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Capturing the Uncertainty of Seismic Activity Rates in Probabilistic Seismic Hazard Assessments

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Abstract A new two-step approach is proposed to quantify the uncertainties of the seismicity rate $\nu(M)$ estimated by Gutenberg-Richter relations of seismic area source zones in logic tree-driven probabilistic seismic hazard analysis. The first step makes use of the possibility that the uncertainties of the Gutenberg-Richter parameters α and β (or a and b), expressed by their covariance matrix $\mathbf{C}(\alpha, \beta)$ (or $\mathbf{C}(a, b)$), can completely be propagated to $\nu(M)$. In the second step, the resulting one-dimensional probability distribution of $\nu(M)$ for any fixed magnitude M is approximated by a finite set of values and weights in a statistically optimal manner (Miller and Rice, 1983). The procedure yields a set of alternative recurrence models which exhaustively approximates the epistemic uncertainties of the seismicity rate.

Introduction

The estimation of the seismicity rates based on cataloged earthquake data is an essential prerequisite for any probabilistic seismic hazard analysis (PSHA). This step also requires the estimation of the uncertainties associated with these input data to quantify the resulting uncertainties in the hazard (Abrahamson and Bommer, 2005; Bommer et al., 2005).

The subject of this paper is an improved statistical description of the uncertainties connected with the Gutenberg-Richter (GR) magnitude-frequency relation as the standard rate model of an area source zone in which it is assumed that the seismicity is uniformly distributed within the source area.

The empirical relation of the total number of earthquakes equal and above magnitude M of a specific region observed in a fixed time period may be quantitatively described by (Gutenberg and Richter, 1944)

$$\nu(M) = \int_M e^{\alpha - \beta m} dm \quad (1)$$

with activity and rate parameters α and β . With respect to hazard calculations, it is usually expressed by the equation

$$\nu(M) = \nu_0 \int_M^{M_{max}} \frac{\beta e^{-\beta m}}{e^{-\beta M_{min}} - e^{-\beta M_{max}}} dm, \quad (2)$$

where $\nu(M)$ is the cumulative annual frequency of earthquakes in the magnitude interval $[M, M_{max}]$ and $\nu_0 = \nu(M_{min})$ is the total frequency equal to and above a minimum level of

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completeness M_{min} . The integral term represents a double-truncated exponential probability distribution with rate β (Page, 1968).

Significant differences exist between employing the GR relation for seismicity studies and for seismic hazard assessments. In the first case, the interest is often restricted to the rate parameter β and its uncertainties, whereas $\nu(M)$ has little significance (Wiemer and Wyss, 2002; Schorlemmer et al., 2004). The appropriate estimators for β go back to the early works of Aki (1965) and Utsu (1965). With the extension to binned magnitude classes (Utsu, 1966; Bender, 1983; Tinti and Mulargia, 1987) and an improved error assessment (Shi and Bold, 1982), these estimators of β are established in seismology, especially for the analysis of usually large numbers of instrumentally-recorded events above a fixed completeness magnitude (Marzocchi and Sandri, 2003). In contrast, the basic target of seismic hazard assessments is the magnitude rate $\nu(M)$ itself. This requires a joint estimation of both parameters, ν_0 and β (or α and β) of the GR relation. Furthermore, the use of historical as well as instrumental seismicity in seismic hazard assessments implies estimation methods for binned magnitude classes but unequal observation periods.

The first suitable method to estimate the GR parameters with respect to hazard assessments was published by Weichert (1980). His maximum-likelihood estimation (MLE) provides expectations and uncertainties for both β and ν_0 based on binned magnitude rates and different completeness periods. For the first time, instrumental and historical data could be jointly analyzed in a statistically correct way. This procedure additionally considers the basic assumption of classical PSHA that the seismic sequence can be described by a stationary Poisson process (Cornell, 1968).

Although methodically prepared in Weichert (1980), a successive rather than simultaneous determination of β and ν_0 is performed. Thus, the variance of β , $\text{var}(\beta)$, is correctly estimated but $\text{var}(\nu_0)$ is only valid if β is fixed in advance. The correlation between β and ν_0 is missing. The joint uncertainties of both parameters were first mentioned as a two-dimensional normal distribution with a covariance matrix $\mathbf{C}(\nu_0, \beta)$ in the EPRI report (Johnston, 1994) of the CEUS-SSC project (Central and Eastern United States Seismic Source Characterization for Nuclear Facilities, EPRI/DOE/NRC, 2012). Since that time, this technique has been used as modified method of Weichert in several comprehensive seismic hazard projects, e.g., PEGASOS (Probabilistic Seismic Hazard Analysis for Swiss Nuclear Power Plant Sites, Coppersmith et al., 2009). It was planned to apply this method also in the frame of the EC project Seismic-Hazard Harmonization in Europe (SHARE) (Giardini et al., 2013).

The propagation of parametric uncertainties to the seismicity rate $\nu(M)$ is usually performed by discretizing the parameter distribution $\mathbf{C}(\nu_0, \beta)$ by a set of (ν_0, β) pairs and appropriate weights. Then, the resulting GR relations are used as epistemic model uncertainties in a logic tree (Johnston, 1994; Coppersmith et al., 2009). This approach has several drawbacks: (1) there is no valid method which approximates a multidimensional probability distribution by a finite set of sampling points and weights in a statistically optimal manner, and (2) a set of k^2 (ν_0, β) pairs ($k = 5$) is recommended (Johnston, 1994) which substantially expands the logic tree of the seismic hazard assessment and can require significant computational power for large hazard projects.

The approach presented here to quantify the uncertainties of $\nu(M)$ in PSHA is based on the traditional GR parameters α and β and their covariance matrix $\mathbf{C}(\alpha, \beta)$. Any estimation technique can be used to provide this input. The advantage of the corresponding formulation is the fact that the uncertainties expressed by $\mathbf{C}(\alpha, \beta)$ can be completely propagated to the linear expression $\alpha - \beta m$ in eq. (1), which provides a probability density of the seismicity rate $\nu(M)$ in a closed analytical form for any fixed magnitude. Furthermore, there is an approved algorithm to discretize the resulting one-dimensional probability density at k sampling points with appropriate weights so that the first $(2k - 1)$ moments of the corresponding discrete distribution are equal to the moments of the original density (Miller and Rice, 1983). This

discretization technique can also be applied if a probability density distribution of the maximum magnitude M_{max} is given, e.g., if M_{max} is estimated by the EPRI approach (Coppersmith, 1994a, b) for stable continental regions. The proposed procedure provides a set of alternative recurrence models which exhaustively approximate the epistemic uncertainties of the seismicity rate in logic tree driven PSHA.

Throughout this paper, the modified MLE of Weichert (1980) is used to determine the input α , β and $\mathbf{C}(\alpha, \beta)$ for the proposed approach. It is a widely accepted technique in the context of classical PSHA. However, any other estimation technique which outputs α , β and $\mathbf{C}(\alpha, \beta)$ is suitable to provide the input for the presented new method. Before the uncertainties of $\nu(M)$ are quantified, a short derivation of the modified MLE is given for the sake of completeness. In addition to the usual small magnitude bin approximation, a description of the MLE for "finite magnitude bins" is specified. This refinement should be used for bins equal to or larger than 0.5 magnitude units especially if recurrence relations uncertainties are to be taken into account. After the proposed method is presented, it is demonstrated for a Central European area source zone as an example. In addition, the total uncertainties of the seismicity rate for the zone are illustrated by combining the quantified uncertainties of the GR relation with a probability density of M_{max} derived by the EPRI approach (Coppersmith, 1994a,b). While only the simplest case of a single area source zone is considered when deriving the GR relation and its uncertainty, Appendix A summarizes the corresponding equations with a prior estimate of β_0 for stabilizing the assessment in regions with limited seismicity (Veneziano and Van Dyck, 1985). Appendix B provides an alternative approach to jointly estimate a common β value but different activity parameters α for an assembly of several low seismicity zones.

Magnitude-Frequency Parameter Estimation

In PSHA, the GR relation is commonly expressed in terms of ν_0 and β (see eq. (2)). Although the parameter ν_0 is more intuitively accessible compared with the original parameter α , this formulation masks the linear structure of the exponent $\alpha - \beta m$ in eq. (1) and hampers a smooth propagation of the parameter uncertainties to the seismicity rate $\nu(M)$. To keep technical considerations as simple as possible, the following derivations are based on the GR relation in form of eq. (1) with parameters α and β . However, both formulations are equivalent and can be converted from one to the other according to

$$e^\alpha = \frac{\beta \nu_0}{e^{-\beta M_{min}} - e^{-\beta M_{max}}} . \quad (3)$$

It is widely accepted that only MLE methods are suitable to estimate the GR parameters and that their uncertainties sufficiently correct. Although still in use, it has already been recognized that the least-squares technique should not be applied to this estimating recurrence (Page, 1968; Bender, 1983; Sandri and Marzocchi, 2007). Therefore, Weichert's (1980) modified MLE (Johnston, 1994) is used in the following to provide expectations and covariances of α and β , but any other estimation procedure for α , β and $\mathbf{C}(\alpha, \beta)$ is suitable as well. This paper describes a new approach where these derived quantities are combined into quantified uncertainties of the seismicity rate $\nu(M)$ independent of any estimation method.

MLE Formulation

According to eq. (1), the magnitude frequency rate $\delta\nu(M)$ of a magnitude bin $[M-dM, M+dM]$ with center M and half bin size dM can be expressed by

$$\delta\nu(M) = \int_{M-dM}^{M+dM} e^{\alpha-\beta m} dm = 2e^{\alpha-\beta M} \frac{\sinh(\beta dM)}{\beta} \approx 2e^{\alpha-\beta M} dM . \quad (4)$$

The last term applies for a small bin size, $2dM < 0.5$, where $\sinh(\beta dM)/\beta \approx dM$. In the following, the two cases $\delta\nu(M) = 2e^{\alpha-\beta M}dM$ and $\delta\nu(M) = 2e^{\alpha-\beta M} \sinh(\beta dM)/\beta$ are distinguished by the concept of small magnitude bin and finite magnitude bin.

Assuming a Poisson process with rate parameter λ , the probability that n earthquakes occur in a magnitude bin $[M - dM, M + dM]$ during time period t is

$$P(n, \lambda) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{where} \quad \lambda = \delta\nu(M) t. \quad (5)$$

Let the seismicity of a single source zone be classified by I magnitude bins $\mathbf{M} = \{M_1, \dots, M_I\}$ with non-constant half bin size $\mathbf{dM} = \{dM_1, \dots, dM_I\}$, appropriate completeness times $\mathbf{t} = \{t_1, \dots, t_I\}$ and number of earthquakes per bin $\mathbf{n} = \{n_1, \dots, n_I\}$ during the respective periods. The probability that the corresponding frequency magnitude rates satisfy a GR relation with parameters α and β is described by the likelihood function

$$\mathfrak{L}(\alpha, \beta) = \prod_{i=1}^I \frac{(\delta\nu(M_i) t_i)^{n_i}}{n_i!} e^{-\delta\nu(M_i) t_i}. \quad (6)$$

The bins $\mathbf{M}$ do not need to completely cover the magnitude interval $[M_{min}, M_{max}]$, but there is a difference between a missing bin and an empty bin. In the first case, nothing is known of the respective magnitude class, whereas an empty bin implies no seismicity in a specified observation period.

Small Magnitude Bins

For small magnitude bins, all $2dM_i < 0.5$, the logarithm of the likelihood $\mathfrak{L}(\alpha, \beta)$ reads (see eq. (4))

$$\ln \mathfrak{L}(\alpha, \beta) = \sum_{i=1}^I \left[n_i(\alpha - \beta M_i + \ln(2t_i dM_i)) - \ln(n_i!) - 2t_i dM_i e^{\alpha - \beta M_i} \right]. \quad (7)$$

The standard procedure to minimize the negative of $\ln \mathfrak{L}(\alpha, \beta)$ yields

$$-\frac{\partial \ln \mathfrak{L}}{\partial \alpha} = \sum_{i=1}^I (2t_i dM_i e^{\alpha - \beta M_i} - n_i) = 0, \quad (8)$$

$$-\frac{\partial \ln \mathfrak{L}}{\partial \beta} = \sum_{i=1}^I (n_i M_i - 2M_i t_i dM_i e^{\alpha - \beta M_i}) = 0. \quad (9)$$

From eq. (8), it follows that the cumulative number of observed earthquakes is equal to the number of cumulative events predicted by the model

$$N = \sum_{i=1}^I n_i = \sum_{i=1}^I 2t_i dM_i e^{\alpha - \beta M_i}. \quad (10)$$

Solving eq. (10) for e^α

$$e^\alpha = \frac{N}{\sum_{i=1}^I 2t_i dM_i e^{-\beta M_i}} \quad (11)$$

and inserting this expression in eq. (9) gives the following nonlinear equation for β , which must be solved numerically

$$\sum_{i=1}^I n_i M_i = N \frac{\sum_{i=1}^I M_i t_i dM_i e^{-\beta M_i}}{\sum_{i=1}^I t_i dM_i e^{-\beta M_i}}. \quad (12)$$

For an overall constant bin size, $dM_i = \text{const}$, eq. (12) corresponds to the estimation of β by Weichert (1980). The covariance matrix $\mathbf{C}$, which describes the uncertainties of the estimated parameters for a sufficiently large number of events, can be approximated by the inverse of the Hessian of $\ln \mathfrak{L}(\alpha, \beta)$

$$\mathbf{C} = \begin{pmatrix} \text{cov}(\alpha, \alpha) & \text{cov}(\alpha, \beta) \\ \text{cov}(\beta, \alpha) & \text{cov}(\beta, \beta) \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \alpha} & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \beta} \end{pmatrix}^{-1}. \quad (13)$$

The derivatives explicitly read

$$\begin{aligned} -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \alpha} &= 2 \sum_{i=1}^I t_i dM_i e^{\alpha - \beta M_i} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \beta} &= -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha} = -2 \sum_{i=1}^I M_i t_i dM_i e^{\alpha - \beta M_i} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \beta} &= 2 \sum_{i=1}^I M_i^2 t_i dM_i e^{\alpha - \beta M_i}. \end{aligned} \quad (14)$$

It is easy to verify that $\text{cov}(\beta, \beta)$ is equal to $\text{var}(\beta)$ as given by Weichert (1980).

Finite Magnitude Bins

For finite magnitude bins, any $2dM_i \geq 0.5$, the equivalent expressions of (11), (12) and (14) are given without their explicit derivatives. The nonlinear equation for β reads now

$$\sum_{i=1}^I n_i [M_i - dM_i \coth(\beta dM_i)] = N \frac{\sum_{i=1}^I t_i \sinh(\beta dM_i) [M_i - dM_i \coth(\beta dM_i)] e^{-\beta M_i}}{\sum_{i=1}^I t_i \sinh(\beta dM_i) e^{-\beta M_i}}. \quad (15)$$

This expression is somewhat more complex than eq. (12), but it can be solved numerically in the same way. The activity parameter α is now given by

$$e^\alpha = \frac{\beta N}{\sum_{i=1}^I 2t_i \sinh(\beta dM_i) e^{-\beta M_i}}. \quad (16)$$

The terms describing the covariance matrix are

$$\begin{aligned} -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \alpha} &= 2 \sum_{i=1}^I t_i \frac{\sinh(\beta dM_i)}{\beta} e^{\alpha - \beta M_i} = N \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \beta} &= -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha} = -2 \sum_{i=1}^I t_i \frac{\sinh(\beta dM_i)}{\beta} [M_i - dM_i \coth(\beta dM_i)] e^{\alpha - \beta M_i} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \beta} &= 2 \sum_{i=1}^I t_i \frac{\sinh(\beta dM_i)}{\beta} [M_i - dM_i \coth(\beta dM_i)]^2 e^{\alpha - \beta M_i}. \end{aligned} \quad (17)$$

Quantified Uncertainties of Seismicity Rate $\nu(M)$

The MLE of the GR parameters provides both expectations $\hat{\alpha}$ and $\hat{\beta}$ and covariances $\mathbf{C}(\alpha, \beta)$ of α and β , which define a two-dimensional normal distribution. These values can be propagated

directly to the expression $\alpha - \beta m$ since it is linearly dependent on α and β . Hence, the uncertainty of $\alpha - \beta m$ is normally distributed for every fixed m , with mean value $\hat{\alpha} - \hat{\beta}m$ and variance $\sigma^2(m)$, i.e., $\alpha - \beta m \sim \mathcal{N}(\hat{\alpha} - \hat{\beta}m, \sigma^2(m))$ with

$$\sigma^2(m) = \begin{pmatrix} 1 \\ -m \end{pmatrix}^T \mathbf{C} \begin{pmatrix} 1 \\ -m \end{pmatrix}. \quad (18)$$

The normalized variable

$$z = \frac{(\alpha - \hat{\alpha}) - (\beta - \hat{\beta})m}{\sigma(m)} \quad (19)$$

transforms $\alpha - \beta m$ and its uncertainties into

$$\alpha - \beta m = \hat{\alpha} - \hat{\beta}m + \sigma(m)z = \hat{\alpha} - \hat{\beta}m + \sigma(m)\mathcal{N}(0, 1). \quad (20)$$

in which the probability in terms of z can be described by the standard normal distribution, $\mathcal{N}(0, 1)$. This implies, for the frequency magnitude rate,

$$\nu(M) = \int_M^{M_{max}} e^{\hat{\alpha} - \hat{\beta}m + \sigma(m)\mathcal{N}(0, 1)} dm. \quad (21)$$

A discretization of $\nu(M)$ including all uncertainties is now reduced to a suitable approximation of $\mathcal{N}(0, 1)$. Any one-dimensional probability distribution can be approximated by a few representative values at sampling points z_i ($i = 1 \dots k$) and associated weights p_i in an optimal sense with the approved simple procedure by Miller and Rice (1983) based on Gaussian quadrature. The criterion for the accuracy of this discrete approximation is that it preserves as many moments of the original distribution as possible. A discretization with k pairs (z_i, p_i) can match at least the first $(2k - 1)$ moments exactly. For the standard normal distribution $\mathcal{N}(0, 1)$, sampling points and weights for $k=3, 4$ and 5 are given in Table 1.

Table 1
Optimal Sampling Point Positions z and Corresponding Weights p of the Standard Normal Distribution for $k=3$ (Above), 4 (Middle) and 5 (Below).

Position of sampling points	$-\sqrt{3}$		0		$-\sqrt{3}$
Weights	$1/6$		$4/6$		$1/6$
Position of sampling points	$-\sqrt{3 + \sqrt{6}}$	$-\sqrt{3 - \sqrt{6}}$		$\sqrt{3 - \sqrt{6}}$	$\sqrt{3 + \sqrt{6}}$
Weights	$(3 - \sqrt{6})/12$	$(3 + \sqrt{6})/12$		$(3 + \sqrt{6})/12$	$(3 - \sqrt{6})/12$
Position of sampling points	$-\sqrt{5 + \sqrt{10}}$	$-\sqrt{5 - \sqrt{10}}$	0	$\sqrt{5 - \sqrt{10}}$	$\sqrt{5 + \sqrt{10}}$
Weights	$(7 - 2\sqrt{10})/60$	$(7 + 2\sqrt{10})/60$	$8/15$	$(7 + 2\sqrt{10})/60$	$(7 - 2\sqrt{10})/60$

The covariance matrix, eq. (13), only approximates the uncertainties of parameters α and β . Particularly for a small number of earthquakes, N , in a studied region, a bootstrapping procedure should be used to model the one-dimensional probability of $\alpha - \beta m$, which now differs from a normal distribution. However, the approximation technique by Miller and Rice (1983) can also be applied to such numerically generated probability densities. Therefore, the offered procedure to capture the uncertainties of frequency magnitude rates by analyzing the distribution of $\alpha - \beta m$ can be transferred to this situation. Further investigations will determine the range of validity where eq. (13) is a sufficient approximation of the parameter uncertainties or where bootstrapping is required.

Application of the new Method

The following example illustrates the proposed method to quantify the uncertainties of the seismicity rate $\nu(M)$ in PSHA. We have chosen a small-scale area seismic source zone (SSZ), the Hohenzollernalb, located in SW Germany north of the Alpine front and east of the Upper Rhine Graben. The seismotectonic constraints to define this SSZ are described by Burkhard and Grünthal (2009). This small SSZ has been Germany's most energetic seismicity area since 1911. It generated the largest instrumentally recorded earthquake in Germany, known as Central European earthquake, with Richter magnitude $M_L = 6.1$ (moment magnitude $M_w = 5.7$), on November 16, 1911. The last $M_w > 5$ event occurred here on September 3, 1978. This small seismic zone was included in most of the hazard projects covering Germany (e.g., Grünthal et al., 1998, 2009) and those involving Central Europe (e.g., Grünthal et al., 1999, 2010a; Burkhard and Grünthal, 2009). In the context of the recently finished European Project SHARE (Giardini et al., 2013) it is labeled as "DEAS140" in the area source branch of the respective logic tree. The characterization of this SSZ in SHARE can be accessed from the project web portal (Giardini et al., 2013). Since the seismicity of this SSZ has been described in detail in the mentioned papers, we can focus here on the demonstration of the new approach.

We have selected the SSZ Hohenzollernalb also because it is one of the few small scale SSZ in Central Europe with sufficient seismicity to estimate $\nu(M)$ without any stabilizing constraints for β , i.e., where the common- b technique (e.g., Burkhard and Grünthal, 2009) or the penalizing β approach (Veneziano and Van Dyck, 1985; Johnston, 1994) are unnecessary.

The seismicity data used here are from the database of the European-Mediterranean earthquake catalog EMEC (Grünthal and Wahlström, 2012), which have been adjusted to have uniform M_w values. This data set is extended to lower magnitudes of $M_w = 2$, according to the corresponding domestic catalogs (cf. section Data and Resources, see also Grünthal, 2014), following the procedures described in Grünthal and Wahlström (2012). The data were declustered by a windowing technique in space and time (Grünthal, 1985; Grünthal et al., 2009; Hakimhashemi and Grünthal, 2012) which has been successfully applied in several projects covering Central Europe (Grünthal et al., 1998, 1999; Burkhard and Grünthal, 2009; Coppersmith et al., 2009) and all of Europe (Grünthal et al., 2010a; Woessner et al., 2013). To estimate the completeness times of different magnitude bins, a statistical method is used based on changes in variances of earthquake inter-event times (Hakimhashemi and Grünthal, 2012). The EPRI method (Coppersmith, 1994a,b) for a non-extended stable continental crust is applied to determine the expected probability distribution for the range of maximum magnitudes that are possible in the study area. An optimal three-point discretization of this distribution after Miller and Rice (1983) provides $M_{max} = 5.7, 6.0$, and 6.6 with the corresponding weights $0.27, 0.60$, and 0.13 .

The 155 independent $M_w \geq 2.0$ events remaining after the described data preprocessing are distributed over eight magnitude classes with a constant bin size of 0.5 magnitude units. The corresponding cumulative rates for $M_{max} = 6.0$ are shown in Fig. 1 as circles together with the estimated GR relation.

The uncertainties of the relation are expressed by a five point discretization at z_i of the resulting distribution, with appropriate weights p_i according to

$$\nu_i(M) = \int_M^{M_{max}} e^{\hat{\alpha} - \hat{\beta}m + \sigma(m)z_i} dm. \quad (22)$$

Fig. 1 illustrates not only the relation for the normally used small bin approximation (black lines) of $\delta\nu(M)$ but also the more accurate results of the finite bin approximation (gray lines). There are no remarkable differences between the mean relations (solid lines) for the chosen bin size of 0.5 magnitude units. Slight differences can be observed for the lower weighted relations which describe the uncertainties. Table 2 quantitatively compares the results of both approximations.

Figure 1: Observed cumulative seismicity rates (circles) and magnitude-frequency graphs with annotated weights p capturing the uncertainties of the estimated Gutenberg-Richter (GR) relation by a five point discretization of the resulting distribution. Black lines correspond to the small magnitude bin approximation and gray lines to the more accurate finite bin calculations respectively.

Table 2
Estimated GR Parameters $\hat{\alpha}$, $\hat{\beta}$ and Their Uncertainties for a Small Bin and a Finite Bin Approximation of $\delta\nu(M)$, Respectively.

approximation	$\hat{\alpha}$	$\hat{\beta}$	$\text{var}(\hat{\alpha}, \hat{\alpha})$	$\text{var}(\hat{\alpha}, \hat{\beta})$	$\text{var}(\hat{\beta}, \hat{\beta})$
small magnitude bin	5.2254	1.7951	0.0990	0.0312	0.0105
finite magnitude bin	5.1920	1.7951	0.0657	0.0249	0.0105

Obviously, $\hat{\beta}$ and $\text{var}(\hat{\beta}, \hat{\beta})$ coincide for both procedures. This fact results from the overall constant bin size dM_i of the current example and can be verified comparing eq. (12) with eq. (15) or the last expressions in eq. (14) with eq. (17).

The non-cumulative representation of the GR relation according to

$$-\frac{d\nu_i(M)}{dM} = e^{\hat{\alpha} - \hat{\beta}m + \sigma(m)z_i} \quad (23)$$

is shown in Fig. 2. Only the mean relation ($z_i = 0$, $p_i = 0.53$) appears as a classical straight line in the logarithmic plot. The graphs of the other four relations reflect the nonlinear dependency of $\sigma(M)$ on M .

Fig. 3 summarizes the results of the proposed approach for the study area combining the uncertainties of the GR relations with the uncertainties of the maximum magnitude distribution. Only the graphs of the small magnitude bin approximation are shown.

Figure 2: Observed non-cumulative seismicity rates (circles) and magnitude-frequency graphs with annotated weights p capturing the uncertainties of the estimated GR relation by a five point discretization of the resulting distribution. Black lines correspond to the small magnitude bin approximation and gray lines to the more accurate finite bin calculations respectively. Note, data are normalized to one magnitude unit compatible to the meaning of α .

Figure 3: Observed cumulative seismicity rates (circles) and magnitude-frequency graphs corresponding to a three point discretization of the M_{max} posterior distribution. The annotated weights p refer to the discretized distribution of M_{max} .

Conclusions

A new approach is given to quantify the uncertainties of the seismicity rate $\nu(M)$ estimated for GR relations of seismic area source zones in a logic tree-driven PSHA. It can be described as a two stage process. The first step is based on the simple observation that the uncertainties of the GR parameters α and β (or a and b) expressed by their covariance matrix $\mathbf{C}(\alpha, \beta)$ (or $\mathbf{C}(a, b)$) can completely be propagated to $\nu(M)$. This does not apply when using the parameters ν_0 and β (or b) of the GR relation, which is the traditional formulation in previous approaches (Weichert, 1980; Johnston, 1994; Coppersmith et al., 2009). In the second step, the resulting one-dimensional probability distribution of $\nu(M)$ is approximated by a finite set of values and weights in a statistically optimal manner (Miller and Rice, 1983). The proposed procedure yields a freely chosen number of alternative recurrence models which exhaustively approximate the epistemic uncertainties of the seismicity rate. The accuracy achieved with different model quantities can be expressed by the number of preserved moments of $\nu(M)$. This first step can be performed independently of the estimation technique providing the GR parameters and their covariances.

In addition, using the technique of Weichert (1980), all numerical derivations are given not only, for the commonly used small magnitude bin approximation of the MLE, but also for its finite magnitude bin extension. It is strongly recommended to use this approximation, especially if magnitude bins are equal to or larger than 0.5 magnitude units and if the uncertainties of the seismicity rate need to be described accurately.

An alternative to the MLE with a prior estimate β_0 (Veneziano and Van Dyck, 1985) for regions with limited seismicity is given in Appendix B. It jointly estimates a common β value for a set of several low seismicity zones and different activity parameters α for each zone. This eliminates some subjective aspects in the choice of β_0 and its reliability expressed by the corresponding weighting factor W_0 .

Data and Resources

The computations were performed with the mathematical software package MATLAB (www.mathworks.com; last accessed April 2014). The corresponding scripts are available from the authors upon request. The data of the European-Mediterranean earthquake catalog EMEC (Grünthal and Wahlström, 2012) can be downloaded under <http://www.gfz-potsdam.de/emec/> (last accessed June 2014). Data for magnitudes $M \leq 3.5$ of the study area since 1995 are available under http://www.bgr.bund.de/DE/Themen/Erdbeben-Gefaehrdungsanalysen/Seismologie/Seismologie/Erdbebenauswertung/Erdbebenkataloge/Kataloge_Bulletins/kataloge_bulletins_node.html (last accessed June 2014).

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Appendix A

Veneziano and Van Dyck (1984) extend the MLE of α and β to include varying completeness times for fixed magnitude bins of large source regions assuming a uniform rate density over the area. They later introduce a prior estimate β_0 stabilizing the assessment in regions with limited seismicity (Veneziano and Van Dyck, 1985).

Let $S = \{S_1, \dots, S_J\}$, a set of individual subzones S_j with equal area density of seismicity rate. Each subzone is characterized by a sequence of I_j magnitude bins $\mathbf{M}_j = \{M_{1j}, \dots, M_{I_jj}\}$ with appropriate half bin width $\mathbf{dM} = \{dM_{1j}, \dots, dM_{I_jj}\}$, completeness times $\mathbf{t} = \{t_{1j}, \dots, t_{I_jj}\}$, and number of earthquakes per bin $\mathbf{n} = \{n_{1j}, \dots, n_{I_jj}\}$. Assuming a prior estimate β_0 , the probability that the magnitude-frequency rates $\delta\nu(m_{ij})$ satisfy a GR relation with parameters α and β reads

$$\mathcal{L}(\alpha, \beta) = \prod_{j=1}^J \prod_{i=1}^{I_j} \frac{(\delta\nu(M_{ij}) t_{ij})^{n_{ij}}}{n_{ij}!} e^{-\delta\nu(m_{ij}) t_{ij}} e^{-W_0(\beta-\beta_0)^2/2}. \quad (\text{A.1})$$

The weighting parameter W_0 of the penalty term $e^{-W_0(\beta-\beta_0)^2/2}$ controls the strength of β_0 and can be considered equivalent to the inverse of the variance of β_0 (Johnston, 1994).

Small Magnitude Bins

The uniform rate density over the area S for a magnitude bin requires a scaling of $\delta\nu$ by a factor W_j

$$\delta\nu(M_{ij}) = 2W_j e^{\alpha-\beta M_{ij}} dM_{ij} \quad \text{with} \quad W_j = A_j / \sum_{k=1}^J A_k, \quad (\text{A.2})$$

in which the index j refers to the different subzones, each with an area A_j . With the same minimization procedure as described above, the equation to calculate β now reads

$$W(\beta - \beta_0) + \sum_{j=1}^J \sum_{i=1}^{I_j} n_{ij} M_{ij} = \sum_{j=1}^J N_j \frac{\sum_{i=1}^{I_j} W_j \sum_{i=1}^{I_j} t_{ij} dM_{ij} M_{ij} e^{-\beta M_{ij}}}{\sum_{i=1}^{I_j} W_j \sum_{i=1}^{I_j} t_{ij} dM_{ij} e^{-\beta M_{ij}}}. \quad (\text{A.3})$$

α can be expressed by

$$e^\alpha = \frac{N}{\sum_{j=1}^J W_j \sum_{i=1}^{I_j} 2t_{ij} dM_{ij} e^{-\beta M_{ij}}} \quad \text{with} \quad N = \sum_{j=1}^J N_j = \sum_{j=1}^J \sum_{i=1}^{I_j} n_{ij} \quad (\text{A.4})$$

and provides the activity parameters, $\alpha_j = \alpha + \ln(W_j)$, for each subzone. The corresponding expressions to estimate the covariance matrix are

$$\begin{aligned} -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \alpha} &= 2 \sum_{j=1}^J W_j \sum_{i=1}^{I_j} t_{ij} dM_{ij} e^{\alpha - \beta M_{ij}} = N \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \beta} &= -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha} = -2 \sum_{j=1}^J W_j \sum_{i=1}^{I_j} M_{ij} t_{ij} dM_{ij} e^{\alpha - \beta M_{ij}} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \beta} &= W_0 + 2 \sum_{j=1}^J W_j \sum_{i=1}^{I_j} M_{ij}^2 t_{ij} dM_{ij} e^{\alpha - \beta M_{ij}}. \end{aligned} \quad (\text{A.5})$$

Finite Magnitude Bins

The frequency magnitude rate $\delta\nu(M_{ij})$ for the finite magnitude bin approximation now reads

$$\delta\nu(M_{ij}) = 2W_j e^{\alpha - \beta M_{ij}} \frac{\sinh(\beta dM_{ij})}{\beta}. \quad (\text{A.6})$$

The appropriate derived equations are

$$\begin{aligned} W(\beta - \beta_0) + \sum_{j=1}^J \sum_{i=1}^{I_j} n_{ij} [M_{ij} - dM_{ij} \coth(\beta dM_{ij})] = \\ N \frac{\sum_{j=1}^J \sum_{i=1}^{I_j} t_{ij} W_j \sinh(\beta dM_{ij}) [m_{ij} - dM_{ij} \coth(\beta dM_{ij})] e^{-\beta M_{ij}}}{\sum_{j=1}^J \sum_{i=1}^{I_j} t_{ij} W_j \sinh(\beta dM_{ij}) e^{-\beta M_{ij}}} \end{aligned} \quad (\text{A.7})$$

for evaluating β ,

$$e^\alpha = \frac{\beta N}{\sum_{j=1}^J \sum_{i=1}^{I_j} 2 W_j t_{ij} \sinh(\beta dM_{ij}) e^{-\beta M_{ij}}} \quad (\text{A.8})$$

for the activity parameter α ,

$$\begin{aligned} -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \alpha} &= 2 \sum_{j=1}^J W_j \sum_{i=1}^{I_j} t_{ij} \frac{\sinh(\beta dM_{ij})}{\beta} e^{\alpha - \beta M_{ij}} = N \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha \partial \beta} &= -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha} = -2 \sum_{j=1}^J W_j \sum_{i=1}^{I_j} t_{ij} \frac{\sinh(\beta dM_{ij})}{\beta} [M_{ij} - dM_{ij} \coth(\beta dM_{ij})] e^{\alpha - \beta M_{ij}} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \beta} &= W_0 + 2 \sum_{j=1}^J W_j \sum_{i=1}^{I_j} t_{ij} \frac{\sinh(\beta dM_{ij})}{\beta} [M_{ij} - dM_{ij} \coth(\beta dM_{ij})]^2 e^{\alpha - \beta M_{ij}} \end{aligned} \quad (\text{A.9})$$

and for calculating the covariance matrix $\mathbf{C}(\alpha, \beta)$.

Appendix B

Alternatively to a prior β_0 for regions with limited seismicity, a common β value for an assembly S of several low seismicity zones S_j can be postulated. In this case all activity parameters α_j

of the subzones must be considered in the Maximum Likelihood function

$$\mathfrak{L}(\alpha_1, \dots, \alpha_J, \beta) = \prod_{j=1}^J \mathfrak{L}_j(\alpha_j, \beta) = \prod_{j=1}^J \prod_{i=1}^{I_j} \frac{(\delta\nu(m_{ij}) t_{ij})^{n_{ij}}}{n_{ij}!} e^{-\delta\nu(m_{ij}) t_{ij}}. \quad (\text{B.1})$$

Small Magnitude Bins

For the small magnitude bin approximation, the frequency magnitude rate $\delta\nu(M_{ij})$ yields

$$\delta\nu(M_{ij}) = 2e^{\alpha_j - \beta M_{ij}} dM_{ij}, \quad (\text{B.2})$$

β is the solution of the non-linear equation

$$\sum_{j=1}^J \sum_{i=1}^{I_j} n_{ij} M_{ij} = \sum_{j=1}^J N_j \frac{\sum_{i=1}^{I_j} t_{ij} dM_{ij} m_{ij} e^{-\beta M_{ij}}}{\sum_{i=1}^{I_j} t_{ij} dM_{ij} e^{-\beta M_{ij}}} \quad (\text{B.3})$$

and the activity parameters α_j of subzones S_j can be expressed by

$$e^{\alpha_j} = \frac{N_j}{\sum_{i=1}^{I_j} 2t_{ij} dM_{ij} e^{-\beta M_{ij}}}. \quad (\text{B.4})$$

The covariance matrix now has the dimension $(J+1) \times (J+1)$

$$\begin{aligned} \mathbf{C} &= \begin{pmatrix} \text{cov}(\alpha_1, \alpha_1) & \text{cov}(\alpha_1, \alpha_2) & \dots & \text{cov}(\alpha_1, \alpha_J) & \text{cov}(\alpha_1, \beta) \\ \text{cov}(\alpha_2, \alpha_1) & \text{cov}(\alpha_2, \alpha_2) & \dots & \text{cov}(\alpha_2, \alpha_J) & \text{cov}(\alpha_2, \beta) \\ \vdots & \vdots & & \vdots & \vdots \\ \text{cov}(\beta, \alpha_1) & \text{cov}(\beta, \alpha_2) & \dots & \text{cov}(\beta, \alpha_J) & \text{cov}(\beta, \beta) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_1 \partial \alpha_1} & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_1 \partial \alpha_2} & \dots & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_1 \partial \alpha_J} & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_1 \partial \beta} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_2 \partial \alpha_1} & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_2 \partial \alpha_2} & \dots & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_2 \partial \alpha_J} & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_2 \partial \beta} \\ \vdots & \vdots & & \vdots & \vdots \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha_1} & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha_2} & \dots & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha_J} & -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \beta} \end{pmatrix}^{-1} \end{aligned} \quad (\text{B.5})$$

and can completely described by the expressions

$$\begin{aligned} -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_i \partial \alpha_j} &= 0 \quad \text{für } i \neq j \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_j \partial \alpha_j} &= 2 \sum_{i=1}^{I_j} t_{ij} dM_{ij} e^{\alpha_j - \beta M_{ij}} = N_j \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_j \partial \beta} &= -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha_j} = -2 \sum_{i=1}^{I_j} M_{ij} t_{ij} dM_{ij} e^{\alpha_j - \beta M_{ij}} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \beta} &= 2 \sum_{j=1}^J \sum_{i=1}^{I_j} m_{ij}^2 t_{ij} dM_{ij} e^{\alpha_j - \beta M_{ij}}. \end{aligned} \quad (\text{B.6})$$

For a subzone S_j , the uncertainties of the GR parameters are determined by the submatrix

$$\mathbf{C}_j = \begin{pmatrix} \text{cov}(\alpha_j, \alpha_j) & \text{cov}(\alpha_j, \beta) \\ \text{cov}(\beta, \alpha_j) & \text{cov}(\beta, \beta) \end{pmatrix}. \quad (\text{B.7})$$

Finite Magnitude Bins

The frequency magnitude rate $\delta\nu(M_{ij})$ for the finite magnitude bin approximation now reads

$$\delta\nu(M_{ij}) = 2e^{\alpha_j - \beta M_{ij}} \frac{\sinh(\beta dM_{ij})}{\beta}. \quad (\text{B.8})$$

the appropriate derived equations are

$$\begin{aligned} \sum_{j=1}^J \sum_{i=1}^{I_j} n_{ij} [M_{ij} - dM_{ij} \coth(\beta dM_{ij})] = \\ \sum_{j=1}^J N_j \frac{\sum_{i=1}^{I_j} t_{ij} \sinh(\beta dM_{ij}) [m_{ij} - dM_{ij} \coth(\beta dM_{ij})] e^{-\beta M_{ij}}}{\sum_{i=1}^{I_j} t_{ij} \sinh(\beta dM_{ij}) e^{-\beta M_{ij}}} \end{aligned} \quad (\text{B.9})$$

for evaluating β ,

$$e^{\alpha_j} = \frac{\beta N_j}{\sum_{i=1}^{I_j} 2 t_{ij} \sinh(\beta dM_{ij}) e^{-\beta M_{ij}}} \quad (\text{B.10})$$

for the activity parameters α_j of subzones S_j ,

$$\begin{aligned} -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_i \partial \alpha_j} &= 0 \quad \text{for } i \neq j \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_j \partial \alpha_j} &= 2 \sum_{i=1}^{I_j} t_{ij} \frac{\sinh(\beta dM_{ij})}{\beta} e^{\alpha_j - \beta M_{ij}} = N_j \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \alpha_j \partial \beta} &= -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \alpha_j} = -2 \sum_{i=1}^{I_j} t_{ij} \frac{\sinh(\beta dM_{ij})}{\beta} [M_{ij} - dM_{ij} \coth(\beta dM_{ij})] e^{\alpha_j - \beta M_{ij}} \\ -\frac{\partial^2 \ln \mathfrak{L}}{\partial \beta \partial \beta} &= 2 \sum_{j=1}^J \sum_{i=1}^{I_j} t_{ij} \frac{\sinh(\beta dM_{ij})}{\beta} [M_{ij} - dM_{ij} \coth(\beta dM_{ij})]^2 e^{\alpha_j - \beta M_{ij}} \end{aligned} \quad (\text{B.11})$$

and for calculating the covariances.