Relative wobble of the Earth’s inner core derived from polar motion and associated gravity variations

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SUMMARY
In this paper an explanation of the observed decadal variations of the polar motion of the Earth is presented. Recent investigations show that the contribution of surface processes is too small to excite the observed magnitudes of the decadal variations of polar motion. After removing the known effects of atmospheric variations from the observed polar motion, we obtain significant residuals that obviously can only be explained by processes in the core. In this paper, we investigate the effect of a relative inner-core rotation. In particular, we assume that the orientation of the figure axis of the inner core changes with respect to the outer core and mantle. Due to the flattening of the inner core and the density difference between the inner and outer core, these changes contribute to variations of the Earth’s polar motion due to internal mass redistributions. The objective of this study is to determine those changes in the orientation of the figure axis of the inner core that produce mass redistributions necessary to excite the decadal variations of the observed polar motion minus the estimated atmospheric influence.

Using polar motion data and the atmospheric excitation function, it is possible to determine the excitation function of the internal process superimposed on the free wobble of the Earth by linear approximation of the Liouville equation. On the other hand, provided that only the contribution of the mass redistributions is significant, we can express the time function obtained in terms of the orientation of the figure axis of the inner core. The final expression then contains the corresponding orientation angles as unknowns. Using this expression, we calculate the orientation angles from the numerically determined values of the excitation function. The calculation results in a mean tilt of 1°, and a mean eastward drift of 0.7° yr⁻¹ of the figure axis of the inner core and quasi-periodic decadal variations of its orientation angles.

The associated changes of the mass geometry in the core due to these variations of the figure axis of the inner core are then used to estimate their influence on the Earth’s outer gravity field. Finally, we compare the magnitude of the resulting gravity field variations with the accuracy of recent and future gravity field models.

Key words: gravity changes, inner core, polar motion.

1 INTRODUCTION
The observed decadal variations of polar motion cannot be explained completely by surface processes. Jochmann (1999) investigated the contributions of atmospheric circulation and ground water storage and concluded that the latter is of minor importance on the decadal timescale, whereas atmospheric excitation cannot be neglected. After removing the atmospherically excited parts, we obtain significant residuals that must be explained by internal processes. Electromagnetic core–mantle coupling is too small because the electrical conductivity of the lower mantle is too low (e.g. Greiner-Mai 1993; Greff-Lefftz & Legros 1995), although it can be responsible for the decadal variations of the length of day, ΔLOD (e.g. Holme 1998a,b). With regard to the topographic coupling, knowledge of the core–mantle topography is insufficient to decide whether this coupling can completely explain decadal variations of polar motion or not. It appears that the topographic coupling torques are too large for ΔLOD if they are sufficient for polar-motion variations (Jault & Le Mouël 1989; Hinderer et al. 1990). Recent investigations show that the topographic coupling is probably too small by a factor 10 (Greff-Lefftz & Legros 1995).
or 5 (Hide et al. 1996) to explain the decadal variations of polar motion. Hulot et al. (1996) concluded that the excitation of polar motion by outer-core processes is a combination of several processes. In recent models of polar-motion excitation by internal processes, the method of coupling between particular parts of the Earth is used (torque approach); the effect of mass redistributions within the core is ignored. Another method applied here is to consider the angular momentum balance of the torque-free whole Earth, which implies that the internal torques are balanced (angular momentum approach), and to investigate the influence of geophysical processes by evaluating their excitation functions (e.g. Lambeck 1980).

Besides external forcing from the Sun and Moon (not considered here), variations of angular momentum of the Earth are caused by relative motions and mass redistributions on and within it. In the polar-motion equation, relative motions are considered in terms of the relative angular momentum, \( h \), changes of mass geometry and density in terms of the products of inertia, \( c \), and their combined effect in terms of the excitation function, \( \psi \). It is well known that in the atmospheric excitation function of polar motion the variations of the relative angular momentum, \( h \), can be neglected compared to the variations of \( c \) caused by mass redistributions. By analogy with atmospheric excitation, we search for mechanisms in the core that are accompanied by mass redistributions. Because of the significant density difference between inner and outer core, we find that one possible mechanism is a relative wobble of the oblate inner core. The crucial points then are how to infer a motion of this kind from an observable quantity and how to model the relative wobble by core dynamics. With respect to the first point, we consider the angular momentum approach and use the polar motion variations as observables. We do not plan to investigate the torque approach; for this approach we refer to Aurnou & Olson (2000), Dehant et al. (1993), Mathews et al. (1991), Szteto & Xu (1997) and Xu & Szteto (1994, 1996, 1998).

In previous investigations, the variations of the geomagnetic dipole axis were used as the observable, from which the relative inner-core motion was derived. Schmutzer (1978) and Smylie et al. (1984) suggested that a variation of the axis of relative rotation of the inner core can possibly be detected by a corresponding variation of the geomagnetic dipole axis. Considering decadal fluctuations in the Earth’s rotation and the geomagnetic dipole field, Jochmann (1989) investigated the consequences of variations of the figure axis of an oblate inner core on the inertia tensor of the Earth and on polar motion by identifying the observed variations of the geomagnetic dipole axis with those of the figure axis of the inner core. He calculated variations of polar motion that approximately agree with the observations. Greiner-Mai et al. (2000) re-examined these calculations by using new geomagnetic field representations and determined the influence of the relative inner-core wobble on gravity variations. As outlined in this paper, the hypothesis about the coincidence of the figure axis of the inner core and the geomagnetic dipole axis was used as an \textit{a priori} assumption, which is not explained by modern dynamo theories. Here, we do not use this hypothesis and derive values for the orientation angles of the figure axis of the inner core from polar motion data.

The objective of our investigation is to determine those variations of the orientation of the figure axis of the inner core that are necessary to excite the observed decadal variations of polar motion, to calculate their influence on the gravity field and to test their detectability by modern gravity measurements.

The investigations in this paper differ from those in Greiner-Mai et al. (2000) in that (i) the orientation angles of the figure axis of the inner core are not derived from those of the geomagnetic dipole axis but instead are derived from polar motion data by an inverse solution of the polar motion equation, (ii) the contribution of atmospheric processes to polar motion is considered by involving their excitation function, and (iii) because of points (i) and (ii), the resulting variations of the figure axis of the inner core and the gravity field differ from those obtained in Greiner-Mai et al. (2000). Nevertheless, the theoretical expression for the excitation function of the inner-core motion and the formalism for computing the gravity field are the same.

In this study, we neglect translational motions of the inner core and changes of its inertia tensor by viscoelastic deformation. Elastic deformation influences the eigenfrequencies (nearly diurnal and Chandler wobbles) of the free wobble of the Earth (e.g. Dehant et al. 1993). Here we are concerned with variations of polar motion, which are excited by geophysical processes at a timescale much larger than the Chandler period, that is, the eigenfrequency of the Earth model. The viscosity of the inner core is poorly determined and would introduce an additional unknown into the formalism. Its neglect implies the assumption that the associated relaxation time is longer than the decadal timescale considered here. Furthermore, we assume that the density of the inner core is constant and its figure is a rotational ellipsoid. The variation of the orientation of the figure axis of the inner core then indicates a relative wobble of the inner core as a solid ellipsoidal body.

Our investigation consists of two steps. First, we investigate those variations of the orientation of the figure axis of the flattened inner core that are \textit{necessary} to excite the observed decadal variations of the polar motion of the Earth. The causative process is contingent on the density jump between the inner and outer core. Then, a variation of the figure axis of the oblate inner core with respect to the polar axis of the Earth changes the internal mass geometry of the Earth and therefore its inertia tensor (hereafter referred to as the internal mass effect). To describe the influence on polar motion, \( m \), we use the polar-motion equation for a torque-free Earth with a viscoelastic mantle. In this model, longer-periodic geophysical processes conventionally considered by their excitation functions, \( \psi \), are superimposed on the free wobble of the viscoelastic earth model. The atmospheric excitation function, \( \psi_{\text{atm}} \), can be derived from air-pressure data (Section 2.2). In the polar-motion equation, \( \psi_{\text{atm}} \) appears therefore as a known quantity (in addition to \( m \)). Provided that only the internal mass effect is significant, the excitation function associated with the variations of the figure axis of the inner core, \( \psi_{\text{r}} \), can then be determined from polar-motion data, \( m \), and \( \psi_{\text{atm}} \). The variations of \( \psi_{\text{r}} \) can be modelled using the orientation angles of the figure axis, \( \psi_{\text{f}} \) (longitude) and \( \psi_{\text{c}} \) (co-latitude) as unknowns; having obtained the final expression of \( \psi_{\text{r}} \) in terms of \( \psi_{\text{f}} \) and \( \psi_{\text{c}} \), we can determine them as a function of time by inserting for \( \psi_{\text{r}} \) the values derived from \( m \) and \( \psi_{\text{atm}} \).

Second, the variations of the orientation of the figure axis of the oblate inner core also influence the gravity field. The detectability of this effect depends on the amplitudes of the associated gravity variations. They can be computed from the orientation angles of the figure axis of the inner core by methods of potential theory and coordinate transformation formulae of spherical harmonic series (Section 3).
point is that the mass redistributions in the shell bounded by the semi-minor and semi-major axes of the ellipsoidal inner core determine the gravity changes. Because of the small flattening of the inner core, this shell is thin and the expected gravity changes are small. Nevertheless, comparing them with the expected accuracy of planned satellite missions, they may be estimable within the next decade.

2 VARIATIONS OF THE FIGURE AXIS OF THE INNER CORE

2.1 Theoretical excitation function of polar motion due to internal mass redistribution

The formalism for the calculation of the excitation function and its dependence on the orientation angles of the figure axis of the inner core is shown in Greiner-Mai et al. (2000). For details of the derivation of eq. (5) below, we refer to this paper. In Greiner-Mai et al. (2000), the atmospheric excitation function was neglected in the polar motion equation, whereas it is retained in this study.

With respect to decadal variations of polar motion, it is sufficient to consider a conventional earth model with a viscoelastic mantle, as was done by Munk & McDonald (1960). In this model, the wobble of the inner core is described by its excitation function, the computation of which is the main objective of this theoretical investigation.

The quantities refer to the Earth-fixed geocentric coordinate system x, y, z, where the z-axis is aligned with the polar axis of the Earth. In this coordinate system, the components of the rotation vector of the Earth, \( \mathbf{w} \), are given by

\[
\mathbf{w} = \omega_0 (m_x, m_y, 1 + m_z), \quad \omega_0 = \frac{2 \pi}{86400 \text{ s}},
\]

where \( m_x, m_y, m_z \ll 1 \). The polar motion is governed by the complex differential equation

\[
\dot{m} + \alpha m = \sigma_{CH} \left( m - \sigma_{EL} \psi_i - \psi_{atm} \right),
\]

where \( m = m_x + j m_y \) are the complex coordinates of the rotation pole of the Earth \( (j = \sqrt{-1}) \), \( \sigma_{CH} = 5.28 \text{ yr}^{-1} \) and \( \sigma_{EL} = 7.46 \text{ yr}^{-1} \) are the Chandler and Eulerian frequencies, \( \alpha = 0.05 \text{ yr}^{-1} \) is the dampening coefficient, and \( \psi_i = \psi_{i_x} + j \psi_{i_y} \) and \( \psi_{atm} = \psi_{atm_x} + j \psi_{atm_y} \) are the complex excitation functions associated with the relative inner-core wobble and atmospheric circulation, respectively. The values of \( \alpha \) and \( \sigma_{CH} \) are chosen according to Jochmann (1981, 1999). Eq. (2) is an approximation valid for decadal and longer-period variations of \( m \) and \( \psi_i, \psi_{atm} \). For \( \psi_i = \psi_{atm} = 0 \) it would describe a rigid earth model with a viscoelastic mantle and corotating atmosphere and core in which the Earth oscillates freely with the frequency \( \sigma_{CH} \). For much larger frequencies, the inertial coupling between mantle and core (and inner and outer core) must be considered, which conventionally leads to models with nearly diurnal core wobbles; these are not considered here.

The calculation of \( \psi_{atm} \) will be shown in the next section. In the following will express \( \psi_i \) in terms of the orientation angles of the figure axis, \( \chi_i \) and \( \beta_i \). A comprehensive derivation of the final expression is given by Greiner-Mai et al. (2000). In the following we give only an outline of it.

The general expression for a complex excitation function \( \psi \) of polar motion for a torque-free earth is given by (e.g., Lambeck 1980)

\[
\psi = \chi - \frac{j}{\omega_0} \dot{\chi}, \quad \chi = \frac{c}{C - A} - \frac{h}{(C - A)\omega_0},
\]

where \( A \) and \( C \) are the temporally constant principal moments of inertia of the Earth, \( c = c_{i_z} + c_{i_x} \) are the complex temporally variable components of the inertia tensor associated with the mass redistribution considered and \( h = h_i + h_t \) is the complex relative angular momentum.

On the decadal timescale considered here, the term \( j\omega_0^{-1} \dot{\chi} \) can be neglected and the excitation function can be approximated by

\[
\psi = \frac{c}{A - C} - \frac{h}{(C - A)\omega_0}.
\]

Greiner-Mai et al. (2000) used a decomposition of the inertia tensor of the Earth into a constant part (represented by its constant principal moments of inertia, \( A \) and \( C \)) and a temporally variable part caused by the relative wobble of the inner core and the density difference between inner and outer core. The temporally variable part was determined using a rotational transformation of the inertia tensor of the inner core by which the temporally varying components, \( c \), of the inertia tensor of the Earth can be expressed in terms of the orientation angles, \( \psi_{\ell} \) and \( \beta_{\ell} \), of the figure axis of the inner core. The resulting expression of the excitation function is

\[
\psi_{\ell}(t) = \frac{c_{\ell}}{C - A} \eta \frac{1}{2} \sin 2\beta_{\ell} \exp(i\psi_{\ell}) - \frac{h}{(C - A)\omega_0},
\]

where \( c_{\ell} \), \( A_{\ell} \) are the constant principal moments of inertia of the inner core and \( \eta = (p_i - p_o) n i^{-1} (p_i \text{ is the density of the inner core and } p_o, \text{ that of the outer core at the inner core–outer core boundary}) \).

The relative angular momentum of the Earth is given by

\[
\mathbf{h} = \int_V \rho \mathbf{r} \times \mathbf{u} dV,
\]

where \( \mathbf{u} \) is the velocity field of motions relative to the earth-fixed coordinate system. The relative atmospheric angular momentum entering into \( \psi_{atm} \) is considered separately (Section 2.2). The volume \( V \) in eq. (6) is therefore that of the Earth without atmosphere.

For an axially symmetric density distribution, only non-axial velocity fields contribute to \( \mathbf{h} \). Velocity fields at the core surface can be determined from geomagnetic variations by inverse solutions of the frozen-field equation. For the calculation of the angular momentum according to eq. (6), the velocity field must be known in the whole core. This requires a dynamic model of the associated core motions, the values of which at the core–mantle boundary must be consistent with those inferred from the geomagnetic field. Unfortunately, the non-axial parts of the velocity field inside the core varying with decadal periods have been poorly determined previously. We therefore assume that the ‘motion term’ of \( \psi \) (see eq. 5) containing \( h \) is negligible small compared with the ‘matter term’ containing \( c \). In the Appendix we give some arguments that seem to indicate that \( h \) can be neglected in \( \psi_i \). Nevertheless, this is an assumption that is ultimately speculative within the scope of our investigation.

For evaluating eq. (5), the flattening of the inner core and the density jump between inner and outer core, \( \Delta p = p_i - p_o \), must be known. Smylie et al. (1984) determined the flattening
according to Clairaut’s equation. The density jump is chosen according to theoretical earth models. In Jochmann (1989) the following excitation function is given:
\[
\psi_i(t) = 4.3787 \times 10^{-5} \sin 2\theta_i \exp(j\phi_i).
\]
(7)

This equation was evaluated using the inner-core flattening given by Smylie et al. (1984) and the density jump given by Bullen & Jeffreys (in Egyed 1969, p. 197). Dziewonski & Anderson (1981) proposed an earth model with a smaller density jump (PREM). The corresponding excitation function is obtained by multiplying eq. (7) by 0.32. Here we use both earth models for showing the influence of the density jump; the gravity variations are then computed with PREM.

2.2 Orientation of the figure axis of the inner core inferred from polar-motion data

In the following we use IERS (EOP97C01) polar-motion data in 0.05 yr intervals for the pole coordinates, \(m\). Because of poor knowledge of the atmospheric winds in the time interval considered (1900–1987) and of the density distribution in the atmosphere, the values for \(\psi_{\text{atm}}\) are derived from air-pressure variations given in Vose et al. (1992).

According to eq. (4), \(\psi_{\text{atm}}\) can be divided into a motion term and a matter term, \(\psi_{\text{atm}} = \psi_{\text{mat}} + \psi_{\text{atm}}\), where the motion term describes the effect of the relative angular momentum, \(h\), and the matter term that of the product of inertia, \(c\). Using eq. (6), we then obtain for the motion term
\[
\psi_{\text{mat}} = \frac{1}{(C-A)v_0} \int_a^{e/2} \int_{\Phi=-\pi/2}^{\pi/2} \int_{\Lambda=0}^{2\pi} r^4 \rho(\Phi, \Lambda, t) \\
\times (v_A \sin \Phi - j\phi) \exp(j\lambda) \cos \Phi d\Phi d\Lambda dr,
\]
(8)

where \(\Phi\) and \(\Lambda\) are the geographical latitude and longitude, respectively, and the velocities \(v_A\), \(\phi\) and the density are expressed by air pressure \(p\) according to the geostrophic approximation. The dependence of the pressure on atmospheric height is derived from the equation of state of an isothermal atmosphere (Jochmann 1987, 1988). Using the same equation of state, we can express the density variation with height in the matter term, \(\psi_{\text{mat}}\), by the surface air pressure. After integration with respect to \(r\), we obtain
\[
\psi_{\text{mat}} = \frac{(a + h_m)^4}{(C-A)g} \int_a^{e/2} \int_{\Phi=-\pi/2}^{\pi/2} \int_{\Lambda=0}^{2\pi} \Delta \rho(\Phi, \Lambda, t) \exp(j\lambda) \\
\times \sin \Phi \cos^2 \Phi d\Phi d\Lambda d\lambda,
\]
(9)

where \(a\) is the mean Earth radius, \(h_m\) is the effective atmospheric height (\(\approx 7000\) m), \(g\) is the mean surface gravity and \(\Delta \rho\) are the air-pressure variations with respect to a long-term local mean. Because of poorly distributed air-pressure values on the oceans in the first decades of this century, surface loading on the oceans is derived from air-pressure variations on the continents, assuming conservation of mass and the inverted barometer principle. The integrals in eqs (8) and (9) are solved numerically by dividing the earth’s surface into a \(\Phi-\Lambda\) grid of \(5.625^\circ \times 5.625^\circ\) and completing the missing pressure values with those derived from a climate model. Numerical calculations show that \(\psi_{\text{mat}}\) is negligible compared with \(\psi_{\text{mat}}\). Assuming \(\psi_{\text{atm}} \approx \psi_{\text{mat}}\), we can then calculate the excitation function associated with relative inner-core motions.

Solving eq. (2) for \(\psi_i\), we obtain the complex equation
\[
\psi_i = \frac{i}{\sigma_{\text{EU}}} [m + i\sigma_{\text{CH}}(m - \psi_{\text{mat}})].
\]
(10)

For decadal variation we use an approximation of eq. (10). With the values for \(\sigma_{\text{CH}}\) and \(x\) given in Section 2.1 and the relation \(|\sigma| \ll |\sigma_{\text{CH}}|\) valid for periods significantly longer than the Chandler period, eq. (10) can be approximated by
\[
\psi_{\text{obs}}^{\text{atm}} = \frac{\sigma_{\text{CH}}}{\sigma_{\text{EU}}} (m - \psi_{\text{mat}}).
\]
(11)

To ensure that the approximation condition \(|\sigma| \ll |\sigma_{\text{CH}}|\) is valid for the data used, we must smooth the short-period parts in the data for \(m\) and \(\psi_{\text{atm}}\), before applying eq. (11). This can be achieved by a conventional harmonic filter procedure. The resulting decadal variations of the polar motion and the atmospheric excitation functions are shown in Fig. 1. We conclude from the figure that the atmospheric contribution is significant, but does not completely explain the observed decadal variations of polar motion. In future investigations, the excitation by hydrospheric processes should be taken into account. This is not possible in this paper because of the lack of sufficiently long time-series.

The orientation angles of the figure axis of the inner core can then be derived from eq. (5), with the last term neglected, and eq. (7) by substituting \(\psi_{\text{obs}}^{\text{atm}}\) for \(\psi_i\), decomposing into real and imaginary parts and solving for the unknowns \(\delta_t\) and \(\psi_t\).

Figure 1. Decadal variations of the \(x\) and \(y\) components (a), (b) of the polar motion, \(m\) (IERS data filtered), and of the atmospheric excitation function, \(\psi_{\text{atm}} = \psi_{\text{atm},x} + j\psi_{\text{atm},y}\) (computed from air pressure), and (c) of the excitation function associated with relative inner-core motions, \(\psi_{\text{obs}}^{\text{atm}} = \psi_{\text{obs},x} + j\psi_{\text{obs},y}\), according to eq. (11).

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obtain

\[ \varphi_f = \arctan \frac{\psi_\text{obs}}{\psi_\text{eq}}, \]

\[ \vartheta_f = \frac{1}{2} \arcsin b \sqrt{\left(\frac{\psi_\text{obs}}{\psi_\text{eq}}\right)^2 + \left(\frac{\psi_\text{eq}}{\psi_\text{obs}}\right)^2}, \]

where \( b = 1.107 \times 10^{-4} \) for the model of Bullen & Jeffreys and \( b = 3.460 \times 10^{-4} \) for the PREM model, if the excitation functions are given in milliarcseconds (mas). Eq. (12) shows that \( \varphi_f \) is independent of the density jump at the inner core–outer core boundary.

The resulting time-series of the orientation angles \( \varphi_f \) and \( \vartheta_f \) on the decadal timescale are shown in Fig. 2. We conclude from this figure that (i) the angle between the figure axis of the inner core and the polar axis of the Earth changes between about 0.1° and 0.5° for the density model of Bullen & Jeffreys and between 0.4° and 1.5° for the PREM model, (ii) the direction of the mean relative rotation of the figure axis indicated by \( \varphi_f(t) \) is eastwards, and (iii) the mean angular velocity of this eastward relative rotation is about 0.7° yr\(^{-1}\).

In addition, the time behaviour indicates that (i) the orientation of the figure axis changes with time at periods of about 70, 30 and 20 yr, and (ii) the colatitude \( \vartheta_f \) oscillates between 1900 and 1940, increases at about 0.04° yr\(^{-1}\) from about 1940 to 1965 and then begins to oscillate again at a larger period. Furthermore, compared with the geomagnetic hypothesis used in Greiner-Mai et al. (2000), which assumes a large tilt of about 10°, the inverse computation of \( \vartheta_f \) from observed polar-motion variations shows that a tilt smaller by one order of magnitude is sufficient to explain the decadal variations of the polar motion by relative motions of the figure axis of the inner core.

Finally, the model shows that a smaller tilt of the figure axis of the inner core results for larger values of the density jump. The model therefore offers the possibility to determine the density jump from gravity measurements if the gravity contributions from different sources can be separated.

3 INFLUENCE ON THE GRAVITY FIELD

Provided that the figure axis of the oblique ellipsoidal inner core moves relative to the Earth, this motion causes changes of the gravity field. Knowing the density difference between inner and outer core, the flattening of the inner core and the time variation of the orientation of its figure axis in terms of \( \varphi_f(t) \) and \( \vartheta_f(t) \) (Fig. 3), we can estimate these changes and compare them with the accuracy of recent gravity field models and with the expected accuracy of planned satellite gravity missions such as CHAMP and GRACE.

To simplify the calculations, we introduce a second geocentric coordinate system, which is fixed to the inner core. Its \( z \)-axis, \( z_f \), is aligned with the figure axis of the ellipsoid; the \( x_f \)-axis is then defined by the \( z-z_f \) plane. The gravity potential difference between an oblique and an aligned ellipsoidal inner core with the density distributions \( \rho_e(\mathbf{r}) \) (outer core) and \( \rho_i(\mathbf{r}) \) (inner core) is

\[ \Delta V = V_f - V_0, \]

with

\[ V_0 = V(\vartheta = 0) = \int_{\Gamma_e} \rho_e(\mathbf{r}) r^{-1} d\mathbf{r} + \int_{\Gamma_i} \rho_i(\mathbf{r}) r^{-1} d\mathbf{r}, \]

and

\[ V_f = V(\varphi, \vartheta) = \int_{\Gamma_e'} \rho_e(\mathbf{r}) r^{-1} d\mathbf{r} + \int_{\Gamma_i'} \rho_i(\mathbf{r}) r^{-1} d\mathbf{r}. \]

In the following, we

(1) show that the calculation of the potentials according to eqs (13), (14) and (15) can be reduced to an integration over a rotational ellipsoid homogeneously filled with mass of density \( \rho = \Delta \rho \),

(2) calculate the potential for an ellipsoidal inner core in the coordinate system fixed to the inner core (Fig. 3), and

(3) transform the solution to the mantle-fixed coordinate system, thus obtaining the associated perturbation of the gravitational potential in the coordinate system conventionally used.

![Figure 3. Aligned (left) and oblique (right) ellipsoidal inner core.](image)

\[ \Gamma_e, \Gamma_i, \Gamma_e' \] and \( \Gamma_i' \) are the domains of the inner and outer core, respectively; \( \vartheta, \varphi \) are the colatitude and azimuth of the figure axis, \( z_c \) of the inner core with respect to the mantle-fixed coordinate system.

\[ \varphi_f, \vartheta_f \]

\[ \text{Figure 2. Resulting orientation angles of the figure axis of the inner core. (a) } \varphi_f \text{ for both density models, (b) } \vartheta_f \text{ for the density model of Bullen & Jeffreys and (c) } \vartheta_f \text{ for the PREM model.} \]

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The assumption in step (1) seems plausible, because an ellipsoidal inner core precessing within a medium with the same density has no effect on the gravity field.

Beginning with step (1), we consider more generally the relation for the difference of the integration over two domains $A$ and $B$:

$$\int_A f(x)dx - \int_B f(x)dx = \int_{A-B} f(x)dx + \int_{A-B} f(x)dx$$

Applying eq. (16) to eqs (13)–(15), we must solve the following integrals over four difference regions (for notation see Fig. 3):

$$\Delta V = \int_{\Gamma_i \Gamma_i^*} \rho_i(r)r^{-1}dr - \int_{\Gamma_i^* \Gamma_i} \rho_i(r)r^{-1}dr + \int_{\Gamma_i^* \Gamma_i^*} \rho_i(r)r^{-1}dr - \int_{\Gamma_i \Gamma_i^*} \rho_i(r)r^{-1}dr.$$  \hfill (17)

Because $\Gamma_i \Gamma_i^* = \Gamma_i \Gamma_i$ and $\Gamma_i^* \Gamma_i^* = \Gamma_i \Gamma_i^*$ holds, we can write

$$\Delta V = \int_{\Gamma_i} (\rho_i(r) - \rho_i(r))r^{-1}dr - \int_{\Gamma_i} (\rho_i(r) - \rho_i(r))r^{-1}dr,$$  \hfill (18)

and with eq. (16) it follows that

$$\Delta V = \int_{\Gamma_i} (\rho_i(r) - \rho_i(r))r^{-1}dr - \int_{\Gamma_i} (\rho_i(r) - \rho_i(r))r^{-1}dr.$$  \hfill (19)

Eq. (18) shows that the potential difference between an oblique and an aligned inner core depends only on the difference in the density distributions $\rho_i(r) - \rho_i(r)$ in the difference domains $\Gamma_i \Gamma_i$ and $\Gamma_i \Gamma_i^*$. Due to the small flattening of the inner core, the integrations in eq. (18) are in fact carried out over a thin shell (thickness $\approx 3$ km, Table 2). This shows that the density jump at the inner core–outer core boundary, $\Delta \rho$, is significant rather than the density distributions in the full domains and that the distribution $\rho_i(r) - \rho_i(r)$ in eq. (19) can be approximated by the constant density $\Delta \rho$. Using eq. (19), the calculations can then be continued according to steps (2) and (3) (see also Section 2.1).

Usually, the geopotential is given in terms of the coefficients $C_{nm}$ and $S_{nm}$ of a spherical harmonic expansion,

$$V(r, \varphi, \vartheta) = \sum_{m=0}^{N} \sum_{n=0}^{m} (C_{nm} \cos m\varphi + S_{nm} \sin m\varphi)P_{nm}(\cos \vartheta).$$  \hfill (20)

With respect to our problem, we must compute the coefficients of the potential $\Delta V$:

$$\Delta C_{nm}(t) = \Delta C_{nm}(\phi_i(t), \theta_i(t)),$$

$$\Delta S_{nm}(t) = \Delta S_{nm}(\phi_i(t), \theta_i(t))$$

in the mantle-fixed coordinate system. For this, we compute the potential coefficients of the oblate inner core with respect to the coordinate system mentioned above fixed to the inner core according to step (2) and transform them into the coordinate system fixed to the mantle according to step (3).

In the coordinate system fixed to the inner core, the fully normalized zonal harmonic coefficients $C_{nm}$ for the potential of a body with rotational symmetry and density $\rho$ are given by

$$C_{nm} = \frac{1}{MR_0^{2n+1}} \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \int_{r=0}^{r_{\text{max}}} \rho^n P_n(\cos \vartheta) r^2 (\sin \vartheta)^{2n+1} dr d\vartheta d\varphi,$$  \hfill (21)

where $M$ is the mass of the body and $R_0$ is the reference radius. In our case, the integral must be evaluated for the region of the ellipsoidal inner core, the density of which is $\Delta \rho$. The solution for $\rho = \Delta \rho = \text{constant}$ is

$$C_{2l,0} = \frac{3}{M} \frac{M_e}{R_0^3} \left( \frac{a^2 - b^2}{2l + 1} \right)^{l-1},$$  \hfill (22)

where $M_e = (4/3)\pi M \Delta \rho a^2$ is the mass of the ellipsoid with axes $a$ and $b$ and density $\Delta \rho$. Since we are interested in the contribution with respect to the potential of the whole Earth, we must take for $M$ the mass of the Earth. Because we assume a rotational ellipsoid as the shape of the inner core, only zonal coefficients $C_{nm}$ of even degree $n$ occur in the coordinate system fixed to the inner core. In practice, it is sufficient to consider the coefficient $C_{20}$.

The transformation of spherical harmonic coefficients $(C_{nm}, S_{nm}) \Rightarrow (C_{np}(x, \beta, \gamma), S_{np}(x, \beta, \gamma))$ with respect to a coordinate system rotated by the angles $x, \beta, \gamma$ is considered by Kautz (1965) and by Ilk (1983). Rearranging those formulae for numerical computation of rotation by the angles $\psi$ and $\delta$ (Fig. 3) gives

$$C_{np}(\psi, \delta) = \sum_{m=0}^{n} (C_{nm} \cos m\varphi + S_{nm} \sin m\varphi)A_{nm}^{pm}(\psi),$$  \hfill (23)

$$S_{np}(\psi, \delta) = \sum_{m=1}^{n} (S_{nm} \cos m\varphi - C_{nm} \sin m\varphi)B_{nm}^{pm}(\psi).$$  \hfill (24)

The transformation parameters $A_{nm}^{m}(\delta)$ and $B_{nm}^{m}(\delta)$ are

$$A_{nm}^{m}(\delta) = (-1)^{p} \left[ (2 - \delta_{0,p}) - (n + m)!/(n - m)! \right]^{1/2} \times \sum_{j=0}^{n} \left[ \frac{m!}{(m-j)!j!(p-j)!} P_{n+p-j}^{m-j}(\cos \delta) \right] \times \frac{(-1 - \cos \delta) + (1 - \cos \delta)}{2 \sin \delta},$$  \hfill (25)

$$B_{nm}^{m}(\delta) = (-1)^{p} \left[ (2 - \delta_{0,p}) - (n + m)!/(n - m)! \right]^{1/2} \times \sum_{j=0}^{n} \left[ \frac{m!}{(m-j)!j!(p-j)!} P_{n+p-j}^{m-j}(\cos \delta) \right] \times \frac{(-1 - \cos \delta) - (1 - \cos \delta)}{2 \sin \delta},$$  \hfill (26)

$j_1 = \max(p+m-n, 0)$ and $j_2 = \min(m, p)$. 

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The numerical values of the parameters used for calculations are given in Table 1 and the values for the orientation angles \( \varphi = \varphi_1 \) and \( \beta = \beta_1 \) of the figure axis of the inner core derived from the observed polar motion are shown in Fig. 2. Using the density jump according to the PREM model, we obtain for \( C_{20} \) (with respect to a coordinate system fixed to the inner core) the value
\[
C_{20} = -1.240 \times 10^{-8},
\]
and for the time variations of the transformed coefficients \( \Delta C_{nm}(\varphi(t), \beta(t)) \) and \( \Delta S_{nm}(\varphi(t), \beta(t)) \) \((n = 2, m = 0, 1, 2)\) the values shown in Fig. 4.

The calculated rates of change of the coefficients in the mantle-fixed coordinate system over the last 10 yr are given in Table 2. Table 3 shows the accuracy of the low-degree spherical harmonic coefficients of present global gravity models, e.g. GRIM4 (Schwintzer et al. 1997), and the expected accuracies of the planned satellite missions CHAMP (Reigber et al. 1997) and GRACE (Tapley 1997). Until now it seems impossible to verify the hypothesis of inner-core precession since the accuracies of the gravity models are too low. The highest accuracies achieved so far are those for the zonal coefficients when analysing satellite laser ranging data applying sophisticated methods (Cheng et al. 1997). However, even for the zonal coefficients the actual uncertainties have the same order as the inner-core effect and furthermore the major difficulty will be the separation of the different influences on the low-degree harmonic coefficients, for example, those due to atmospheric and oceanic fluctuations. Nevertheless, the expected improvements in gravity field determination by the new missions (Table 3) along with a combined analysis of all affected coefficients could give us the chance to verify the hypothesis during the next decade.

### Table 1. Numerical values for calculating the influence of the relative inner-core precession on the gravity field.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of the Earth</td>
<td>( M = 5.973698995 \times 10^{27} ) kg</td>
<td>IERS standards</td>
</tr>
<tr>
<td>Reference radius of spherical harmonics</td>
<td>( R_0 = 6378136.49 ) m</td>
<td>IERS standards</td>
</tr>
<tr>
<td>Major semi-axis of inner core</td>
<td>( a = 1229.5 ) km</td>
<td>Dziewonski &amp; Anderson (1981)</td>
</tr>
<tr>
<td>Flattening of inner core</td>
<td>( f = 1/415.78 )</td>
<td>Smylie et al. (1984)</td>
</tr>
<tr>
<td>Minor semi-axis of inner core</td>
<td>( b = a(1-f)^{1/2} = 1226.54 ) ( ) km</td>
<td></td>
</tr>
<tr>
<td>Density jump at inner core–outer core boundary</td>
<td>( \Delta \rho = 0.5973 ) ( ) g cm(^{-3} )</td>
<td>Dziewonski &amp; Anderson (1981)</td>
</tr>
</tbody>
</table>

**DISCUSSION**

In this section we discuss our results in Section 2.2 in order to formulate some ideas for a possible future interpretation by an internal torque balance, which may be found by analogy to recent investigations of relative rotation about the polar axis using electromagnetic (EM) torques.

Glatzmaier & Roberts (1996) showed that an angular velocity of an axial relative rotation of the inner core of \( 1^\circ \) \( \) yr\(^{-1} \) or larger is consistent with recent dynamo models and inner core–outer core EM coupling. The relative rotation is then maintained by EM coupling between the inner core and the flow of the overlying liquid in the outer core (e.g. the thermal wind). This relative rotation would cause a wobble of the figure axis of the inner core about the polar axis of the Earth with a frequency given by that of the axial rotation if the angle between both axes is not zero. In Section 2.2, a mean wobble frequency of \( 0.7^\circ \) \( \) yr\(^{-1} \) is determined according to the linear trend in \( \varphi \), which is of the order of the axial relative rotation mentioned above. Nevertheless, the decadal variations shown in Fig. 2 are superimposed on this mean wobble and cannot be explained by a temporally constant axial rotation.

Of particular interest to Aurnou & Olson (2000) were attenuated oscillations of \( \varphi \) following a long-term trend caused by the combined effects of axial gravitational and EM torques, if the magnitude of the EM torque dominates that of the gravitational torque only by a small amount. In their paper, they assumed that the relative rotation of the inner core is parallel to the polar axis of the Earth and is maintained by EM torques. For evaluating the restoring gravitational torque, they assumed that the inner-core shape is a given triaxial ellipsoid.

### Table 2. Predicted rates of change of the normalized degree-two spherical harmonic coefficients of the Earth’s gravity field caused by the relative precession of the inner core averaged over the last 10 yr.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \frac{d}{dt} C_{20} [\text{yr}^{-1}] )</th>
<th>( \frac{d}{dt} S_{20} [\text{yr}^{-1}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( +0.7 \times 10^{-12} )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( -3.4 \times 10^{-12} )</td>
<td>( +16.3 \times 10^{-12} )</td>
</tr>
<tr>
<td>2</td>
<td>( +0.4 \times 10^{-12} )</td>
<td>( +0.1 \times 10^{-12} )</td>
</tr>
</tbody>
</table>

### Table 3. Estimated standard deviations of the low-degree \((n<5)\) harmonic coefficients and their rate of change for present and future gravity field models; the estimates for CHAMP and GRACE are based on 1 yr time-series of data for \( \sigma(C/S) \) and 5 yr time-series for \( \sigma\left( \frac{d}{dt}(C/S) \right) \).

<table>
<thead>
<tr>
<th>Model</th>
<th>( \sigma(C/S) )</th>
<th>( \sigma\left( \frac{d}{dt}(C/S) \right) [\text{yr}^{-1}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRIM4</td>
<td>( 2 \times 10^{-10} )</td>
<td>( 4 \times 10^{-12} )</td>
</tr>
<tr>
<td>CHAMP</td>
<td>( 2 \times 10^{-11} )</td>
<td>( 1 \times 10^{-12} )</td>
</tr>
<tr>
<td>GRACE</td>
<td>( 2 \times 10^{-12} )</td>
<td>( 1 \times 10^{-13} )</td>
</tr>
</tbody>
</table>
Nevertheless, the maintenance of the tilt, $\beta_f$, of the figure axis throughout several decades and its variations shown in Fig. 2 cannot be explained without assumptions about the non-axial torque components. The temporal structure of $\beta_f$ seems to have elements similar to those given by Aurnou & Olson (2000) for $\Omega_q$. By analogy with this axially symmetric case, we speculate that a combination of non-axial EM and gravitational torques may cause this behaviour of $\beta_f$. Besides the effect of non-axial motions in the outer core, these EM torques can be maintained by the interaction between the axial relative rotation and the non-axial part of the geomagnetic field in the core. To get an idea of their magnitude, we use the non-axial gravitational torque given by Smylie et al. (1984), the magnitude of which is $1.28 \times 10^{24}$ N m sin $2\beta_q$, and find that the EM torque necessary to produce variations of $\beta_f$ with amplitudes of about 0.5° should be about $10^{22}$ N m. However, the excitation of non-axial EM torques of this magnitude is an unsolved problem for future investigation.

In eq. (2), the short-period variations are omitted because of the long timescale considered. In recent literature, besides the Chandler period, nearly diurnal periods of a three-component model consisting of a viscoelastic mantle, a fluid outer core and a viscoelastic inner core have been obtained as eigenperiods (e.g. Dehant et al. 1993), which are neglected in our investigation for the decadal interval. In these models, the outer core is coupled to the inner core and the mantle by inertial torques (pressure torques including the self-gravitational potential). In general, the amplitudes of the eigenperiods are not quantified by the solutions of the rotation equations and must be determined by observations or by additional assumptions concerning, for example, the tilt of the axis of the relative rotation and its magnitude. Therefore, it may also be possible to examine our model assumptions for short-period variations provided that the contribution of the relative wobble of the inner core to short-period polar motion and the gravity variations can be detected by observations.

5 CONCLUSIONS

The inner core, with its small inertia, is a sensitive detector of processes in the outer core and associated changes in the angular momentum. It responds very quickly to changes in the internal torque balance. In this paper, we have described such a hypothetical response of the inner core without knowing the processes causing it, derived the decadal variations of the orientation of the figure axis of the inner core from polar motion and atmospheric data, and finally studied the associated gravity variations.

The results of our kinematic model show that the influence of the relative motion of the inner core inferred from polar motion may cause observable decadal changes of the gravity field (for example, in the Stokes coefficients $C_{21}$ and $S_{21}$). High-accuracy gravity observations in the near future should be able to prove whether or not the model used is realistic.

Finally, because the assumption of an axial relative rotation of the inner core often made in literature dealing with inner-core motions is an exceptional case, the possibility of a non-axial rigid relative rotation of the inner core in outer-core dynamics should be examined in the future, in particular with respect to short-period oscillations in polar motion.
ACKNOWLEDGMENTS

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REFERENCES


APPENDIX A: THE RELATIVE ANGULAR MOMENTUM OF THE INNER CORE

From eq. (6) in Section 2.1 we obtain for the complex equatorial component $h$ the integral expression

$$h = h_x + jh_y = -\int_{V} \rho r (u_\theta \cos \vartheta - j u_\phi) e^{j\varphi} dV,$$

(A1)
where $V_c$ is the core volume, $u_q$ is the velocity of the relative west–east flow and $u_c$ is that of the north–south flow in the core. These flow velocities can be determined from the geomagnetic secular variation as core surface flow by inverting the frozen-field equation of the geomagnetic field. For deeper parts of the core, the velocity field must be described by a flow model in which $u(r, \theta, \varphi, t)$ is uniquely related to its surface values, $u(r_c, \theta, \varphi, t)$. An example is given in the literature with respect to the axial angular momentum, $h_z$, for which only zonal axial fields are significant: using Taylor’s theorem, Jault et al. (1998a) found that the zonal velocity field determined in the core can be described by the model of rigidly rotating axial cylinder annuli throughout the whole core, the relative motions of which are uniquely related to zonal velocity fields at the core surface. By the angular momentum balance, changes of the mantle rotation can then be derived from the geomagnetic secular variation; these agree fairly well with ALOD (Jackson et al. 1993; Holme 1998a). In these models, the inner core corotates with the innermost cylinder or is ignored by assuming a constant density throughout the core. Within the scope of this hypothesis, disturbances of the axial symmetry of its relative rotation must therefore be negligible and their effect on the fluid flow in the outer core must be absorbed in a small shell at the inner core–outer core boundary.

According to eq. (A1), axially symmetric flows have no effect on $h$ if $\rho$ does not depend on $\varphi$. Non-axial flows can be determined by inversion of the frozen-field equation at or near the core surface (e.g. Gire et al. 1986). Parts of them (mainly with quadrupole structure) can influence polar motion by topographic coupling (Hinderer et al. 1987), but their continuation into deeper parts of the core is unknown.

In the angular momentum approach, the main problem is to find a process maintaining a non-axial distribution of the velocity field in the deeper layers of the core. Here we only estimate the contribution of the relative inner-core wobble to the angular momentum of the core and test its effect on the mantle rotation by ignoring changes of the non-axial outer core flow.

To obtain $h$ in the excitation function due to the inner-core wobble, we estimate the contribution of an assumed relative inner-core rotation of about $1^\circ$ yr$^{-1}$, which is accepted in modern dynamo theories (Section 4). The relative angular momentum of the inner core is given by $\mathbf{h}(t) = \mathbf{I}(t) \Delta \mathbf{w}(t)$, where $\mathbf{I}$ is the temporal variable tensor of inertia of the inner core given according to Greiner-Mai et al. (2000) by

$$ I(t) = I_\theta + (C_1 - A_1) $$

where $I_\theta$ is the constant diagonal inertia tensor of the inner core. Furthermore, we assume that the vector of relative rotation, $\Delta \mathbf{w}(t)$, is misaligned with respect to the polar axis of the Earth by the angle $\pi(t)$ and rotated eastwards to the $x$–$z$ plane by the angle $\beta(t)$. We can derive its components from the vector $\Delta \mathbf{w}_a = (0, 0, \Delta w_a)$ aligned with the $z$-axis by multiplying this vector with the transformation matrix, $\mathbf{R}(t)$, given in Greiner-Mai et al. (2000) (eq. 10), in which $\beta$ and $\varphi$ must be replaced by $x$ and $\beta$, respectively. We obtain

$$ \Delta \mathbf{w}(t) = \Delta \mathbf{w}_a \sin \beta \cos \beta, \sin \beta \sin \beta, \cos \beta \cos \beta. $$

Using the expressions (A2) and (A3), we obtain the complex equatorial component of the relative angular momentum,

$$ h = \Delta \mathbf{w}_a \left[ A_1 \sin \beta \cos \beta + (C_1 - A_1) \right] $$

$$ \times \left[ \sin^2 \varphi \sin \varphi \sin \varphi \sin \varphi \cos \varphi \cos \varphi \cos \varphi \cos \varphi + \frac{1}{2} \sin 2 \varphi \cos \beta \right]. $$

With $(C_1 - A_1) = 0.015 \times 10^{41}$ g cm$^2$, $(C - A) = 0.0261 \times 10^{44}$ g cm$^2$ and $\Delta w_a = 1^\circ$ yr$^{-1}$, we obtain for the motion part of the excitation function in eq. (4)

$$ \psi_{\text{Lmot}} = 7.6 \times 10^{-6} \left[ 0.24 \sin \beta \cos \beta + 5.75 \times 10^{-4} \right] $$

$$ \times \left[ \sin \beta \sin \beta \sin \varphi \sin \varphi \cos \varphi \cos \varphi \cos \varphi \cos \varphi + \frac{1}{2} \sin 2 \varphi \cos \beta \right]. $$

In general, we can assume that $\beta$ is not larger than $\pi/2$. Consequently, $\psi_{\text{Lmot}}$ can be approximated by

$$ \psi_{\text{Lmot}} = 1.8 \times 10^{-6} \sin \beta \cos \beta, $$

which shows that $\psi_{\text{Lmot}}$ is one order of magnitude smaller than $\psi_t$ derived from the internal mass effect (eq. 7).

If the axis of relative rotation is closer to the $z$-axis than the figure axis, the contribution of the relative angular momentum is even smaller. For negligible values of $\beta$, the term $(1/2) \sin \beta \cos \beta$ dominates in the expression for the relative angular momentum, but its contribution to $\psi_t$ is then smaller than the mass term by five orders of magnitude. Therefore, we can neglect the relative angular momentum in eq. (5), provided that non-axial fluid motions in the outer core have an insignificant effect on $h$. 

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