

Heuser, H., 1993. Lehrbuch der Analysis, Teil 2. B.G. Teubner, Stuttgart.

Kertz, W., 1995. Einführung in die Geophysik. Spektrum Akademischer Verlag, Heidelberg, Berlin, Oxford.

Martinec, Z., Hagedoorn, J., 2005. Time-domain approach to linearized rotational response of a three-dimensional viscoelastic earth model induced by glacial isostatic adjustment: I. Inertia-tensor perturbations. Geophys. J. Int. 163, 443–462.

Morelli, A., Dziewonski, A., 1987. Topography of the core-mantle boundary and lateral homogeneity of the liquid core. Nature 325, 678–683.

Pěč, K., Martinec, Z., 1984. Constraints to the three-dimensional non-hydrostatic density distribution in the Earth. Studia geoph. et geod. 28, 364–380.

Pěč, K., Martinec, Z., 1988. Gravitational potential inside 3-D inhomogeneous Earth: a boundary-value problem for the Poisson equation. Studia geoph. et geod. 32, 32–46.

Smirnow, S., 1964. Lehrgang der höheren Mathematik. Band 2. VEB Deutscher Verlag der Wissenschaft, Berlin.

Smylie, D., Szeto, A., Rochester, M., 1984. The dynamics of the Earth's inner and outer cores. Rep. Prog. Phys. 47, 855–906.

Tanaka, S., 2010. Constraints on the core-mantle boundary topography from P4KP-PcP differential travel times. J. Geophys. Res. 115, doi:10.1029/2009JB006563.

Varshalovich, D., Moskalev, A., Khersonskii, V., 1989. Quantum Theory of Angular Momentum. World Scientific, Singapore.

Spherical harmonics and related derivations | A

A.1 Definition of scalar spherical harmonics and basic relations

In this section, we shortly summarize the definition and basic relations of spherical harmonics (SH). The most definitions are according to Varshalovich et al. (1989, Chap. 5), and we follow their notation, where Y_{jm} are the scalar spherical harmonics of degree j and order m, which are defind by:

$$Y_{jm}(\Omega) := P_{jm}(\cos\vartheta)e^{im\varphi}. \tag{A.1}$$

Here are $i = \sqrt{-1}$, $\Omega = (\vartheta, \varphi)$ and P_{jm} the associated Legendre functions, which are defined as follows:

$$P_{jm}(\cos\vartheta) := (-1)^m \sqrt{\frac{2j+1}{4\pi}\frac{(j-m)!}{(j+m)!}}(\sin\vartheta)^m \frac{d^m}{(d\cos\vartheta)^m} P_j(\cos\vartheta), \tag{A.2}$$

$$P_j(\cos\vartheta) := \frac{1}{2^j j!}\frac{d^j(\cos^2\vartheta - 1)^j}{d(\cos\vartheta)^j}. \tag{A.3}$$

For the degree and order of the SH and Legendre functions the co-domains

$$j = 0, 1, 2, 3, \ldots, \infty, \tag{A.4}$$

$$m = -j, \ldots, 0, \ldots, j \tag{A.5}$$

are valid. The SH basis functions are orthonormal on the unit sphere (Varshalovich et al., 1989, Chap. 5):

$$\int\limits_{\Omega_0} Y_{jm}(\Omega)Y^*_{j'm'}(\Omega)\,\mathrm{d}\Omega = \delta_{jj'}\delta_{mm'}. \tag{A.6}$$

Here, $*$ denotes the complex conjugate, and δ_{ij} is Kronecker's symbol. For the complex conjugate SH basis functions the following relation holds:

$$Y^*_{jm}(\Omega) = (-1)^m Y_{j\,-m}(\Omega). \tag{A.7}$$

We summarize here also the differential operators in spherical coordinates and the related splitting into angular and radial parts, which are given for the Nabla operator by

$$\nabla = \left[\frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\nabla_\Omega\right], \tag{A.8}$$

$$\nabla_\Omega = \left[\frac{\partial}{\partial\vartheta}\boldsymbol{e}_\vartheta + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi\right], \tag{A.9}$$

and for the Laplace operator by

$$\Delta = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \Delta_\Omega\right], \tag{A.10}$$

$$\Delta_\Omega = \left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right] \tag{A.11}$$

The SH basis functions are eigenfunctions of the angular part of the Laplace operator, so that

$$\Delta_\Omega\, Y_{jm}(\Omega) = -j(j+1)Y_{jm}(\Omega) \tag{A.12}$$

is valid.

Moreover, we show here the used derivative of SH with respect to the spherical coordinates φ and ϑ (Varshalovich et al., 1989, Sec. 5.8):

$$\frac{\partial}{\partial\varphi} Y_{jm}(\Omega) = imY_{jm}(\Omega), \tag{A.13}$$

$$\frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) = \frac{1}{2}\Bigg[\sqrt{j(j+1)-m(m+1)}\, Y_{j\,(m+1)}(\Omega)\, e^{-i\varphi} - \sqrt{j(j+1)-m(m-1)}\, Y_{j\,(m-1)}(\Omega)\, e^{i\varphi}\Bigg]. \tag{A.14}$$

In addition, we present here the recursion formula for the following product of a trigonometric function and a SH as given in Varshalovich et al. (eq. (1) 1989, Sec. 5.7)

$$\cot\vartheta Y_{jm}(\Omega) = -\frac{1}{2\,m}\Bigg[\sqrt{j(j+1)-m(m+1)}\, Y_{j\,(m+1)}(\Omega)\, e^{-i\varphi} + \sqrt{j(j+1)-m(m-1)}\, Y_{j\,(m-1)}(\Omega)\, e^{i\varphi}\Bigg]. \tag{A.15}$$

A.2 Vector spherical harmonics

The vector spherical harmonics (VSH), we have chosen here, are defined as follows (Varshalovich et al., 1989)

$$\boldsymbol{S}^{(-1)}_{jm}(\Omega) = \boldsymbol{e}_r Y_{jm}(\Omega)\,, \tag{A.16}$$

$$\boldsymbol{S}^{(+1)}_{jm}(\Omega) = \nabla_\Omega\, Y_{jm}(\Omega)\,, \tag{A.17}$$

$$\boldsymbol{S}^{(0)}_{jm}(\Omega) = \mathrm{L}_\Omega\, Y_{jm}(\Omega)\,, \tag{A.18}$$

where $\boldsymbol{e}_r$, $\boldsymbol{e}_\vartheta$ and $\boldsymbol{e}_\varphi$ are the spherical basis vectors and the differential operator ∇_Ω is given in eq. (A.9). The operator L_Ω is defined by

$$\mathrm{L}_\Omega = \boldsymbol{e}_r \times \nabla_\Omega = \left[\frac{\partial}{\partial\vartheta}\boldsymbol{e}_\varphi - \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\vartheta\right]. \tag{A.19}$$

Two vector spherical harmonics $\boldsymbol{S}^{(\lambda)}_{jm}(\Omega)$ and $\boldsymbol{S}^{(\lambda')}_{j'm'}(\Omega)$, with different degree, $j \neq j'$, and different order, $m \neq m'$, as well as with different indices, $\lambda \neq \lambda'$, are orthogonal, expressed by

$$\int\limits_{\Omega_0} \boldsymbol{S}^{(\lambda)}_{jm}(\Omega) \cdot \left[\boldsymbol{S}^{(\lambda')}_{j'm'}(\Omega)\right]^* \mathrm{d}\Omega = 0, \tag{A.20}$$

where the dot denotes the scalar product of vectors. Moreover, we summarize the following expressions for orthogonal vector spherical harmonics:

$$\int\limits_{\Omega_0} \boldsymbol{S}^{(-1)}_{jm}(\Omega) \cdot \left[\boldsymbol{S}^{(-1)}_{j'm'}(\Omega)\right]^* \mathrm{d}\Omega = \delta_{jj'}\delta_{mm'}, \tag{A.21}$$

$$\int\limits_{\Omega_0} \boldsymbol{S}^{(1)}_{jm}(\Omega) \cdot \left[\boldsymbol{S}^{(1)}_{j'm'}(\Omega)\right]^* \mathrm{d}\Omega = j(j+1)\delta_{jj'}\delta_{mm'}, \tag{A.22}$$

$$\int\limits_{\Omega_0} \boldsymbol{S}^{(0)}_{jm}(\Omega) \cdot \left[\boldsymbol{S}^{(0)}_{j'm'}(\Omega)\right]^* \mathrm{d}\Omega = j(j+1)\delta_{jj'}\delta_{mm'}. \tag{A.23}$$

For the vector spherical harmonics, as defined here, we can easily derive the following expressions:

$$\boldsymbol{S}^{(\lambda)}_{j\,-m}(\Omega) = (-1)^m \left[\boldsymbol{S}^{(\lambda)}_{jm}\right]^*, \tag{A.24}$$

$$\boldsymbol{e}_r \times \boldsymbol{S}^{(0)}_{jm}(\Omega) = -\boldsymbol{S}^{(1)}_{jm}(\Omega), \tag{A.25}$$

$$\boldsymbol{e}_r \times \boldsymbol{S}^{(-1)}_{jm}(\Omega) = 0, \tag{A.26}$$

$$\boldsymbol{e}_r \times \boldsymbol{S}^{(1)}_{jm}(\Omega) = \boldsymbol{S}^{(0)}_{jm}(\Omega). \tag{A.27}$$

Additional derivation for the calculation of the torques | B

B.1 Surface integral with respect to the unit sphere

In eq. (2.5), the gravitational coupling torque is given by a surface integral with the kernel $\boldsymbol{r}_{\text{ICB}}(\Omega) \times \boldsymbol{N}(\Omega)$. For the further derivation, we have to execute the vector product considering the definition in eqs. (2.3) and (2.4). For the outward normal of the ICB we find

$$\begin{aligned}
\boldsymbol{N}(\Omega) &= \frac{\partial}{\partial\vartheta}\Big[\boldsymbol{e}_r a_{\text{ICB}} E(\Omega)\Big] \times \frac{\partial}{\partial\varphi}\Big[\boldsymbol{e}_r a_{\text{ICB}} E(\Omega)\Big] \\
\boldsymbol{N}(\Omega) &= a_{\text{ICB}}^2 \left[\left(\frac{\partial}{\partial\vartheta}\boldsymbol{e}_r\right)E(\Omega) + \boldsymbol{e}_r\left(\frac{\partial}{\partial\vartheta}E(\Omega)\right)\right] \times \left[\left(\frac{\partial}{\partial\varphi}\boldsymbol{e}_r\right)E(\Omega) + \boldsymbol{e}_r\left(\frac{\partial}{\partial\varphi}E(\Omega)\right)\right] \\
\boldsymbol{N}(\Omega) &= a_{\text{ICB}}^2\, E(\Omega) \left[\boldsymbol{e}_r \sin\vartheta E(\Omega) - \boldsymbol{e}_\vartheta \sin\vartheta \frac{\partial}{\partial\vartheta}E(\Omega) - \boldsymbol{e}_\varphi \frac{\partial}{\partial\varphi}E(\Omega)\right].
\end{aligned} \tag{B.1}$$

For the surface integral of the gravitational torque, applying these expressions and considering $\boldsymbol{r}_{\text{ICB}}(\Omega) = \boldsymbol{e}_r a_{\text{ICB}} E(\Omega)$ yield

$$\boldsymbol{L} = -\rho^\Delta a_{\text{ICB}}^3 \int_\Omega E(\Omega)\, E(\Omega)\, \phi(\Omega)\, \boldsymbol{e}_r \times \left[\boldsymbol{e}_r E(\Omega) - \boldsymbol{e}_\vartheta \frac{\partial}{\partial\vartheta}E(\Omega) - \boldsymbol{e}_\varphi \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}E(\Omega)\right] \sin\vartheta\, \mathrm{d}\vartheta \mathrm{d}\varphi\,.$$

We apply the vector product of the spherical base vectors and consider the definition of the surface element of the unit sphere $\mathrm{d}\Omega = \sin\vartheta\, \mathrm{d}\vartheta \mathrm{d}\varphi$, which leads to the expression

$$\boldsymbol{L} = -\rho^\Delta a_{\text{ICB}}^3 \int_\Omega E(\Omega)\, E(\Omega)\, \phi(\Omega) \left[-\boldsymbol{e}_\varphi \frac{\partial}{\partial\vartheta}E(\Omega) + \boldsymbol{e}_\vartheta \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}E(\Omega)\right] \mathrm{d}\Omega\,.$$

This can by simplified using the definition of the angular part ∇_Ω as defined in eq. (A.9) to

$$\boldsymbol{L} = \rho^\Delta a_{\text{ICB}}^3 \int_\Omega E(\Omega)\, E(\Omega)\, \phi(\Omega)\, \boldsymbol{e}_r \times \nabla_\Omega\, E(\Omega)\, \mathrm{d}\Omega\,. \tag{B.2}$$

B.2 Derivation of the combined coefficient

In eq. (2.9) the gravitational torque is given as a product of four quantities expressed by scalar and vector spherical harmonics. The triple product of scalar spherical harmonics can be combined by applying a special relation for triple product of spherical harmonics. We rewrite eq. (2.9)

$$\boldsymbol{L} = \rho^\Delta a_{\text{ICB}}^3 \int_\Omega \sum_{\substack{j_1 m_1 j_2 m_2 \\ j_3 m_3 j m}} E_{j_1 m_1} Y_{j_1 m_1}(\Omega)\, E_{j_2 m_2} Y_{j_2 m_2}(\Omega)\, \Phi_{j_3 m_3} Y_{j_3 m_3}(\Omega)\, E_{jm} \boldsymbol{S}_{jm}^{(0)}(\Omega)\, \mathrm{d}\Omega\,, \tag{B.3}$$

and consider eq. (11) from Varshalovich et al. (1989, Sec. 5.6.2), which is repeated here,

$$\begin{aligned}
Y_{j_1 m_1}(\Omega)\, Y_{j_2 m_2}(\Omega)\, Y_{j_3 m_3}(\Omega) = &\sum_{j_4 m_4\, j' m'} \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{(4\pi)^2(2j'+1)}} \\
&\cdot \mathbf{C}_{j_1 0\, j_2 0}^{j_4 0}\, \mathbf{C}_{j_4 0\, j_3 0}^{j' 0}\, \mathbf{C}_{j_1 m_1\, j_2 m_2}^{j_4 m_4}\, \mathbf{C}_{j_4 m_4\, j_3 m_3}^{j' m'}\, Y_{j' m'}(\Omega)\,,
\end{aligned} \tag{B.4}$$

and leads to

$$\boldsymbol{L} = \rho^{\Delta} a_{\mathrm{ICB}}^3 \int_{\Omega} \sum_{\substack{j_1 m_1 j_2 m_2 \\ j_3 m_3 j_4 m_4 \\ j' m' j m}} E_{j_1 m_1}\, E_{j_2 m_2}\, \Phi_{j_3 m_3} \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{(4\pi)^2(2j'+1)}} \cdot \mathbf{C}^{j_4 0}_{j_1 0\, j_2 0}\, \mathbf{C}^{j' 0}_{j_4 0\, j_3 0}\, \mathbf{C}^{j_4 m_4}_{j_1 m_1\, j_2 m_2}\, \mathbf{C}^{j' m'}_{j_4 m_4\, j_3 m_3}\, Y_{j'm'}(\Omega)\, E_{jm} \boldsymbol{S}^{(0)}_{jm}(\Omega)\, \mathrm{d}\Omega \,. \tag{B.5}$$

If we now introduce the combined coefficients

$$\Theta_{j'm'} = \sum_{\substack{j_1 m_1 j_2 m_2 \\ j_3 m_3 j_4 m_4}} E_{j_1 m_1} E_{j_2 m_2} \Phi_{j_3 m_3} \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{(4\pi)^2(2j'+1)}} \cdot \mathbf{C}^{j_4 0}_{j_1 0\, j_2 0}\, \mathbf{C}^{j' 0}_{j_4 0\, j_3 0}\, \mathbf{C}^{j_4 m_4}_{j_1 m_1\, j_2 m_2}\, \mathbf{C}^{j' m'}_{j_4 m_4\, j_3 m_3} \,,$$

as in eq. (2.10), the integral for the gravitational torque becomes

$$\boldsymbol{L} = \rho^{\Delta} a_{\mathrm{ICB}}^3 \int_{\Omega} \sum_{j'm'\, jm} \Theta_{j'm'}\, E_{jm}\, Y_{j'm'}(\Omega)\, \boldsymbol{S}^{(0)}_{jm}(\Omega)\, \mathrm{d}\Omega \,.$$

This relation is identical with eq. (2.11) in Sec. 2.3.

B.3 Derivation of the Cartesian components

Based on the definition of the vector spherical harmonic $\boldsymbol{S}^{(0)}_{jm}(\Omega)$ given in eq. (A.18), its Cartesian components are deduced by considering the relation between spherical and Cartesian basis vectors in eqs. (2.18) and (2.19). In a first step we find for the vector spherical harmonic

$$\begin{aligned}\boldsymbol{S}^{(0)}_{jm}(\Omega) = \boldsymbol{e}_x &\left[-\sin\vartheta \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) - \cos\vartheta\cos\varphi \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} Y_{jm}(\Omega)\right] \\ &+ \boldsymbol{e}_y\left[\cos\varphi \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) - \cos\vartheta\sin\varphi \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} Y_{jm}(\Omega)\right] + \boldsymbol{e}_z\left[\frac{\sin\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi} Y_{jm}(\Omega)\right] .\end{aligned} \tag{B.6}$$

In the next step, we substitute this expression into the integral equation of the gravitational coupling torque, given in eq. (2.11):

$$\begin{aligned}\boldsymbol{L} = \rho^{\Delta} a_{\mathrm{ICB}}^3 \int_{\Omega} \sum_{j'm'\, jm} \Theta_{j'm'}\, E_{jm}\, Y_{j'm'}(\Omega) \cdot \Bigg\{ \boldsymbol{e}_x &\left[-\sin\vartheta \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) - \cos\vartheta\cos\varphi \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} Y_{jm}(\Omega)\right] \\ + \boldsymbol{e}_y\left[\cos\varphi \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) - \cos\vartheta\sin\varphi \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} Y_{jm}(\Omega)\right] &+ \boldsymbol{e}_z\left[\frac{\partial}{\partial\varphi} Y_{jm}(\Omega)\right]\Bigg\}\, \mathrm{d}\Omega \,.\end{aligned} \tag{B.7}$$

Splitting this expression into the three Cartesian components leads to eqs. (2.20)–(2.22).

B.4 Derivation of equatorial coupling torque

The complex combination of the equatorial components of the coupling torque, given in eq. (2.27), contains products of exponential functions and partial derivatives of scalar spherical harmonics. The related part of the integral kernel can be transformed into

$$\begin{aligned}-i\, e^{i\varphi} \frac{\partial}{\partial\vartheta} Y_{jm}(\Omega) + e^{i\varphi} \cot\vartheta \frac{\partial}{\partial\varphi} Y_{jm}(\Omega) = e^{i\varphi} \Bigg[-\frac{i}{2}\Bigg(&\sqrt{j(j+1) - m(m+1)}\, Y_{j(m+1)}(\Omega) e^{-i\varphi} \\ &- \sqrt{j(j+1) - m(m-1)}\, Y_{j(m-1)}(\Omega) e^{i\varphi}\Bigg) + \cot\vartheta\, i\, m Y_{jm}(\Omega)\Bigg] ,\end{aligned}$$

by considering the partial derivatives of spherical harmonics given in eqs (A.13) and (A.14). Moreover, we apply the recursion formula for $\cot\vartheta$, given in eq. (A.15),

$$= i\,e^{i\,\varphi}\Bigg[-\frac{1}{2}\bigg(\sqrt{j(j+1)-m(m+1)}\,Y_{j(m+1)}(\Omega)e^{-i\,\varphi}-\sqrt{j(j+1)-m(m-1)}\,Y_{j(m-1)}(\Omega)e^{i\,\varphi}\bigg)$$
$$+\,m\frac{-1}{2m}\bigg(\sqrt{j(j+1)-m(m+1)}\,Y_{j(m+1)}(\Omega)e^{-i\,\varphi}+\sqrt{j(j+1)-m(m-1)}\,Y_{j(m-1)}(\Omega)e^{i\,\varphi}\bigg)\Bigg]\,,$$

which can be simplified to

$$= -i\,\sqrt{j(j+1)-m(m+1)}\,Y_{j(m+1)}(\Omega)\,. \tag{B.8}$$

Inserting this relation in eq. (2.27) results in eq. (2.28).

Solution for the j-depending integral in eq. (3.24) | C

In Section 3.2.2 the SH coefficients of the gravitation potential, caused by the mass distribution in $B_\circ$, are determined. For this purpose, the j-depending integrals in eq. (3.24) have to be executed.

For $j = 0$, we have to evaluate the following expression

$$I_0 = \int\limits_\Omega \int\limits_0^1 \big[(r_n - r_{n-1})x + r_{n-1}\big](r_n - r_{n-1})\, Y_{00}^*(\Omega)\,\mathrm{d}x\,\mathrm{d}\Omega\,,$$

$$I_0 = \int\limits_\Omega \int\limits_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\big[(r_n - r_{n-1})x + r_{n-1}\big]^2\right] Y_{00}^*(\Omega)\,\mathrm{d}x\,\mathrm{d}\Omega\,.$$

First, we integrate over x, which leads to

$$I_0 = \frac{1}{2}\int\limits_\Omega \left[r_n^2 - r_{n-1}^2\right] Y_{00}^*(\Omega)\,\mathrm{d}\Omega\,,$$

where we have to consider the angular dependency of the radial distances r_n, expressed by SH coefficients according to eqs. (3.25)–(3.26),

$$I_0 = \frac{1}{2}\int\limits_\Omega \sum_{j'm'}\left[a_n^2 B_{j'm'}^{\langle 2\rangle}(a_n) Y_{j'm'}(\Omega) - a_{n-1}^2 B_{j'm'}^{\langle 2\rangle}(a_{n-1}) Y_{j'm'}(\Omega)\right] Y_{00}^*(\Omega)\,\mathrm{d}\Omega\,.$$

Applying the orthogonality condition for SH basis functions, given in eq (A.6)

$$I_0 = \frac{1}{2}\big[a_n^2 B_{00}^{\langle 2\rangle}(a_n) - a_{n-1}^2 B_{00}^{\langle 2\rangle}(a_{n-1})\big]\,. \tag{C.1}$$

For $j = 1$, the integral reads

$$I_1 = \int\limits_\Omega \int\limits_0^1 (r_n - r_{n-1})\, Y_{1m}^*(\Omega')\,\mathrm{d}x\,\mathrm{d}\Omega'\,,$$

where the execution of the integral over x leads to

$$I_1 = \int\limits_\Omega (r_n - r_{n-1})\, Y_{1m}^*(\Omega)\,\mathrm{d}\Omega\,.$$

We express the angular depending radial distances r_n by SH coefficients according to eq. (3.25),

$$I_1 = \int\limits_\Omega \sum_{j'm'}\left[a_n B_{j'm'}(a_n) Y_{j'm'}(\Omega) - a_{n-1} B_{j'm'}(a_{n-1}) Y_{j'm'}(\Omega)\right] Y_{1m}^*(\Omega)\,\mathrm{d}\Omega\,,$$

and consider orthogonality condition given in eq (A.6), and we find

$$I_1 = \big[a_n B_{1m}(a_n) - a_{n-1} B_{1m}(a_{n-1})\big]\,. \tag{C.2}$$

For $j = 2$, an analytical solution of the integral can not be found. In the following, we summarize the derivation of an approximate solution up to the fourth order of the normalized radial distance (this is elucidated by the approximation of $\ln(1+x)$ by a series). The integral for $j = 2$ reads

$$
\begin{aligned}
I_2 &= \int_\Omega \int_0^1 \left[(r_n - r_{n-1})x + r_{n-1}\right]^{-1} (r_n - r_{n-1})\,\mathrm{d}x\, Y^*_{2m}(\Omega)\,\mathrm{d}\Omega\,, \\
I_2 &= \int_\Omega \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\big[(r_n - r_{n-1})x + r_{n-1}\big]\right]\mathrm{d}x\, Y^*_{2m}(\Omega)\,\mathrm{d}\Omega\,, \\
I_2 &= \int_\Omega \left[\ln\big(r_n(\Omega)\big) - \ln\big(r_{n-1}(\Omega)\big)\right] Y^*_{2m}(\Omega)\,\mathrm{d}\Omega\,.
\end{aligned} \tag{C.3}
$$

The remaining problem is the integration over the full solid angle of the products of SH basis functions and the logarithm of the angular depending radial distances. For the further derivation we introduce an SH representation of the radial distance, which differs from those in eq. (3.25)

$$
r_n(\Omega) = a_n \sum_{j'm'} B_{j'm'}(a_n) Y_{j'm'}(\Omega) = a_n B_{00}(a_n) Y_{00}\left(1 + \frac{1}{B_{00}(a_n)Y_{00}} \sum_{j'\geq 1\, m'} B_{j'm'}(a_n) Y_{j'm'}(\Omega)\right).
$$

We use this representation to split the logarithm as follows

$$
\ln\big(r_n(\Omega)\big) = \ln\big(a_n B_{00}(a_n) Y_{00}\big) + \ln\left(1 + \frac{1}{B_{00}(a_n)Y_{00}} \sum_{j'\geq 1\, m'} B_{j'm'}(a_n) Y_{j'm'}(\Omega)\right), \tag{C.4}
$$

and apply the series expansion (e.g. Bronstein et al., 1997, Tab. 21.3)

$$
\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \mathcal{O}(5)
$$

to the second term in eq. (C.4)

$$
\begin{aligned}
\ln\left(1 + \frac{1}{B_{00}(a_n)Y_{00}} \sum_{j'\geq 1\, m'} B_{j'm'}(a_n) Y_{j'm'}(\Omega)\right) = {} & \frac{1}{B_{00}(a_n)Y_{00}} \sum_{j'\geq 1\, m'} B_{j'm'}(a_n) Y_{j'm'}(\Omega) \\
& - \frac{1}{2}\left(\frac{1}{B_{00}(a_n)Y_{00}}\right)^2 \sum_{j'\geq 1\, m'} B^{\langle 2\rangle}_{j'm'}(a_n) Y_{j'm'}(\Omega) \\
& + \frac{1}{3}\left(\frac{1}{B_{00}(a_n)Y_{00}}\right)^3 \sum_{j'\geq 1\, m'} B^{\langle 3\rangle}_{j'm'}(a_n) Y_{j'm'}(\Omega) \\
& - \frac{1}{4}\left(\frac{1}{B_{00}(a_n)Y_{00}}\right)^4 \sum_{j'\geq 1\, m'} B^{\langle 4\rangle}_{j'm'}(a_n) Y_{j'm'}(\Omega) + \mathcal{O}(5)\,.
\end{aligned} \tag{C.5}
$$

We reformulate I_2, as given in eq. (C.3),

$$
I_2 = \int_\Omega \ln\big(r_n(\Omega)\big)\, Y^*_{2m}(\Omega)\,\mathrm{d}\Omega - \int_\Omega \ln\big(r_{n-1}(\Omega)\big) Y^*_{2m}(\Omega)\,\mathrm{d}\Omega\,. \tag{C.6}
$$

Herein, it is visible that the remaining problem is the determination of integrals of the type

$$
I = \int_\Omega \ln\big(r_n(\Omega)\big)\, Y^*_{2m}(\Omega)\,\mathrm{d}\Omega\,,
$$

which reads with eq. (C.4) and the approximation in eq. (C.5)

$$
\begin{aligned}
I = &\int_\Omega \ln\big(a_n B_{00}(a_n) Y_{00}\big)\, Y_{2m}^*(\Omega)\,\mathrm{d}\Omega + \frac{1}{B_{00}(a_n)Y_{00}} \int_\Omega \Big(\sum_{j'\geq 1\, m'} B_{j'm'}(a_n) Y_{j'm'}(\Omega)\Big)\, Y_{2m}^*(\Omega)\,\mathrm{d}\Omega \\
&- \frac{1}{2}\bigg(\frac{1}{B_{00}(a_n)Y_{00}}\bigg)^2 \int_\Omega \Big(\sum_{j'\geq 1\, m'} B_{j'm'}^{\langle 2\rangle}(a_n) Y_{j'm'}(\Omega)\Big)\, Y_{2m}^*(\Omega)\,\mathrm{d}\Omega \\
&+ \frac{1}{3}\bigg(\frac{1}{B_{00}(a_n)Y_{00}}\bigg)^3 \int_\Omega \Big(\sum_{j'\geq 1\, m'} B_{j'm'}^{\langle 3\rangle}(a_n) Y_{j'm'}(\Omega)\Big)\, Y_{2m}^*(\Omega)\,\mathrm{d}\Omega \\
&- \frac{1}{4}\bigg(\frac{1}{B_{00}(a_n)Y_{00}}\bigg)^4 \int_\Omega \Big(\sum_{j'\geq 1\, m'} B_{j'm'}^{\langle 4\rangle}(a_n) Y_{j'm'}(\Omega)\Big)\, Y_{2m}^*(\Omega)\,\mathrm{d}\Omega + \mathcal{O}(5)\,.
\end{aligned} \tag{C.7}
$$

The first integral in eq. (C.7) vanishes according to Varshalovich et al. (1989, Sec. 5.9.1 eq. 1), because of the integration of Y_{2m} over the full solid angle and $\ln(a_n B_{00} Y_{00}) = \text{const}$. For the remaining four terms, we find by considering the orthogonality condition in eq. (A.6) the approximation up to the fourth order

$$
I \approx \sum_{p=1}^{4} \frac{(-1)^{p+1}}{p} \bigg(\frac{1}{B_{00}(a_n)Y_{00}}\bigg)^p B_{2m}^{\langle p\rangle}(a_n)\,. \tag{C.8}
$$

Furthermore, we take into account the reformulation of I_2 in eq. (C.6), which leads to the final expression for the approximate solution of I_2

$$
I_2 \approx \sum_{p=1}^{4} \frac{(-1)^{p+1}}{p} \Bigg[\bigg(\frac{1}{B_{00}(a_n)Y_{00}}\bigg)^p B_{2m}^{\langle p\rangle}(a_n) - \bigg(\frac{1}{B_{00}(a_{n-1})Y_{00}}\bigg)^p B_{2m}^{\langle p\rangle}(a_{n-1})\Bigg]\,. \tag{C.9}
$$

This expression is used in Section 3.2.2 to derive the A_{2m}° coefficients as given in eq. (3.29).

For $j \geq 3$, an analytical solution for I_j can be derived, which is summarized in the following. The integral I_j reads for all $j \geq 3$

$$
\begin{aligned}
I_j &= \int_\Omega \int_0^1 \big[(r_n - r_{n-1})x + r_{n-1}\big]^{-(j-1)} (r_n - r_{n-1})\, Y_{jm}^*(\Omega)\,\mathrm{d}x\,\mathrm{d}\Omega\,, \\
I_j &= \int_\Omega \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\bigg[-\frac{1}{j-2}\big[(r_n - r_{n-1})x + r_{n-1}\big]^{-(j-2)}\bigg]\, Y_{jm}^*(\Omega)\,\mathrm{d}x\,\mathrm{d}\Omega\,,
\end{aligned}
$$

where the intergration over x leads to

$$
I_j = \int_\Omega -\frac{1}{j-2}\Big[\big(r_n(\Omega)\big)^{-(j-2)} - \big(r_{n-1}(\Omega)\big)^{-(j-2)}\Big]\, Y_{jm}^*(\Omega)\,\mathrm{d}\Omega\,.
$$

Here, we express the angular depending radial distances r_n and its pth power by the SH representation given in eqs. (3.25) and (3.26), which leads to

$$
I_j = \sum_{j'm'} -\frac{1}{j-2}\Big[a_n^{-(j-2)} B_{j'm'}^{\langle -(j-2)\rangle}(a_n) - a_{n-1}^{-(j-2)} B_{j'm'}^{\langle -(j-2)\rangle}(a_{n-1})\Big] \int_\Omega Y_{j'm'}(\Omega)\, Y_{jm}^*(\Omega)\,\mathrm{d}\Omega\,.
$$

We consider additionally the orthogonality condition in eq. (A.6), which leads to the final expression used in eq. (3.30)

$$
I_j = -\frac{1}{j-2}\Big[a_n^{-(j-2)} B_{jm}^{\langle -(j-2)\rangle}(a_n) - a_{n-1}^{-(j-2)} B_{jm}^{\langle -(j-2)\rangle}(a_{n-1})\Big]\,. \tag{C.10}
$$

Concept of derivation of the p-th power of the radial distance | D

In Section 3.3 the p-th power of the radial distance is expressed by the SH representation in eq. (3.26) introducing the new coefficients $B_{jm}^{\langle p \rangle}$. In this Appendix, we present the concept for the approximate determination of these coefficients and summarize the detailed derivation given by Pěč & Martinec (1984) and Pěč & Martinec (1988).

The basic idea is not to determine the coefficients $B_{jm}^{\langle p \rangle}$ precisely, but restrict an approximate solution to terms equal or less than the fourth power of the Earth's flattening α. The largest term in the expansion of the radial distance is $B_{00}Y_{00} \sim 1$. In the order of their magnitude follow $B_{20} \sim \mathcal{O}(\alpha)$, $B_{22} \sim \mathcal{O}(\alpha^2)$, $B_{30} \sim B_{31} \sim \ldots \sim B_{53} \sim \mathcal{O}(\alpha^3)$ and $B_{1\pm1} \sim B_{1\pm2} \sim B_{jm} \sim \mathcal{O}(\alpha^4)\, \forall j \geq 5$. If the dominant terms B_{00} and B_{20} considered separately the binomial expansion for the pth power accurate to $\mathcal{O}(\alpha^4)$ is given by (see Pěč & Martinec, 1984, eq. (16))

$$r_n^p(\Omega) \approx a_n^p \Bigg[B_{00}Y_{00} + \sum_{\substack{jm \\ j \geq 1}} B_{jm}(a_n) Y_{jm} \Bigg]^p .$$

Here and in the following the angular arguments (Ω) are suppressed. The binomial expansion leads to

$$\begin{aligned} r_n^p(\Omega) \approx a_n^p \Bigg\{ & (B_{00}Y_{00})^p + (B_{00}Y_{00})^{p-1} \binom{p}{1} \sum_{\substack{jm \\ j \geq 1}} B_{jm}Y_{jm} \\ & + (B_{00}Y_{00})^{p-2} \binom{p}{2} \Bigg[B_{20}Y_{20} + \sum_{\substack{jm \\ \neq (2,0)}} B_{jm}Y_{jm} \Bigg] \Bigg[B_{20}Y_{20} + \sum_{\substack{j_1m_1 \\ \neq (2,0)}} B_{j_1m_1}Y_{j_1m_1} \Bigg] \\ & + (B_{00}Y_{00})^{p-3} \binom{p}{3} \Bigg[B_{20}Y_{20} + \sum_{\substack{jm \\ \neq (2,0)}} B_{jm}Y_{jm} \Bigg] \Bigg[B_{20}Y_{20} + \sum_{\substack{j_1m_1 \\ \neq (2,0)}} B_{j_1m_1}Y_{j_1m_1} \Bigg] \\ & \cdot \Bigg[B_{20}Y_{20} + \sum_{\substack{j_2m_2 \\ \neq (2,0)}} B_{j_2m_2}Y_{j_2m_2} \Bigg] + (B_{00}Y_{00})^{p-4} \binom{p}{4} B_{20}^4 Y_{20}^{\langle 4 \rangle} \Bigg\} . \end{aligned} \tag{D.1}$$

For the next step, the terms in eq. (D.1) are rearranged according to its order of magnitude in such a way that the first terms in the parentheses are $\mathcal{O}(\alpha)$, the second terms are $\mathcal{O}(\alpha^2)$ and so on. This rearrangement yields

$$\begin{aligned} r_n^p(\Omega) \approx a_n^p \Bigg\{ & (B_{00}Y_{00})^p + p\,(B_{00}Y_{00})^{p-1} \sum_{\substack{jm \\ j \geq 1}} B_{jm}Y_{jm} + \binom{p}{2} (B_{00}Y_{00})^{p-2} \sum_{\substack{jmj_1m_1 \\ j \geq 2, j_1 \geq 2}} B_{jm}B_{j_1m_1}Y_{jm}Y_{j_1m_1} \\ & + \binom{p}{3} (B_{00}Y_{00})^{p-3} \Bigg[B_{20}^3 Y_{20}^{\langle 3 \rangle} + 3\, B_{20}^2 Y_{20}^{\langle 2 \rangle} \sum_{\substack{jm \\ \neq (2,0)}} B_{jm}Y_{jm} \Bigg] + \binom{p}{4} (B_{00}Y_{00})^{p-4} B_{20}^4 Y_{20}^{\langle 4 \rangle} \Bigg\} . \end{aligned} \tag{D.2}$$

Here, $Y_{20}^{\langle p \rangle}$ denotes the result of the product of p degree-two SH base functions. To derive the $\mathcal{B}^{\text{i}}$ in eqs. (3.37)–(3.41), we have to reformulate eq. (D.2) in terms of a SH expansion of only a single Y_{jm}. This requires to represent products of SH base functions according to eq. (3.35) in terms of $\mathbf{Q}_{j_1m_1j_2m_2}^{jm}$

as defined in eq. (3.34). We find for the second order term (with renamed indices)

$$\sum_{\substack{j_1m_1j_2m_2\\ j_1\geq 2, j_2\geq 2}} B_{j_1m_1}B_{j_2m_2}Y_{j_1m_1}Y_{j_2m_2} = \sum_{\substack{jm\\ j_1m_1j_2m_2\\ j_1\geq 2, j_2\geq 2}} B_{j_1m_1}B_{j_2m_2}\mathbf{Q}^{jm}_{j_1m_1j_2m_2}Y_{jm}\,, \tag{D.3}$$

For the third order terms, we first derive the second summand

$$B_{20}^2 Y_{20}^{\langle 2\rangle} \sum_{\substack{j_2m_2\\ \neq(2,0)}} B_{j_2m_2}Y_{j_2m_2} = B_{20}^2 Y_{20}Y_{20}\Bigg(\sum_{\substack{j_2m_2\\ \neq(2,0)}} B_{j_2m_2}\Bigg)\,,$$

which transforms with eq. (3.35) applied to degree-two SH base functions to

$$B_{20}^2 Y_{20}^{\langle 2\rangle} \sum_{\substack{j_2m_2\\ \neq(2,0)}} B_{j_2m_2}Y_{j_2m_2} = B_{20}^2\Bigg(\sum_{j_1m_1}\mathbf{Q}^{j_1m_1}_{2020}Y_{j_1m_1}\Bigg)\Bigg(\sum_{\substack{j_2m_2\\ \neq(2,0)}} B_{j_2m_2}\Bigg)\,. \tag{D.4}$$

Only for $m_1 = 0$ is valid $\mathbf{Q}^{j_1m_1}_{2020} \neq 0$, and a second time applying eq. (3.35) leads to

$$B_{20}^2 Y_{20}^{\langle 2\rangle} \sum_{\substack{j_2m_2\\ \neq(2,0)}} B_{j_2m_2}Y_{j_2m_2} = B_{20}^2 \sum_{\substack{jm\\ j_1j_2m_2\\ j_2\neq 2}} B_{j_2m_2}\mathbf{Q}^{j_10}_{2020}\mathbf{Q}^{jm}_{j_10j_2m_2}Y_{jm}\,. \tag{D.5}$$

For the first summand of the third order term, the triple product of Y_{20} can be derived from eq. (D.4) in the following way

$$\begin{aligned} Y_{20}^{\langle 3\rangle} &= \sum_{j_1}\mathbf{Q}^{j_10}_{2020}Y_{j_10}Y_{20}\,,\\ Y_{20}^{\langle 3\rangle} &= \sum_{j_1jm}\mathbf{Q}^{j_10}_{2020}\mathbf{Q}^{jm}_{j_1020}Y_{jm}\,. \end{aligned} \tag{D.6}$$

In the last term in eq. (D.2) we have still to consider $Y_{20}^{\langle 4\rangle}$, which solution can easily be obtained by the fourth multiplication of Y_{20} to eq. (D.6)

$$Y_{20}^{\langle 4\rangle} = \sum_{j_1j_2m_2}\mathbf{Q}^{j_10}_{2020}\mathbf{Q}^{j_2m_2}_{j_1020}Y_{j_2m_2}Y_{20}\,,$$

where we apply again eq. (3.35)

$$Y_{20}^{\langle 4\rangle} = \sum_{\substack{jm\\ j_1j_2m_2}}\mathbf{Q}^{j_10}_{2020}\mathbf{Q}^{j_2m_2}_{j_1020}\mathbf{Q}^{jm}_{j_2020}Y_{jm}\,.$$

From the second coupling coefficient it is visible, that it is $\mathbf{Q}^{j_2m_2}_{j_1020} \neq 0$ for $m_2 = 0$, and the final expression reads

$$Y_{20}^{\langle 4\rangle} = \sum_{\substack{jm\\ j_1j_2}}\mathbf{Q}^{j_10}_{2020}\mathbf{Q}^{j_20}_{j_1020}\mathbf{Q}^{jm}_{j_2020}Y_{jm}\,. \tag{D.7}$$

Based on eq. (D.2) it is possible to derive eq. (3.36) considering the relations in eqs. (D.3)–(D.7) and recalling that $B_{jm}^{\langle p\rangle}$ is defined for an individual base function Y_{jm}.

List of symbols | L

Symbol	Explanation	Page
a_{ICB}	radius related to the ICB	4
A^{i}_{jm}	SH coefficient of the contribution to ϕ^{i}	10
A^{o}_{jm}	SH coefficient of the contribution to ϕ^{i}	12
B	volume of the Earth separated into B_{i} and B_{o}	9
B_{i}	volume of the inner core	9
B_{o}	remaining volume outside the inner core	9
$\mathbf{C}^{jm}_{klst}$	Clebsch-Gordan coefficients defined in Varshalovich et al. (1989, Sec. 8.1)	4
$\mathrm{d}\boldsymbol{S}_{\mathsf{ICB}}$	orientated surface element of the ICB	3
$\mathrm{d}V$	infinitesimal volume element	3
∂B_n	material interface between layers with homogeneous density	9
$\mathbf{D}^{j}_{nm}$	Wigner D-function defined in Varshalovich et al. (1989, Sec. 4.3 eq. (1))	5
$\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$	Cartesian basis vectors	6
$\boldsymbol{e}_r, \boldsymbol{e}_\vartheta, \boldsymbol{e}_\varphi$	spherical basis vectors	6
E	ICB deviation from a sphere	4
$\widetilde{E}$	alternative representation of the ICB	17
E_{jm}	SH coefficient of the representation of the ICB	11
$E^{\langle p\rangle}_{jm}$	SH coefficient of the p th power of the radial distance of the ICB	11
$\widetilde{E}_{jm}$	SH coefficient of the alternative representation of the ICB	17
$\widetilde{E}^{\mathsf{REF}}_{jm}$	$\widetilde{E}_{jm}$ coefficient in the reference state	18
$\boldsymbol{L}$	gravitational coupling torque	3
$\boldsymbol{N}$	outward normal vector	3
$\mathbf{Q}^{jm}_{klst}$	coupling coefficient for the product of two scalar SH basis function	14
$\boldsymbol{r}$	position vector	3
$\boldsymbol{r}_{\mathsf{ICB}}$	position vector of point on ICB	4
$\boldsymbol{S}_{\mathsf{ICB}}$	inner-core boundary	4
$\boldsymbol{S}^{(\lambda)}_{jm}(\Omega)$	vector SH basis function, defined in eqs. (A.16)–(A.18)	4
Y_{jm}	scalar SH basis function	4
α	first Euler angle (see Fig. 2.1)	5
β	second Euler angle (see Fig. 2.1)	5
γ	third Euler angle (see Fig. 2.1)	5
$\boldsymbol{\Gamma}^{jm}_{klst}$	uniform coupling coefficient defined in eq. (2.30)	7
ρ_{IC}	volume-mass density of the IC	3
ρ_{OC}	volume-mass density of the OC	4
ρ^{Δ}	difference of volume-mass densities between IC and OC	4
ϕ	gravitational potential	3
ϕ^{i}	gravitational potential caused by masses in B_{i}	10
ϕ^{o}	gravitational potential caused by masses in B_{o}	10
Φ^{i}_{jm}	SH coefficient of the gravitational potential ϕ^{i}	11
Φ^{o}_{jm}	SH coefficient of the gravitational potential ϕ^{o}	13
Φ_{jm}	combined SH coefficient of the gravitational potential	13
$\mathrm{d}\Omega$	infinitesimal surface element of the unit sphere	4
ψ	centrifugal potential	15
Ψ_{jm}	SH coefficient of the centrifugal potential	15
Θ_{jm}	combined SH coefficient defined in eq. (2.10)	4

ISSN 1610-0956