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Core-Mantle Coupling

Part IV:
Axial component of the core angular momentum

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Abstract | A

It is well known that the decadal variations in the length of day (LOD) can be well explained by the axial component of the relative angular momentum of the core. This quantity depends only on the axial symmetric toroidal part of the fluid velocity. For velocities in the Taylor-Proudman state, through the core, moreover holds equatorial symmetry.

For velocities which fulfill the Taylor-Proudman state, we derive an analytical solution which is described for the φ-component by a free function (characterized by a special auxiliary variable) which can be connected with boundary values for given estimated core surface fluid-flow motions. The other (r and ϑ) fluid-flow components vanish.

We integrate the axial core angular momentum component for the axial symmetric toroidal velocity part, thereby using different coordinates, and find that it only depends on **one** zonal toroidal velocity mode, (t_1^0).

This result contradicts the outcome of Jault (1990) (see also Jault et al., 1988) which is based on an alternative approach for the integration. Herein the integration over the sphere is implemented by by integrating over infinitesimal thin cylinder barrels. This derivation shows that the axial core angular momentum component is governed by the **two** zonal velocity modes (t_1^0) and (t_3^0), an result which was included up to now in numerous LOD studies.

We discuss this latter approach and can relate it to our derivation for the axial core angular momentum. The cause for the theoretical discrepancy is found in the analytical integration. As a consequence, the mentioned additional term (t_3^0) has no relevance for the axial core angular momentum and thus also has to be omitted in LOD estimations.

Table of Contents

Abstract i

Table of Contents iii

1 Introduction 1

2 Relative angular momentum of the fluid core 3
2.1 Angular momentum $\boldsymbol{h}$ for a spherically symmetric core and spherical core-mantle boundary 3
2.2 Components of $\boldsymbol{h}$ with respect to a Cartesian coordinate system . . . 4

3 Model of the velocity in the Taylor-Proudman state 7
3.1 Differential equations derived from geostrophy . . . 7
3.2 Poloidal and toroidal scalar functions . . . 8
3.3 Governing equations for the axially symmetric velocity fields in the Taylor-Proudman state 9
3.4 Structure of the axial velocity component . . . 10

4 Analytical solution for the velocity in the Taylor-Proudman state 13
4.1 Transformations . . . 13
4.2 Properties of the auxiliary variable . . . 14

5 The relative angular momentum for a core in the Taylor-Proudman state 17
5.1 Spherical coordinates . . . 17
5.2 The tangential cylinder approach . . . 18
5.3 Cylinder coordinates . . . 20
5.4 Integration towards to the toroidal velocity modes after Jault . . . 21
5.5 Discussion . . . 22

6 Conclusions 25

References 29

Introduction | 1

This report continues a series of Scientific Technical Reports, in which the theoretical description of the electromagnetic (EM, see Hagedoorn & Greiner-Mai, 2008), topographic (TOP, see Greiner-Mai & Hagedoorn, 2008) and gravitational (GRAV, see Hagedoorn et al., 2012) core-mantle coupling torques are presented in detail. Based on these theoretical descriptions numerical codes were developed to compute individual coupling torques. The investigation of the toroidal geomagnetic field at the CMB and results for EM coupling torques were published in Hagedoorn et al. (2010).

The background of part IV of these Scientific Technical Report series comprehends the following two aspects:

1. The parameters like electric conductivity of the lowermost mantle necessary for our torque computations are not very well constraint by other investigations, e.g. EM induction studies. The EM coupling torque determination allows us to assume parameter values which are located somewhere in a certain physically plausible co-domain. Therefore, it is necessary to verify the resulting torque values with observed changes of polar motion and length of day, respectively (hereafter called Earth rotation parameters, ERP). For this, we have to reduce ERP variations by contributions of surface processes, i.e. of dynamics of oceans and atmospher as well as for the influence of continental hydrology, obtaining those parts of ERP variations caused by internal coupling processes ('residuals'). The surface contributions are conventionally given by their excitation functions (χ), i.e. angular momentum functions. Although the ('observed') residuals can be expressed by ('necessary') torques, we will use the more conventional concept of excitation functions (e.g. in IERS products) too, which is helpful for e.g. ocean modelers, when they will use our results for assimilation of ERP data in their models (Moritz & Mueller, 1987; Lambeck, 1988; Thomas & Sündermann, 2001; Dobslaw & Thomas, 2007; Dobslaw et al., 2010).

2. The relative (with respect to the mantle) axial core angular momentum (CAM) explains the decadal variations of the length of day very well by angular momentum conservation (e.g. Jackson et al., 1993). It is based on the investigation of Jault (1990), who used the assumption that the fluid motion within the core is organized in coaxially nested rigidly rotating cylinder annuli to determine the relative angular momentum, $\boldsymbol{h}$, of the fluid core, which is a consequence of the Taylor-Proudman theorem formulated in cylinder coordinates. The determination of h_z by these cylinder models can be seen as a refined description of the axial part of the CAM problem compared with the assumption of a pure rigid rotation of a fluid shell near the CMB e.g. by Greiner-Mai (1989), who has also considered the non-axial case (Greiner-Mai, 1990).

The structure of the report is as follows: The derivations in Chap. 2 are made at first for the complete vector $\boldsymbol{h}$. This should be a step towards the determination of its non-axially symmetric components in future. Thereafter, we will change to the special cases of the axially symmetric component h_z. Chapter 3 considers the fluid velocity in the Taylor-Proudman state, derives the corresponding differential equation systems in different coordinate systems and considers the poloidal-toroidal and spherical-harmonic decomposition. An analytical solution for the velocity in the Taylor-Proudman state is presented in Chap. 4 with some discussion on its functional dependency. The axial angular momentum component h_z is integrated for the velocities in the Taylor-Proudman state using different integration approaches (Chap. 5). The integration with the tangential cylinder approach of Jault (1990) produces a second fluid flow mode which does not appear in our result. Considering this derivation in detail, we show the cause for this deviation and can finally, after correction, establish the relationship to our result.

Relative angular momentum of the fluid core: decomposition | 2

We consider a two-component earth model consisting of mantle and core. The last one either can be modelled by a sphere or more precisely by a spherical shell. The rotation of this model ('the earth rotation') is described by means of the angular momenta of both parts including coupling torques. The angular momentum of the core shall be referred to the mantle and shall be caused by core motions. It is conventionally defined by

$$\boldsymbol{h} = \int_{V_c} \boldsymbol{r} \times \rho \boldsymbol{u} \, \mathrm{d}V \,, \tag{2.1}$$

where V_c and ρ are the core volume and its density, respectively, and $\boldsymbol{u}$ is the vector of the velocity field of the core motion in a mantle-fixed coordinate system.

The main problem to calculate the core-angular momentum is that we need the velocity field throughout the fluid core (see eq. (2.9) below), in contrast to the torque approach in which the torque expressions can be represented by surface integrals for all considered kinds of torques (see e.g. preceding reports part I to III: Hagedoorn & Greiner-Mai, 2008; Greiner-Mai & Hagedoorn, 2008; Hagedoorn et al., 2012).

The velocity field, $\boldsymbol{u}$, can be decomposed into toroidal and poloidal parts. Its decomposition for a non divergence-free field is given by

$$\boldsymbol{u} = \mathrm{rot}\,(\boldsymbol{r}Q) + \boldsymbol{r}\,V + \nabla W \,, \tag{2.2}$$

where two of the three defining scalar functions Q, V and W are normed on the surface of the unit sphere Ω by

$$\oint_{\Omega} \ldots \sin\vartheta \mathrm{d}\vartheta \, \mathrm{d}\Omega = 0 \tag{2.3}$$

(conventionally Q and V). The first summand represents the toroidal part. More details of this splitting are given e.g. in Krause & Rädler (1980, Chap. 13).

2.1 Angular momentum h for a spherically symmetric core and spherical core-mantle boundary

We study the parts of $\boldsymbol{h}$ accordingly to (2.2) decomposed into the three terms:

$$\boldsymbol{h} = \boldsymbol{h}^W + \boldsymbol{h}^V + \boldsymbol{h}^Q \tag{2.4}$$

The W-term

For $\rho = \rho(r)$, we obtain because of $\big(\boldsymbol{r} \times \boldsymbol{e}_r = 0\big)$

$$\boldsymbol{r} \times \rho \, \nabla W = \boldsymbol{r} \times \nabla\big(\rho W\big) \tag{2.5}$$

and (e.g. Smirnow, 1964, p. 296)

$$\boldsymbol{h}^W = \int_{V_c} \boldsymbol{r} \times \nabla\big(\rho W\big) \, \mathrm{d}V = -\int_{V_c} \mathrm{rot}\big(\boldsymbol{r}\rho\, W\big) \, \mathrm{d}V = \oint_{\Omega_{\mathrm{CMB}}} \rho\, W \big(\boldsymbol{r} \times \boldsymbol{n}\big) \, \mathrm{d}S \,, \tag{2.6}$$

where the unit normal vector $\boldsymbol{n}$ is $\boldsymbol{e}_r$ because of the assumed spherical symmetry so that

$$\boldsymbol{h}^W = 0 \tag{2.7}$$

follows. The scalar W has no influence on $\boldsymbol{h}$ for spherical symmetry.

The V-term

The V-term does not contribute to $\boldsymbol{h}$ in any case because of $\boldsymbol{r} \times \boldsymbol{r} \equiv 0$, and therefore, it is

$$\boldsymbol{h}^V = 0 \tag{2.8}$$

The Q-term

We finally conclude that for spherical symmetry the relative angular momentum of the core is only produced by the *toroidal part* of the velocity field, i.e.

$$\boldsymbol{h} = \boldsymbol{h}^Q = \int_{V_\mathsf{c}} \boldsymbol{r} \times \rho(r)\, \mathrm{rot}\big(\boldsymbol{r}Q\big)\, \mathrm{d}V\,. \tag{2.9}$$

Because of

$$\boldsymbol{r} \times \mathrm{rot}\big(\boldsymbol{r}Q\big) = -\boldsymbol{r} \times \big(\boldsymbol{r} \times \nabla Q\big) = -\boldsymbol{r}\, r\, \frac{\partial Q}{\partial r} + r^2 \nabla Q = r\, \nabla_\mathsf{H}\, Q \tag{2.10}$$

with (r, ϑ, φ are the spherical coordinates and $\boldsymbol{e}_{\ldots}$ are the respective unit vectors)

$$\nabla_\mathsf{H}\, Q := \frac{\partial Q}{\partial \vartheta}\, \boldsymbol{e}_\vartheta + \frac{1}{\sin\vartheta}\frac{\partial Q}{\partial \varphi}\, \boldsymbol{e}_\varphi \tag{2.11}$$

and $\rho = \rho(r)$, we obtain

$$\boldsymbol{h} = \int_{V_\mathsf{c}} \nabla_\mathsf{H}\big(r\rho\, Q\big)\, \mathrm{d}V\,. \tag{2.12}$$

2.2 Components of h with respect to a Cartesian coordinate system

With the unit vectors in Cartesian coordinates (x, y, z) we obtain

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \int_{V_\mathsf{c}} r\rho \left[\frac{\partial Q}{\partial \vartheta} \begin{pmatrix} \boldsymbol{e}_x \boldsymbol{e}_\vartheta \\ \boldsymbol{e}_y \boldsymbol{e}_\vartheta \\ \boldsymbol{e}_z \boldsymbol{e}_\vartheta \end{pmatrix} + \frac{1}{\sin\vartheta}\, \frac{\partial Q}{\partial \varphi} \begin{pmatrix} \boldsymbol{e}_x \boldsymbol{e}_\varphi \\ \boldsymbol{e}_y \boldsymbol{e}_\varphi \\ \boldsymbol{e}_z \boldsymbol{e}_\varphi \end{pmatrix} \right] \mathrm{d}V\,. \tag{2.13}$$

Evaluating the scalar products by using the relations of between the cartesian and spherical unit vectors, e.g. shown in Greiner-Mai & Hagedoorn (2008, eq. (2.30)) gives

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \int_{V_\mathsf{c}} r\rho \left[\frac{\partial Q}{\partial \vartheta} \begin{pmatrix} \cos\vartheta\, \cos\varphi \\ \cos\vartheta\, \sin\varphi \\ -\sin\vartheta \end{pmatrix} + \frac{1}{\sin\vartheta}\, \frac{\partial Q}{\partial \varphi} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \right] \mathrm{d}V\,. \tag{2.14}$$

Axially symmetric motions of u

The x, y components of $\boldsymbol{h}$ vanish for axially symmetric motions of u, because in the first and second parentheses in eq. (2.14) holds

$$\int_0^{2\pi} \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix} \mathrm{d}\varphi = 0 \quad \text{and} \quad \frac{\partial Q}{\partial \varphi} \doteq 0\,, \tag{2.15}$$

and we obtain

$$\boldsymbol{h} = \begin{pmatrix} 0 \\ 0 \\ h_z \end{pmatrix}, \qquad h_z = -\int_{V_\mathsf{c}} r\,\rho\, \frac{\partial Q}{\partial \vartheta}\, \sin\vartheta\, \mathrm{d}V\,, \tag{2.16}$$

i.e.

$$h_z = -2\pi \int_{R_\mathrm{i}}^{R_\mathrm{c}} r^3 \, \rho(r) \left[\int_0^\pi \frac{\partial Q(r,\vartheta,t)}{\partial \vartheta} \, \sin^2 \vartheta \, \mathrm{d}\vartheta \right] \mathrm{d}r \tag{2.17}$$

where R_i and R_c are the inner-core and outer-core (CMB) radii, respectively. In contrast, Jackson et al. (1993) have ignored the inner core so that they would have $R_\mathrm{i} = 0$.

Model of the velocity $\boldsymbol{u}$ in the Taylor-Proudman state | 3

3.1 Differential equations derived from geostrophy

The modelling of the velocity field within the core used here is based on the oversimplifying assumption of geostrophy throughout the fluid core. This assumption is also used as a physical constraint in the frozen-flux hypothesis of the geomagnetic secular variation (SV) to make the inversion of the frozen-flux equation with respect to $\boldsymbol{u}$ less ambiguous (e.g. Wardinski, 2005). To assume geostrophy is thoroughly consistent with the derivation of $\boldsymbol{u}$ from the geomagnetic SV at the CMB using the same constraint.

The geostrophic approximation of the Navier-Stokes equation is given by

$$2\,\Omega\,\boldsymbol{e}_z \times \boldsymbol{u} = -\frac{1}{\rho_0}\nabla p\,, \tag{3.1}$$

where Ω and p are the (axial) angular velocity and fluid pressure, respectively. Some reasoning respectively the involved approximations to derive eq. (3.1) from the Navier-Stokes equation are given in Greiner-Mai & Hagedoorn (2008). Assuming that ρ_0 is constant, the application of the rot-operator to eq. (3.1) gives for the left hand side four terms of which two ones remain

$$\big(\boldsymbol{e}_z \cdot \mathrm{grad}\big)\,\boldsymbol{u} - \boldsymbol{e}_z\ \mathrm{div}\ \boldsymbol{u}\ =\ 0. \tag{3.2}$$

Assuming $\mathrm{div}\,\boldsymbol{u} = 0$, we obtain

$$\big(\boldsymbol{e}_z \cdot \mathrm{grad}\big)\,\boldsymbol{u} = 0 \qquad \text{or equivalently} \tag{3.3}$$

$$\frac{\partial \boldsymbol{u}}{\partial z} = 0\,. \tag{3.4}$$

Remark: At this place, it can easily be seen that a (differential) rigid rotation of the fluid with the angular velocity $\omega(s)$, $\boldsymbol{u} = \boldsymbol{e}_\varphi s\,\omega(s)$, is a solution of eq. (3.3), which will used later on to check some further derivations. In this connection, it should be mentioned that the problem to solve for the dependency of $\boldsymbol{u}$ or ω on e.g. s is under-determined by eq. (3.4).

Cartesian coordinates – (x, y, z)

Eq. (3.4) is immediately equivalent to the scalar differential equations

$$\frac{\partial u_x}{\partial z} = 0\,, \quad \frac{\partial u_y}{\partial z} = 0\,, \quad \frac{\partial u_z}{\partial z} = 0\,. \tag{3.5}$$

Cylindrical coordinates – (s, φ, z)

Because of

$$\frac{\partial \boldsymbol{e}_s}{\partial z} = 0\,, \quad \frac{\partial \boldsymbol{e}_\varphi}{\partial z} = 0\,, \quad \frac{\partial \boldsymbol{e}_z}{\partial z} = 0\,, \tag{3.6}$$

the Taylor-Proudman theorem in cylindrical coordinates reads, like in Cartesian coordinates, as

$$\frac{\partial u_s}{\partial z} = 0\,, \quad \frac{\partial u_\varphi}{\partial z} = 0\,, \quad \frac{\partial u_z}{\partial z} = 0\,. \tag{3.7}$$

Spherical coordinates – (r, ϑ, φ)

To use the frozen-flux values of $\boldsymbol{u}$ at the CMB, which are given by the coefficients of the spherical harmonic expansion of its defining scalars, as boundary values for the problem (3.3), we rewrite (3.3) in spherical coordinates, and will try to find a solution in Chap. 4 below. Using

$$\boldsymbol{e}_z = \cos\vartheta\, \boldsymbol{e}_r - \sin\vartheta\, \boldsymbol{e}_\vartheta \tag{3.8}$$

we obtain from eq. (3.3)

$$\cos\vartheta \frac{\partial \boldsymbol{u}}{\partial r} - \frac{\sin\vartheta}{r} \frac{\partial \boldsymbol{u}}{\partial \vartheta} = 0 \tag{3.9}$$

or

$$\frac{\partial \boldsymbol{u}}{\partial r} = \frac{\tan\vartheta}{r} \frac{\partial \boldsymbol{u}}{\partial \vartheta}\,. \tag{3.10}$$

With

$$\frac{\partial \boldsymbol{e}_r}{\partial r} = 0, \quad \frac{\partial \boldsymbol{e}_r}{\partial \vartheta} = \boldsymbol{e}_\vartheta, \quad \frac{\partial \boldsymbol{e}_\vartheta}{\partial r} = 0, \quad \frac{\partial \boldsymbol{e}_\vartheta}{\partial \vartheta} = -\boldsymbol{e}_r, \quad \frac{\partial \boldsymbol{e}_\varphi}{\partial r} = 0, \quad \frac{\partial \boldsymbol{e}_\varphi}{\partial \vartheta} = 0, \tag{3.11}$$

the full boundary value (**bv**) problem is given by (*ff* denotes frozen-flux values)

$$\left.\begin{aligned} \frac{\partial u_\varphi}{\partial r} &= \frac{\tan\vartheta}{r}\frac{\partial u_\varphi}{\partial \vartheta} \\ \frac{\partial u_r}{\partial r} &= \frac{\tan\vartheta}{r}\left(\frac{\partial u_r}{\partial \vartheta} - u_\vartheta\right) \\ \frac{\partial u_\vartheta}{\partial r} &= \frac{\tan\vartheta}{r}\left(\frac{\partial u_\vartheta}{\partial \vartheta} + u_r\right) \end{aligned}\right\} \quad \textbf{bv} = \begin{cases} u_\varphi(R_\mathrm{c}) &= u_\varphi(\mathit{ff}) \\ u_r(R_\mathrm{c}) &= 0 \\ u_\vartheta(R_\mathrm{c}) &= u_\vartheta(\mathit{ff}) \end{cases} \tag{3.12}$$

An interesting conclusion from the last two equations is

$$\left[\frac{\partial}{\partial r} - \frac{\tan\vartheta}{r}\frac{\partial}{\partial \vartheta}\right](u_r^2 + u_\vartheta^2) = 0\,. \tag{3.13}$$

which looks like a constraint of the spatial distribution of the energy of the non-meridional part of the velocity field. Further, we can see that the differential equation for u_φ is decoupled from those of u_ϑ and u_r, whereas the latter two are coupled with each other.

3.2 Poloidal and toroidal scalar decomposition of $\boldsymbol{u}$ for $\operatorname{div}\boldsymbol{u} = 0$

For a divergence free $\boldsymbol{u}$, its decomposition into poloidal and toroidal parts is given by

$$\boldsymbol{u} = \operatorname{rot}\left\{\operatorname{rot}\left[\boldsymbol{r}\, S(r,\vartheta,\varphi,t)\right] + \boldsymbol{r}\, Q(r,\vartheta,\varphi,t)\right\}. \tag{3.14}$$

At first we will show that eq. (2.9) also can be derived for this special case. The second term in eq. (3.14) leads to eq. (2.9). The first term can be expressed by (e.g. Krause & Rädler, 1980)

$$\operatorname{rot}\,\operatorname{rot}\,\boldsymbol{r}\, S = -\boldsymbol{r}\,\Delta\, S + \nabla\left(\frac{\partial}{\partial r}(rS)\right),$$

where Δ is the Laplacean. Applying this term to $\boldsymbol{u}$ in eq. (2.1) for $\boldsymbol{h}$, the first part vanishes identically and the second can be handled as ∇W in eq. (2.6).

The components of $\boldsymbol{u}$ in eq. (3.14) are given by

$$\begin{aligned} u_\varphi &= -\frac{\partial Q}{\partial \vartheta} + \frac{1}{r\,\sin\vartheta}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial \varphi}\right) \\ u_\vartheta &= \frac{1}{r\,\sin\vartheta}\frac{\partial Q}{\partial \varphi} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial \vartheta}\right) \\ u_r &= -\frac{\Omega S}{r}\,, \end{aligned} \tag{3.15}$$

where Ω is the Laplacean on the sphere

$$\Omega S = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial S}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 S}{\partial\varphi^2}\,. \tag{3.16}$$

Because for the associated Legendre functions $Y_{nm}(\vartheta,\varphi)$ holds

$$\Omega Y_{nm} = -n(n+1)\,Y_{nm}\,, \tag{3.17}$$

the boundary condition $u_r(R_c) = 0$ (see last line in eq. (3.15)) leads to that of the harmonic modes $S_{nm}(r,t)$ of $S(r,\vartheta,\varphi)$, $S(r,\vartheta,\varphi) = \sum_{nm} S_{nm}(r,t)Y_{nm}(\vartheta,\varphi)$

$$S_{nm}(R_c,t) = 0 \qquad \Rightarrow \qquad S(R_c,\vartheta,\varphi,t) = 0\,. \tag{3.18}$$

Introducing a new scalar V as

$$V = \frac{\partial}{\partial r}\,(r\,S)\,, \tag{3.19}$$

we obtain from eqs. (3.12) and (3.15) the following system

$$\begin{aligned}
-\frac{\partial^2 Q}{\partial\vartheta\partial r} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial V}{\partial\varphi}\right) &= \frac{\tan\vartheta}{r}\left[-\frac{\partial^2 Q}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial V}{\partial\varphi}\right)\right]\\
\frac{\partial}{\partial r}\left(\frac{\Omega\,S}{r}\right) &= \frac{\tan\vartheta}{r^2}\left[\frac{\partial(\Omega S)}{\partial\vartheta} + \frac{r}{\sin\vartheta}\frac{\partial Q}{\partial\varphi} + \frac{\partial V}{\partial\vartheta}\right]\\
\frac{1}{\sin\vartheta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial Q}{\partial\varphi}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial V}{\partial\vartheta}\right) &= \frac{\tan\vartheta}{r^2}\left[\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial Q}{\partial\varphi}\right) + \frac{\partial^2 V}{\partial\vartheta^2} - \Omega S\right].
\end{aligned} \tag{3.20}$$

Later on, we want to specify the system for the axially symmetric case. To prepare this, we summarize the expressions with $\partial/\partial\varphi$ and obtain after some manipulations with partial derivations

$$\sin^2\vartheta\,\frac{\partial^2 Q}{\partial\vartheta^2} - r\sin\vartheta\,\cos\vartheta\,\frac{\partial^2 Q}{\partial r\partial\vartheta} = \frac{\partial}{\partial\varphi}\left[\frac{\sin\vartheta}{r}\frac{\partial V}{\partial\vartheta} - \cos\vartheta\,\frac{\partial V}{\partial r}\right] \tag{3.21}$$

$$\cos\vartheta\,\Omega\left(S - r\,\frac{\partial S}{\partial r}\right) + \sin\vartheta\left(\frac{\partial(\Omega S)}{\partial\vartheta} + \frac{\partial V}{\partial\vartheta}\right) = -r\frac{\partial Q}{\partial\varphi} \tag{3.22}$$

$$\sin\vartheta\left(\Omega S - \frac{\partial^2 V}{\partial\vartheta^2}\right) + \cos\vartheta\left(r\,\frac{\partial^2 V}{\partial r\partial\vartheta} - \frac{\partial V}{\partial\vartheta}\right) = \frac{\partial}{\partial\varphi}\left[\frac{\partial Q}{\partial\vartheta} - r\,\cot\vartheta\,\frac{\partial Q}{\partial r}\right]. \tag{3.23}$$

In contrast to eqs. (3.12), these equations are fully coupled, which is a consequence of the splitting of $\boldsymbol{u}$ into poloidal and toroidal parts (see, e.g. the coupling of the components in eqs. (3.15)).

3.3 Governing equations for the axially symmetric velocity fields in the Taylor-Proudman state

In the following, we will prepare a solution for the axially symmetric case only. With

$$\frac{\partial}{\partial\varphi}(\ldots) = 0\,, \quad T = \frac{\partial Q}{\partial\vartheta} \quad \text{poloidal}, \quad U = \frac{\partial S}{\partial\vartheta} \quad \text{toroidal} \tag{3.24}$$

we obtain from eqs. (3.21)–(3.23) the system of differential equations

$$\begin{aligned}
\frac{\tan\vartheta}{r}\frac{\partial T}{\partial\vartheta} - \frac{\partial T}{\partial r} &= 0 \qquad \text{toroidal}\\
\frac{\tan\vartheta}{r}\frac{\partial^2 U}{\partial\vartheta^2} - \frac{\partial^2 U}{\partial\vartheta\partial r} + \frac{2}{r}\frac{\partial U}{\partial\vartheta} + \frac{\sin^2\vartheta - \cos^2\vartheta}{\sin\vartheta\,\cos\vartheta}\frac{\partial U}{\partial r} &= 0 \qquad \text{poloidal}\\
\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{\tan\vartheta}{r}\frac{\partial^2 U}{\partial\vartheta\partial r} &= 0 \quad \text{poloidal}
\end{aligned} \tag{3.25}$$

which shows that the poloidal and toroidal fields are now decoupled.

Further, we will show in the following that it is sufficient to assume axis symmetry of Q, and that the poloidal field is then also axially symmetric. Equation (3.21) can be written formally as

$$F(r,\vartheta,\varphi,t) = \frac{\partial G(r,\vartheta,\varphi,t)}{\partial\varphi}\,, \tag{3.26}$$

where F and G are continuously differentiable functions. The integral over a closed φ-circuit is given by

$$\oint_{(\varphi)} F(r,\vartheta,\varphi,t)\,\mathrm{d}\varphi = G(r,\vartheta,2\pi,t) - G(r,\vartheta,0,t) = 0\,, \tag{3.27}$$

where the zero value is a result of uniqueness independently on the symmetry of F. If Q is axially symmetric as assumed for eqs. (3.25), F is it too by its definition as the left-hand side of eq. (3.21) by Q, and, by use of eq. (3.27), the closed integral results

$$0 \overset{eq.\ (3.27)}{=} \oint F(r,\vartheta,\varphi,t)\,\mathrm{d}\varphi = 2\pi\,F(r,\vartheta,t) \quad\Rightarrow\quad F(\ldots)=0 \quad\Rightarrow\quad \frac{\partial G(\ldots)}{\partial\varphi} = 0\,. \tag{3.28}$$

That means the right hand side of the equation (3.21) defined by V does not depend on φ, i.e. G and, therefore, V or S cannot depend on φ. We have shown that S is axially symmetric if Q is it.

3.4 Structure of the axial velocity component u_φ

In the following we keep the axially symmetry for the velocity field. For the computation of h_z according to eq. (2.16), we only need the toroidal scalar function Q. Fortunately, the differential equation for Q is decoupled from that for S in the axially symmetric case, i.e. we can compute $Q(r,\vartheta,t)$ without knowledge about the poloidal scalar function S. The first equation of the system (3.25) gives for Q

$$\sin\vartheta\,\frac{\partial^2 Q}{\partial\vartheta^2} - r\,\cos\vartheta\,\frac{\partial^2 Q}{\partial r\partial\vartheta} = 0\,. \tag{3.29}$$

It can clearly be seen (e.g. eq. (3.15), $u_\varphi = -\partial Q/\partial\vartheta$) that a **rigid** rotation of the fluid with $\boldsymbol{u} = \boldsymbol{e}_\varphi\,\omega\,r\sin\vartheta$ ($s = r\sin\vartheta$) gives $Q = -\omega r\cos\vartheta$, which is a solution of (3.29), as it was found earlier that the rigid rotation is a solution of eq. (3.3).

In the following, we will **expand** Q into a series of **spherical harmonic (SH) functions** to connect it continuously with its boundary value at the CMB, represented in this SH representation, too. The axially symmetric case ($m=0$) allows it to use Legendre polynomials from the beginning. The representation of Q by Legendre polynomials P_n^m is given by

$$Q = \sum_{n=1}^{\infty} q_n(r,t)\,P_n^0(\cos\vartheta)\,. \tag{3.30}$$

A further constraint of u_φ can be derived from eqs. (3.3)–(3.4) which gives

$$\int_z \frac{\partial\boldsymbol{u}}{\partial z}\,\mathrm{d}z = 0 = \boldsymbol{u}(z) - \boldsymbol{u}(-z) \quad\Rightarrow\quad \boldsymbol{u}(z) = \boldsymbol{u}(-z) \tag{3.31}$$

i.e. **the velocity field must be equatorially symmetric**. This can be expressed by the polar angle ϑ as

$$\boldsymbol{u}(\vartheta) = \boldsymbol{u}(\pi-\vartheta), \tag{3.32}$$

i.e. the **even** coefficients in the SH expansion of Q (eq.(3.30)) must vanish. This can be seen by

$$u_\varphi = -\frac{\partial Q}{\partial\vartheta} = \sum_n q_n P_n^1(\cos\vartheta) \doteq \sum_n q_n P_n^1(\cos[\pi-\vartheta]) = \sum_n q_n P_n^1(-\cos\vartheta) \tag{3.33}$$

and (e.g. Kautzleben, 1965, eq.(224))

$$P_n^m(-\cos\vartheta) = (-1)^{n-m} P_n^m(\cos\vartheta)\,, \tag{3.34}$$

which gives for $m = 1$

$$\sum_n q_n P_n^1(\cos\vartheta)[1 - (-1)^{n-1}] \doteq 0 \quad \Rightarrow \quad q_n[1 + (-1)^n] = 0\,, \tag{3.35}$$

from which the follows that $q_n = 0$ for even n. Thus for u_φ finally holds

$$u_\varphi = \sum_{n\ odd} q_n(r,t) P_n^1(\cos\vartheta). \tag{3.36}$$

4 Analytical solution for the velocity $\boldsymbol{u}$ in the Taylor-Proudman state

4.1 Transformations

In this section, we will derive a general solution of the system of differential equations (3.5) respectively (3.12) for the components of $\boldsymbol{u}$ as a function of (x, y, z, t) respectively $(r, \vartheta, \varphi, t)$ (abandoning the assumption of axial symmetry). In Cartesian coordinates, eq. (3.5) has the general solution

$$u_x = C_1(x, y)\,, \qquad u_y = C_2(x, y)\,, \qquad u_z = C_3(x, y)\,, \tag{4.1}$$

where C_i are arbitrary functions of x, y. Changing to spherical coordinates

$$x = r\,\sin\vartheta\,\cos\varphi\,, \qquad y = r\,\sin\vartheta\,\sin\varphi\,, \qquad z = r\,\cos\vartheta\,, \tag{4.2}$$

we obtain

$$\begin{aligned}
u_r &= u_x\,\sin\vartheta\,\cos\varphi + u_y\,\sin\vartheta\,\sin\varphi + u_z\,\cos\vartheta \\
u_\vartheta &= u_x\,\cos\vartheta\,\cos\varphi + u_y\,\cos\vartheta\,\sin\varphi - u_z\,\sin\vartheta \\
u_\varphi &= -u_x\,\sin\varphi + u_y\,\cos\varphi
\end{aligned} \tag{4.3}$$

from which follows

$$\begin{aligned}
u_r &= C_1(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi)\,\sin\vartheta\,\cos\varphi \\
&\quad + C_2(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi)\,\sin\vartheta\,\sin\varphi \\
&\quad + C_3(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi)\,\cos\vartheta \\
u_\vartheta &= C_1(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi)\,\cos\vartheta\,\cos\varphi \\
&\quad + C_2(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi)\,\cos\vartheta\,\sin\varphi \\
&\quad + C_3(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi)\,\sin\vartheta \\
u_\varphi &= -C_1(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi)\,\sin\varphi \\
&\quad + \; C_2(r\,\sin\vartheta\,\cos\varphi, r\,\sin\vartheta\,\sin\varphi)\,\cos\varphi\,.
\end{aligned} \tag{4.4}$$

Further, after the equivalent transformation of the differential equation, we have obtained the differential equation (3.12) shown in Sect. 3.1. Applying the partial derivatives, appearing there, to the velocity field in eqs. (4.4) yields for u_φ as an example (arguments in C_j are suppressed)

$$\begin{aligned}
\frac{\partial u_\varphi}{\partial r} &= -\frac{\partial C_1}{\partial r}\,\sin\varphi + \frac{\partial C_2}{\partial r}\,\cos\varphi \\
\frac{\partial u_\varphi}{\partial \vartheta} &= -\frac{\partial C_1}{\partial \vartheta}\,\sin\varphi + \frac{\partial C_2}{\partial \vartheta}\,\cos\varphi \\
\frac{\partial C_{1,2}}{\partial r} &= \frac{\partial C_{1,2}}{\partial x}\,\frac{\partial x}{\partial r} + \frac{\partial C_{1,2}}{\partial y}\,\frac{\partial y}{\partial r} \\
&= \frac{\partial C_{1,2}}{\partial x}\,\sin\vartheta\,\cos\varphi + \frac{\partial C_{1,2}}{\partial y}\,\sin\vartheta\,\sin\varphi \\
\frac{\partial C_{1,2}}{\partial \vartheta} &= \frac{\partial C_{1,2}}{\partial x}\,\frac{\partial x}{\partial \vartheta} + \frac{\partial C_{1,2}}{\partial y}\,\frac{\partial y}{\partial \vartheta} \\
&= \frac{\partial C_{1,2}}{\partial x}\,r\,\cos\vartheta\,\cos\varphi + \frac{\partial C_{1,2}}{\partial y}\,r\,\cos\vartheta\,\sin\varphi\,.
\end{aligned} \tag{4.5}$$

We can see that the velocity field in eqs. (4.4) is a solution of eq. (3.12) in spherical coordinates (e.g. $r\cos\vartheta$ in (4.5) is cancelled by the factor $\tan\vartheta/r$ in (3.12)). Its boundary values at $r = R_c$ are given (compare with their special formulation in eq. (3.12)) in both coordinate systems by

$$
\begin{aligned}
u_r(R_c,\vartheta,\varphi) &= 0 \\
u_\vartheta(R_c,\vartheta,\varphi) &= f_2(\vartheta,\varphi) \\
u_\varphi(R_c,\vartheta,\varphi) &= f_1(\vartheta,\varphi)
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
u_x(x,y,z) &= C_1(R_c\sin\vartheta\cos\varphi, R_c\sin\vartheta\sin\varphi) \\
&= -f_1(\vartheta,\varphi)\sin\varphi + f_2(\vartheta,\varphi)\; cos\varphi\cos\vartheta \\
u_y(x,y,z) &= C_2(R_c\sin\vartheta\cos\varphi, R_c\sin\vartheta\sin\varphi) \\
&= \quad f_1(\vartheta,\varphi)\cos\varphi + f_2(\vartheta,\varphi)\sin\varphi\cos\vartheta \\
u_z(x,y,z) &= C_3(R_c\sin\vartheta\cos\varphi, R_c\sin\vartheta\sin\varphi) \\
&= -f_2(\vartheta,\varphi)\sin\vartheta\,,
\end{aligned}
\tag{4.7}
$$

where the inverse transformations to eq. (4.3) are used. Then, we have to find a specific solution of eq. (3.12) which fulfills these boundary conditions. The construction of such solutions of partial differential equations of first order (Cauchy problems) is described in textbooks (e.g. Zwillinger, 1997, see Sec. 99.). The solution of this kind is given by

$$
\begin{aligned}
u_r(r,\vartheta,\varphi) &= f_2(\hat\vartheta,\varphi)\left[\sqrt{1-\left(\frac{r}{R_c}\right)^2\sin^2\vartheta}\,\sin\vartheta - \frac{r}{R_c}\sin\vartheta\cos\vartheta\right] \\
u_\vartheta(r,\vartheta,\varphi) &= f_2(\hat\vartheta,\varphi)\left[\sqrt{1-\left(\frac{r}{R_c}\right)^2\sin^2\vartheta}\,\cos\vartheta + \frac{r}{R_c}\sin^2\vartheta\right] \\
u_\varphi(r,\vartheta,\varphi) &= f_1(\hat\vartheta,\varphi)
\end{aligned}
\tag{4.8}
$$

with

$$
\hat\vartheta(\vartheta,r) = \arctan\left(\frac{\sin\vartheta}{\sqrt{\left(\frac{R_c}{r}\right)^2 - \sin^2\vartheta}}\right)\,,
\tag{4.9}
$$

which can be checked by inserting it in eq. (3.12) respectively in the boundary conditions at $r = R_c$, where $\hat\vartheta$ typically changes to ϑ. The appearance of arbitrary functions $f_{1,2}$ reflects again the circumstance that $\boldsymbol{u}$ is under-determined by the initial equation (3.3) and (3.4). Also typical is that u_φ is decoupled from u_ϑ and u_r, whereas the latter are coupled with each other by f_2 (cf. eqs. (3.12)).

4.2 Properties of the auxiliary variable $\hat\vartheta$

Now, we will look for properties and consequences of the auxiliary variable $\hat\vartheta$.

1) From the last equation of (4.8) results that $\hat\vartheta = \vartheta$ for $r = R_c$, i.e. $\hat\vartheta$ passes into the conventional spherical coordinate, ϑ, measured as pole distance on the sphere approximating the CMB. As seen by eqs. (4.8), u_r is zero at the CMB, where f_2 must not necessarily vanish. This implies that (i) u_r must not be necessarily zero for $r < R_c$ and (ii) u_ϑ is not zero for $r \leq R_c$ in the most general case. Because we only need u_φ for computing h_z, which is decoupled from u_ϑ, we can continue with a further evaluation of u_φ.

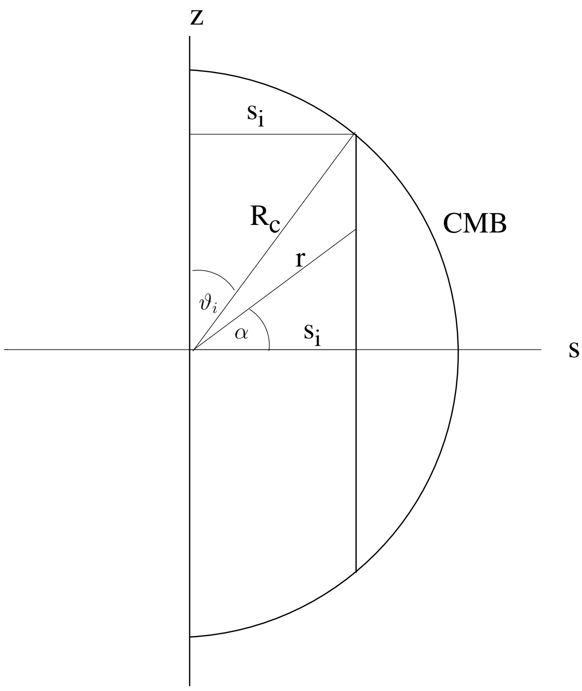

Figure 4.1: Scheme to demonstrate the angles ϑ_i and α related to a cylindrical shell in a spherical core.

2) We have found in Sec. 2.2 that $u_\varphi = u_\varphi(s)$ is a solution of eq. (3.3), i.e. $u_\varphi(s)$ is constant on any coaxial cylinder surface $s = \mathrm{const} = s_i$ and is equatorially symmetric. This does not mean that $\hat{\vartheta}$ is automatically constant there too, and we will look for the special r-dependence of $\hat{\vartheta}$ in $u_\varphi = f_1(\hat{\vartheta})$ in eq. (4.8) on this surface. According to the sketch in Fig. (4.1), we can see that

$$s_i = R_c \sin\vartheta_i = \mathrm{const} = r\cos\alpha \tag{4.10}$$

with

$$\vartheta_i \in \left[0, \frac{\pi}{2}\right] \quad \Rightarrow \quad r \in [s_i, R_c]\,, \quad \alpha \in \left[0, \frac{\pi}{2} - \vartheta_i\right], \tag{4.11}$$

i.e. r varies according to $(r\cos\alpha) = \mathrm{const}$ on the cylinder surface $s = s_i$, giving the value of this constant as radius of the cylinder under consideration. From (4.10) also follows that on those surfaces

$$\hat{\vartheta}(\vartheta_i, r) = \arctan\left(\frac{\sin\vartheta_i}{\sqrt{\left(\frac{R_c}{r}\right)^2 - \sin^2\vartheta_i}}\right) = \arctan\left(\frac{1}{\sqrt{\left(\frac{R_c}{s_i}\right)^4 \cos^2\alpha - 1}}\right) \tag{4.12}$$

is valid. From eqs. (4.12) and (4.10) follows that

$$\hat{\vartheta} = \vartheta_i \quad \text{for} \quad \alpha = \frac{\pi}{2} - \vartheta_i\,,$$

as expected from the definition of $\hat{\vartheta}$ above.

3) From eq. (4.9) follows that $\hat{\vartheta}(r,\vartheta,t) = \hat{\vartheta}(r,\pi-\vartheta,t)$, i.e. $\hat{\vartheta}(r,\vartheta,t)$ is equatorially symmetric. Then, the functions $f_{1,2}(\hat{\vartheta},r,\varphi,t)$ in eq. (4.8) are equatorially symmetric too.

4) At the end, we can only state that $\hat{\vartheta}$ varies on $s = s_i$ not as an usual angle like α, which can be underlined by its value at the other bound of α

$$\hat{\vartheta} = \arctan\left(\frac{s_i^2}{\sqrt{R_c^4 - s_i^4}}\right) \quad \text{for} \quad \alpha = 0\,,$$

and connects r and ϑ in a complicated non-linear way. It is furthermore important to know that the radius variable, r, in the integrand of h_z, e.g. in eqs. (2.14) or (2.16), is not constant on those cylinder surfaces although the other part, u_φ, of the integrand is it. This is not the problem if we work in spherical coordinates from the beginning, but it becomes significant if we choose such cylinder surfaces as finite parts of the volume element (as demonstrated below in Chap. 5).

5) With the constraint of equatorial symmetry of $\boldsymbol{u}$, given by $\boldsymbol{u}(\vartheta) = \boldsymbol{u}(\pi - \vartheta)$, and that of $f_{1,2}$ (see point 3 above), we can derive from eq. (4.8) for any pair of equatorially symmetric points (r, ϑ, φ), $(r, \pi - \vartheta, \varphi)$ the relations

$$u_r(\vartheta) - u_r(\pi - \vartheta) \doteq 0 = -2\, f_2 \cdot \frac{r}{R_\mathrm{c}}\, \sin\vartheta\, \cos\vartheta\,, \tag{4.13}$$

$$u_\vartheta(\vartheta) - u_\vartheta(\pi - \vartheta) \doteq 0 = +2\, f_2 \sqrt{1 - \left(\frac{r}{R_\mathrm{c}}\right)^2 \sin^2\vartheta}\, \cos\vartheta\,. \tag{4.14}$$

from which the condition

$$f_2(\hat{\vartheta}, \varphi, r, t) = 0 \tag{4.15}$$

follows, i.e. the function f_2 must vanish, and therefore also the velocity components

$$\boldsymbol{u}_\vartheta = \boldsymbol{u}_r = 0. \tag{4.16}$$

Thus, the original system (4.8) is substantially simplified.

5 The relative angular momentum for a core in the Taylor-Proudman state

5.1 Spherical coordinates

To compute the axial part h_z of the relative angular momentum, eq. (2.17)

$$h_z = -2\pi \int_{R_i}^{R_c} r^3\,\rho(r) \left[\int_0^\pi \frac{\partial Q(r,\vartheta,t)}{\partial\vartheta}\,\sin^2\vartheta\,\mathrm{d}\vartheta \right] \mathrm{d}r \tag{5.1}$$

has to be rewritten. With the spherical harmonic (SH) expansion of Q considering Legendre polynomials given by eq. (3.30) we obtain

$$h_z = -2\pi \sum_{n=1}^{N} \int_{R_i}^{R_c} r^3\,\rho(r)\,q_n(r,t) \left[\int_0^\pi \frac{\partial P_n^0(\vartheta)}{\partial\vartheta}\,\sin^2\vartheta\,\mathrm{d}\vartheta \right] \mathrm{d}r\,. \tag{5.2}$$

According to the relation

$$\frac{\partial P_n^0(\cos\vartheta)}{\partial\vartheta} = -P_n^1(\cos\vartheta) \tag{5.3}$$

(e.g. see Kautzleben, 1965, p.34,eq. (260a)) we obtain for the ϑ-integration

$$\int_0^\pi \left(\frac{\partial P_n^0}{\partial\vartheta}\,\sin\vartheta \right) \sin\vartheta\,\mathrm{d}\vartheta = -\int_0^\pi \left(P_n^1\,\sin\vartheta \right) \sin\vartheta\,\mathrm{d}\vartheta = -\int_0^\pi \left(P_n^1 P_1^1 \right) \sin\vartheta\,\mathrm{d}\vartheta\,.$$

In Ferrers-Neumann ($P_n^{m\,(FN)}$) normalization we obtain

$$-\int_0^\pi \left(P_n^1 P_1^1 \right) \sin\vartheta\,\mathrm{d}\vartheta = -\frac{2}{2n+1}\,\frac{(n+1)!}{(n-1)!}\,\delta_{1n}\,,$$

where δ_{mn} is the Kronecker symbol. Because the SH coefficients of the velocity field at the CMB are conventionally determined in Schmidt's normalization ($P_n^{m\,(S)}$), the following relation (e.g. Kautzleben, 1965) has to be considered

$$P_n^{m\,(S)} = \sqrt{(2-\delta_{0m})\frac{(n-m)!}{(n+m)!}}\;P_n^{m\,(FN)}\,, \tag{5.4}$$

which gives no change in the case $n=1, m=1$. We finally obtain

$$\int_0^\pi \left(\frac{\partial P_n^0}{\partial\vartheta}\,\sin\vartheta \right) \sin\vartheta\,\mathrm{d}\vartheta = -\frac{2\sqrt{2}}{2n+1}\sqrt{\frac{(n+1)!}{(n-1)!}}\,\delta_{1n} = -\frac{4}{3}\,\delta_{1n}\,. \tag{5.5}$$

With eqs. (5.2) and (5.5), h_z is then given by

$$h_z = \frac{8\pi}{3} \int_{R_i}^{R_c} q_1(r,t)\,r^3\,\rho(r)\,\mathrm{d}r\,. \tag{5.6}$$

According to eq. (3.14), the defining scalar Q and, therefore, q_1 too, have the physical unit of a velocity, i.e. m/sec. The unit of h_z then has the unit of Kg m^2/sec, i.e. the correct (physical) unit.

In the **special case of a rigidly rotating fluid throughout the whole core with constant density**, q_1 is given by $\omega\, r$, where $\omega(t)$ is the time variable angular velocity of the relative rotation of the fluid. In this case, the angular momentum h_z is then given by

$$h_z = \frac{8\pi}{3}\,\omega \int\limits_0^{R_c} \rho\, r^4\, \mathrm{d}r = \frac{8\pi}{15}\,\rho_0\, R_c^5\, \omega\,, \tag{5.7}$$

where the factor of ω is the well-known moment of inertia of the sphere with radius R_c. The same expression is obtained, if we insert $Q = -\omega r \cos\vartheta$ in eq. (2.17).

Using eq. (5.12)(see below), we obtain for the analytical solution (4.8) in Sec. 4.1.

$$h_z = \int\limits_{V_c} r\,\rho\, u_\varphi\, \sin\vartheta\, \mathrm{d}V = \int\limits_{V_c} r\,\rho\, f_1(r,\vartheta,\varphi,t)\, \sin\vartheta\, \mathrm{d}V\,. \tag{5.8}$$

5.2 The tangential cylinder approach

In the following, we will re-derive the expression for the axial relative angular momentum h_z given e.g. by Jault (1990) or Jackson et al. (1993) for a cylinder model of the fluid core motion, because there appears a second term for n = 3 which cannot be produced in our derivation of eq. (5.6). To be close to their derivations, we use a scheme which is slightly different to that in Chap. 2 and Sec.5.1.

The angular momentum, $\boldsymbol{h}$, of the core motion relative to the mantle is defined by eq. (2.1) as follows

$$\boldsymbol{h} = \int\limits_{V_c} \boldsymbol{r} \times \rho \boldsymbol{u}\, \mathrm{d}V \tag{5.9}$$

($\boldsymbol{u}$ velocity field, V_c volume of the fluid core). The cross product gives

$$\boldsymbol{h} = \int\limits_{V_c} r\,\rho\big(-u_\varphi\, \boldsymbol{e}_\vartheta + u_\vartheta\, \boldsymbol{e}_\varphi\big)\, \mathrm{d}V \tag{5.10}$$

where r, ϑ, φ are the polar coordinates and $\boldsymbol{e}_{\dots}$ is their unit vector. For the derivation of the z component we use the relation between the Cartesian unit vector $\boldsymbol{i}_z$ and $\boldsymbol{e}_{\dots}$

$$\boldsymbol{i}_z = \boldsymbol{e}_r\, \cos\vartheta - \boldsymbol{e}_\vartheta\, \sin\vartheta\,. \tag{5.11}$$

The z component of $\boldsymbol{h}$, responsible for the variation of length-of-day (Δ LOD), is then given by

$$h_z = \boldsymbol{i}_z \cdot \boldsymbol{h} = \int\limits_{V_c} \boldsymbol{i}_z \cdot (\boldsymbol{r} \times \rho \boldsymbol{u})\, \mathrm{d}V = \int\limits_{V_c} r\,\rho\, u_\varphi\, \sin\vartheta\, \mathrm{d}V \tag{5.12}$$

which is equivalent to eq. (2.16). In Jault (1990) and Jackson et al. (1993) the following assumptions are made:

1. The density is constant throughout the whole core.
2. The inner core is replaced by the outer-core liquid (does not exist).
3. The zonal part of fluid motion is organized in cylinder annulies, which rotate rigidly about the z axis of the Earth.

According to these assumptions, the spherical core is divided into infinitesimal small coaxial cylinder annulies which end at the spherical CMB with their top and bottom circles (see Fig. 5.1). This hypothesis allows the cylinder annulies to rotate rigidly with s-dependent angular velocity, which fulfills the Taylor-Proudman theorem or its mathematical pendant $\partial \boldsymbol{u}/\partial z = 0$, respectively. The volume element of the integral (5.9) about the sphere is assumed to be the volume of such an infinitesimal thin cylinder annulus. In other words: the angular momentum density $\mathrm{d}h$ per volume unit $\mathrm{d}V$ is assumed constant in such a

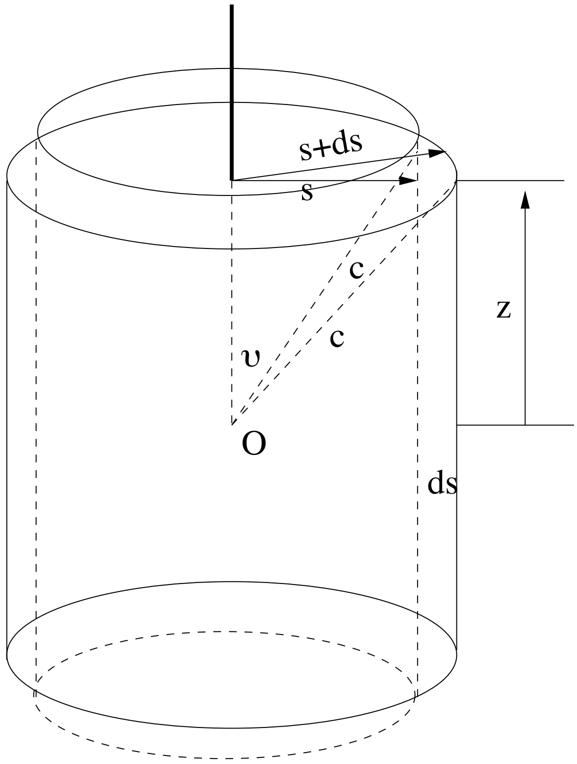

Figure 5.1: Model of coaxial cylinder annulies.

cylinder annulus throughout the core and variates only with ϑ (or primarily with s) like u_φ from annulus to annulus. Because u_φ is described by a smooth function of ϑ, the thickness $\mathrm{d}s$ of the annulus can be made infinitesimally small.

For an infinitesimal thickness of an annulus, the difference $\mathrm{d}z$ between the inner and outer cylinder heights can be neglected for this volume element. According to Fig. 5.1, the following expression can be derived for $\mathrm{d}V$:

$$\mathrm{d}V = V_{s+\mathrm{d}s} - V_s = \pi(s+\mathrm{d}s)^2\,2\,l(s) - \pi s^2\,2\,l(s) \approx 2\pi\,s\,2\,l(s)\,\mathrm{d}s\,. \tag{5.13}$$

$l(s)$ is the half height of a cylinder and given by $\sqrt{c^2-s^2}$, from which follows

$$\mathrm{d}V = 2\pi\,s\,2\,\sqrt{c^2-s^2}\,\mathrm{d}s \tag{5.14}$$

(see Jackson et al., 1993, who have left out $\mathrm{d}s$). The volume element in eq. (5.14) is only infinitesimal in the s-dimension. The factors 2π and $2\,l(s)$ signalize that the φ- and z-integrations are already done, although nothing is assumed about the integrand in eq. (5.12) except that u_φ fulfills the Taylor-Proudman theorem, i.e. the integration supposes that $\rho = \mathrm{const}$ and the integrand is assumed to be equatorially symmetric, respectively.

Because the cylinders are tangential to the sphere of radius c, s and l can be expressed by the co-latitude of the tangential circle

$$s = c\,\sin\vartheta\,, \quad l = c\,\cos\vartheta = c\,\sqrt{1-\sin^2\vartheta} \tag{5.15}$$

In the last part of eqs. (5.15), it is not considered that the square root has two signs. The applied positive sign then holds for an equatorially symmetric integrand. The method of tangential cylinders ensures that ϑ is the only variable, and varies between 0 and $\pi/2$. $\mathrm{d}V$ can then be written as

$$\mathrm{d}V = 4\pi\,c^3\,\sin\vartheta\sqrt{1-\sin^2\vartheta}\,\cos\vartheta\,\mathrm{d}\vartheta = 4\pi\,c^3\,\sin\vartheta\,\cos^2\vartheta\,\mathrm{d}\vartheta\,. \tag{5.16}$$

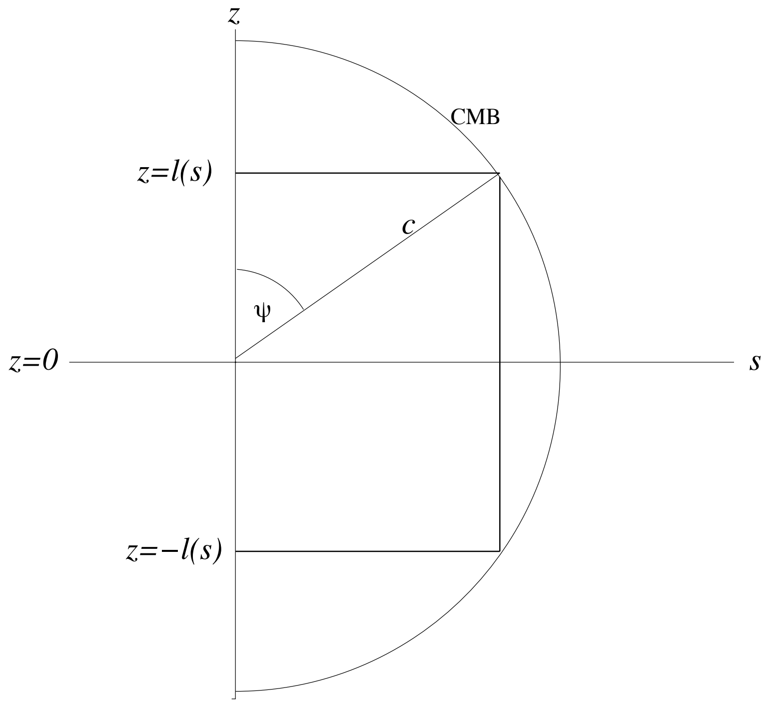

Figure 5.2: Schema of cylinder coordinates and polar angle ψ

For our own belief, we integrate $\mathrm{d}V$ from $\vartheta = 0$ to $\vartheta = \pi/2$, which gives exactly the volume of a sphere with radius c. With eq. (5.16), the integral in eq. (5.12) then reduces to

$$h_z = \int_V r\,\rho\,u_\varphi\,\sin\vartheta\,\mathrm{d}V = 4\pi\rho c^4 \int_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} \sin^2\vartheta\,\cos^2\vartheta\,u_\varphi\,\mathrm{d}\vartheta\,. \tag{5.17}$$

The expression in Jackson et al. (1993, their eq. (21)) is obtained, if we use $x = \cos\vartheta$ as variable ($\mathrm{d}x = -\sin\vartheta\,\mathrm{d}\vartheta$, the lower boundary $\vartheta = 0$ changes to $x = 1$ etc.). Eq. (5.17) clearly shows that the co-domain of ϑ is $[0, \pi/2]$. This can be checked again for the special case of a rigid rotation with $u_\varphi = \omega(t)\,r\,\sin\vartheta$, for which eq. (5.17) gives eq. (5.7) in that co-domain.

5.3 Cylinder coordinates

This result can be derived more transparently, if we use cylinder coordinates s, z, φ from the beginning and introduce a polar angle $\psi \in [0, \pi/2]$ instead of ϑ, as shown in Fig. 5.2. In cylinder coordinates, eq. (5.12) gives for a constant density $\rho = \rho_0$

$$h_z = \rho_0 \int_V s\,u_\varphi\,\mathrm{d}V\,. \tag{5.18}$$

The associated volume element is conventionally given by $\mathrm{d}V = s\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}\varphi$. For an axially symmetric velocity field, the φ-integration gives a factor 2π so that

$$h_z = 2\pi\,\rho_0 \int_V s^2\,u_\varphi(s,t)\,\mathrm{d}s\mathrm{d}z\,. \tag{5.19}$$

According to the applied Taylor-Proudman theorem, u_φ not depends on z. Consequently, the z-integration is possible and simply gives z itself. According to Fig. 5.2, showing the bounds of integration for z, we obtain

$$h_z = 2\pi\,\rho_0 \int_{[s]} s^2 u_\varphi(s,t)\,\big[l(s) - (-l(s))\big]\,\mathrm{d}s = 4\pi\rho_0 \int_{s=0}^{s=c} s^2 u_\varphi(s,t)\,l(s)\,\mathrm{d}s\,, \tag{5.20}$$

where the bounds of integration with respect to s are the pole ($s = 0$) and the equator ($s = c$), respectively.

Then, the following hypothesis is used by Jault (1990),

$$u_\varphi(s) = u_\varphi(c\,\sin\psi)\,, \tag{5.21}$$

i.e. that u_φ at any cylinder surface $s = s_i$ within the core takes the value of $u_\varphi(s_i = c\sin\psi_i)$ at the CMB, where the cylinder tangents this sphere. Next, we use this hypothesis expressed by eq. (5.21), to rewrite the integration in eq. (5.20) as follows. After the transformations

$$\begin{aligned} s &= c\,\sin\psi \qquad \Rightarrow \quad s \in [0, c] \quad \Longleftrightarrow \quad \psi \in [0, \frac{\pi}{2}] \\ l(s) &= c\,\cos\psi \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.22) \\ \mathrm{d}s &= c\,\cos\psi\,\mathrm{d}\psi\,, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.23) \end{aligned}$$

we obtain

$$h_z = 4\pi\rho c^4 \int\limits_{\psi=0}^{\psi=\frac{\pi}{2}} \sin^2\psi\,\cos^2\psi\,u_\varphi(c\,\sin\psi, t)\,\mathrm{d}\psi\,. \tag{5.24}$$

Eq. (5.24) is equivalent to eq. (5.17), but formulates the problem with a polar angle, the co-domain of which is well defined in $[0, \pi/2]$ from the beginning. So far, the derivations of eqs. (5.17) or (5.24), respectively, can be clearly understood.

5.4 Integration towards to the toroidal velocity modes after Jault

In the next step of our re-examination, we will try to follow the derivation of the final expression used in Jault (1990) for h_z, i.e. how to transfer $u_\varphi(c\,\sin\psi)$ defined in the cylinder model to the frozen-flux-u_φ given by a spherical harmonic (SH) expansion at the spherical CMB.

First, we will continue with the derivation in literature. To apply the normalization condition in later integrations about ϑ, an assumption is used (Jault, 1990, see Sec. 2.5.2):

$$h_z = 4\pi\rho c^4 \int\limits_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} \sin^2\vartheta\,\cos^2\vartheta\,u_\varphi(\vartheta, t)\,\mathrm{d}\vartheta \doteq \rho c^4 \int\limits_{r=c} \sin\vartheta\,\cos^2\vartheta\,u_\varphi(\vartheta, t)\,\mathrm{d}S \tag{5.25}$$

where $\mathrm{d}S = \mathrm{d}\varphi\sin\vartheta\mathrm{d}\vartheta$, i.e. the last integral represents an integration about the sphere $r = c$, **which implies that the integration boundaries of ϑ are 0 and π and not $\pi/2$** as in eq. (5.17) or the first term of (5.25), respectively. Because the symbol $\int\limits_{r=c}$ not well defines the accurate way of integration, this is first only our interpretation, which can be proved for the special case of the rigid rotation, but this is at last a necessary condition and not a sufficient one. Later on, we will show that the equality in eq. (5.25) is the critical point in this derivation.

The geostrophic constraint leads for u_φ to a zonal toroidal velocity field at the core surface, given by

$$u_\varphi = \mathrm{rot}_\varphi(\boldsymbol{r}\,Q) = -\frac{\partial Q}{\partial\vartheta}\,. \tag{5.26}$$

In the spherical harmonic (SH) expansion of Q, zonality means that $m = 0$, i.e.

$$u_\varphi = -\sum_n q_n^0\,\frac{\mathrm{d}P_n^0}{\mathrm{d}\vartheta} = \sum_n q_n^0\,P_n^1\,. \tag{5.27}$$

The angular momentum in eq. (5.25), second term, then reads

$$h_z = \rho c^4 \sum_n q_n^0 \int\limits_{r=c} \sin\vartheta\,\cos^2\vartheta\,P_n^1\,\mathrm{d}S\,. \tag{5.28}$$

Using Jault's definition of angular velocity modes t_n^0, our velocity modes q_n^0 must be replaced by $c\,t_n^0$. Further the integration about φ results a factor 2π. Eq. (5.28) then gives

$$h_z = 2\pi\rho c^5 \sum_n t_n^0 \int\limits_0^\pi \sin\vartheta\,\cos^2\vartheta\,P_n^1\,\sin\vartheta\,\mathrm{d}\vartheta\,, \tag{5.29}$$

We can solve the integral, I, replacing $\sin\vartheta\,\cos^2\vartheta$ by P_1^1, P_3^1, i.e.

$$I = \int\limits_0^\pi \left(\sin\vartheta\,\cos^2\vartheta\,P_n^1\right)\,\sin\vartheta\,\mathrm{d}\vartheta = \int\limits_0^\pi \left(\frac{1}{5}P_1^1\,P_n^1 + \frac{2}{15}\,P_3^1\,P_n^1\right)\,\sin\vartheta\,\mathrm{d}\vartheta\,. \tag{5.30}$$

and applying the Ferrers-Neumann's normalization condition for the spherical harmonics:

$$I = \left(\frac{1}{5}\,\delta_n^1\,\frac{2}{3}\,\frac{2!}{0!} + \frac{2}{15}\,\delta_n^3\,\frac{2}{7}\,\frac{4!}{2!}\right) = \frac{4}{15}\left(\delta_n^1 + \frac{12}{7}\,\delta_n^3\right)\,. \tag{5.31}$$

Here, we will stress that **the application of these normalization and orthogonality is only valid for the SH functions $P_n^m(\cos)$ in the domain $\vartheta \in [0,\pi]$ and is not valid in the half domain** (or "double half" domain). Based on eq. (5.31), we obtain

$$h_z = \frac{8}{15}\pi\rho c^5\left(t_1^0 + \frac{12}{7}\,t_3^0\right) \tag{5.32}$$

which gives Jault (1990) final result, which is clearly a consequence of the assumption in the background eq. (5.25).

5.5 Discussion

Two problems remain to be solved:

1. At the beginning, Jault has used an integration in the half sphere $[0,\pi/2]$, but later on he changed to an integration about the whole sphere $[0,\pi]$ by eq. (5.25)(right hand side) without comprehensible consequences for the integrand.

2. It is unclear, why the integration over the spherical core, carried out in Sec. 5.1 from the beginning, gives a contribution of q_1 to h_z only (eq. (5.6)), but the use of the assumptions of cylinder annulies a second term with q_3.

We suggest that both questions have something to do with each other. We will, therefore, discuss the critical point of the derivation, i.e. to begin with cylindrical coordinates and to change to spherical coordinates, which is necessary to combine the cylinder motions with the 'known' velocity field at the CMB.

The reconstruction of the derivation of the final expression (5.32) includes a step, where anything has been reduced to a half-sphere and cylinder surfaces, and another step, where this was withdrawn. Because, later on, the derivation continues with the whole sphere (introducing $\int_c \ldots$), we suggest that this has something to do with a bisection of the co-domain of ϑ, which is not compatible with the use of conventional SH functions. We will at first show that we will reach the weighting function $\sin^2\ldots\cos^2\ldots$ in the integrals (5.17), (5.24) or (5.25) by a transformation of the integration angle.

The evaluation of eq. (5.2) with use of the relation (5.3) gives for constant $\rho = \rho_0$ and axially symmetric u_φ

$$h_z = 2\pi\rho_0 \sum_n \int\limits_{r=0}^{c} q_n(r,t)\,r^3\left[\int\limits_{\vartheta=0}^{\pi} P_n^1(\cos\vartheta)\,\sin^2\vartheta\,\mathrm{d}\vartheta\right]\mathrm{d}r\,, \tag{5.33}$$

where, according to the used symbols and the 2^{nd} item of Sec. 4.2, we set $R_c = c$ and $R_i = 0$. After the transformation $\vartheta = 2\,\psi$ ($\mathrm{d}\vartheta = 2\mathrm{d}\psi$), we obtain

$$h_z = 4\pi\rho_0 \sum_n \int\limits_{r=0}^{c} q_n(r,t)\,r^3\left[\int\limits_{\psi=0}^{\frac{\pi}{2}} P_n^1(\cos 2\psi)\,\sin^2 2\psi\,\mathrm{d}\psi\right]\mathrm{d}r\,. \tag{5.34}$$

Applying $\sin 2\psi = 2\sin\psi\,\cos\psi$, we obtain

$$h_z = 16\pi\rho_0 \sum_n \int\limits_{r=0}^{c} q_n(r,t)\, r^3 \Bigg[\int\limits_{\psi=0}^{\frac{\pi}{2}} P_n^1(\cos 2\psi)\, \sin^2\psi\;\; cos^2\psi\, \mathrm{d}\psi \Bigg] \mathrm{d}r\,. \tag{5.35}$$

which shows that the inner integral has the same weighting functions of ϑ (or ψ) and the same integration bounds as seen in eqs. (5.17), (5.24) or (5.25), respectively. But it also clearly shows that the SH functions appearing there are not those for which the expression in e.q. (5.30) and the normalization in eq. (5.31) can be applied. The correct expression for h_z is, therefore, given in eq. (5.7) in Sec. 5.1.

Next, we will show that we can obtain the expression (21) in Jackson et al. (1993), if we assume this transformation $\vartheta = 2\psi$ with $\psi = \arcsin(s/c)$, seen in Fig. 5.2, i.e.

$$\begin{aligned} \vartheta = 2\,\arcsin\frac{s}{c} \quad &\text{with} \qquad \vartheta = 0 \quad \Rightarrow \quad s = 0 \\ &\text{and} \qquad\; \vartheta = \pi \quad \Rightarrow \quad s = c\,. \end{aligned} \tag{5.36}$$

For constant density ρ_0 and an axially symmetric velocity field, eq. (5.12), last term, can be written as

$$h_z = 2\pi\rho_0 \int\limits_{r=0}^{c} r^3 \Bigg[\int\limits_{\vartheta=0}^{\pi} u_\varphi(\vartheta,t)\, \sin^2\vartheta\, \mathrm{d}\vartheta \Bigg] \mathrm{d}r\,. \tag{5.37}$$

From $\psi = \arcsin(s/c)$ and (5.36), we further obtain

$$\begin{aligned} \sin\vartheta &= \sin 2\psi = 2\sin\psi\,\cos\psi = 2\frac{s}{c}\sqrt{1-\frac{s^2}{c^2}}\,, \\ \mathrm{d}\vartheta &= \frac{2}{c\sqrt{1-\frac{s^2}{c^2}}}\,\mathrm{d}s\,. \end{aligned} \tag{5.38}$$

With these findings, eq. (5.37) gives

$$h_z = 16\pi\rho_0 \int\limits_{r=0}^{c} r^3 \left[\int\limits_{s=0}^{c} u_\varphi(s,t)\, \frac{s^2}{c^2} \sqrt{1-\frac{s^2}{c^2}}\, \frac{\mathrm{d}s}{c} \right] \mathrm{d}r\,, \tag{5.39}$$

With the transformation $s' = s/c$, we can derive

$$h_z = 16\pi\rho_0 \int\limits_{r=0}^{c} r^3 \left[\int\limits_{s'=0}^{1} u_\varphi(s',t)\,(s')^2 \sqrt{1-(s')^2}\, \mathrm{d}s' \right] \mathrm{d}r\,. \tag{5.40}$$

If we assume that the inner integral about s' can be solved independently on the r-integration, giving a factor $\frac{c^4}{4}$, then we obtain eq. (21) in Jackson et al. (1993). Like stated before, this also means that we have to develop u_φ into a series of functions of ψ instead of ϑ or to use SH functions defined for the double angle 2ψ, respectively. Both is not compatible with the use of conventional SH representation of the axially symmetric surface field $u_\varphi(c,\vartheta,t)$, i.e. the decomposition in eq. (5.30) derived from eq. (5.25) cannot be applied. So it remains to solve the integral in eq. (5.24) on another way.

Summing up, this analysis shows that it is possible to **return to relation (5.25), but now corrected** in a compatible way by considering a change in the integrands argument as follows

$$h_z = 4\pi\rho c^4 \int\limits_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} \sin^2\vartheta\,\cos^2\vartheta\, u_\varphi(\vartheta,t)\,\mathrm{d}\vartheta \doteq 2\rho c^4 \int\limits_{\varphi=0}^{\varphi=2\pi} \int\limits_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} \sin\vartheta\,\cos^2\vartheta\, u_\varphi(2\vartheta,t)\,\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \tag{5.41}$$

Applying the substitution $2\vartheta = \psi$ we obtain the relation

$$h_z = 4\pi\rho c^4 \int\limits_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} \sin^2\vartheta\,\cos^2\vartheta\, u_\varphi(2\vartheta,t)\,\mathrm{d}\vartheta = 2\pi\rho c^4 \int\limits_{\psi=0}^{\psi=\pi} \sin^2\frac{\psi}{2}\,\cos^2\frac{\psi}{2}\, u_\varphi(\psi,t)\,\mathrm{d}\psi\,. \tag{5.42}$$

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After a trigonometric exchange the final integral expression

$$h_z = \frac{\pi}{2}\rho c^4 \int\limits_{\psi=0}^{\psi=\pi} \sin^2\psi \; u_\varphi(\psi,t)\,\mathrm{d}\psi \tag{5.43}$$

appears, which is identical with the formerly derived result in spherical coordinates (5.1) if we replace as follows $R_i = 0$, $R_c = c$, $\rho(r) = \rho$ and according to (5.26) $u_\varphi = -\dfrac{\partial Q}{\partial \vartheta}$.

Conclusions | 6

To understand decadal LOD variations, which are generally assigned to outer core motions, core surface flows in the frozen flux approximation are determined from earth surface geomagnetic data. This achieved velocity distribution at the core surface has to be continued anyway into the core interior. Processing in this way represents an essential precondition to be able to determine the relative angular momentum of the core which – on the other hand – can be related with the LOD variations. Thereby, the approach here addressed is only a minimal assumption that the derived interior velocity for the core fullfills the consequences resulting from a distinctly shortened Navier-Stokes equation as e.g. for the velocities in the Taylor-Proudman state (which are thought as organized on nested central cylinders inside the core).

For a spherical symmetric core and axial symmetric core velocities, the angular momentum vector $\boldsymbol{h}$ is reduced to the component h_z as the only non-vanishing component. For the core velocity $\boldsymbol{u}$ in Taylor-Proudman state, which is equatorial symmetric, we find an analytical solution. This solution of the underdetermined vector differential equation of first order, has only one non-vanishing component: the phi-component, which refers to a free function depending on the three spherical coordinates. Thus a connection to boundary values from the magnetic core surface frozen flux approximation is possible.

The main result of this report is the determination the angular momentum component h_z for the velocity u in the Taylor-Proudman state. It shows that it is governed only by **one** toroidal velocity **mode**, usually designated as t_1^0.

Contrary to our result, the derivation of Jault Jault (1990); Jault et al. (1988); Jault & Le Mouël (1991) led to the **two modes** t_1^0, t_3^0, where another way, based on the tangential cylinder approach, was used for the derivation. This often cited and applied result for comparing with the decadal LOD variations, therefore is not correct and has to be reduced by the second velocity mode. Scrutinizing the integration in different coordinates, we could find the discrepancy in the analytical integration and could bring the derivation after correcting in agreement with our integration.

As mentioned in Chap. 3 and 5, the assumption that the core motion is organized in coaxially rigid rotating cylinder annulies (surfaces) is a solution of eqs. (3.3)–(3.4) of the kind $u_\varphi = s\omega(s)$, in which the dependence of $\omega(s)$ on the distance to the z-axis is otherwise no more closely determined by the Taylor-Proudman-Theorem. Jault (1990) has solved this problem by identifying u_φ at any cylinder surface with its value on the associated tangential circle at the CMB, which strongly requires that the values of u_φ are the same at each point of a fixed circle of latitude, which is an exception if we look at one of the well known published graphs of $\boldsymbol{u}$ at the CMB. At the end, the way to identify it mathematically with the frozen-flux field $u_\varphi(\mathrm{CMB})$ holds only for core motions which are exactly equatorially symmetric and not only in the sense of an approximation.

Nevertheless, Jault's idea to divide the fluid core into small coaxially rigid rotating cylinder annulies as an approximation of its mean motion turns out as a promising way, and can be used otherwise to solve the core angular momentum, e.g. numerically, like done by Hide et al. (2000).

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