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Determining the magnetic field in the core–mantle boundary zone by non-harmonic downward continuation

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SUMMARY
The length of day and the geomagnetic field are clearly correlated over decadal periods (10–100 yr). Provided the electrical conductivity of the lower mantle is sufficiently high, a considerable part of this correlation can be explained by electromagnetic core–mantle coupling. Investigating the associated core–mantle coupling torque and fluid velocity fields near the core surface, as well as the interpretation of the observed time lag between length of day and geomagnetic field variations requires the calculation of the temporally variable magnetic field near both sides of and on the core–mantle boundary by solving the magnetic induction equation. Such a solution presents a downward field continuation that has a non-harmonic character if the electrical conductivity is accounted for.

In this paper the Earth’s mantle is assumed to be a two-layer spherical shell, whose inner layer is electrically conductive. We only consider the poloidal part of the magnetic field with boundary values that are conventionally given by a spherical harmonic expansion of the observed geomagnetic potential field at the Earth’s surface. Thus, we are concerned with a one-side-data supported problem, analogous to the inverse heat conduction problem (or sideways heat equation problem), well-known as an ill-posed inverse boundary value problem for a parabolic partial differential equation (diffusion equation).

We develop a several-step solution procedure for this inverse problem in its integral form and use a special regularization method for the final solution. The capabilities of this downward continuation method (which includes varying the mantle conductivity model, quality of data approximation in the regularization and two different depths) are presented and discussed in comparison with the perturbation approach and the usual harmonic downward continuation. The data series used are the single magnetic field components (Gauss coefficients) of the spherical harmonic field expansion (Bloxham & Jackson 1992, J. geophys. Res., 97, 19 537–19 563) beginning from the year 1840 to 1990. In addition, to investigate the spectral effects (changing amplitudes, phase shifting), an artificial data series is used.

The main result is the downward continuation of the radial component of a global (8,8) field onto the core–mantle boundary, produced using the different methods and for the two epochs, 1910 and 1960. Comparing the results with the perturbation solution reveals temporally and locally variable differences up to the order of 5000 nT, while the difference to the harmonic downward continuation amounts to 15000 nT.

Key words: core–mantle boundary, diffusion equation, downward continuation, electrical conductivity, geomagnetic field, inverse boundary value problem.

1 INTRODUCTION
For various geophysical problems such as core–mantle coupling, dynamo modelling and electromagnetic (EM) mantle induction, knowledge of the geomagnetic field within the mantle and on the core–mantle boundary (CMB) is required. A particular problem, for which the magnetic field behaviour is studied in this paper, is the excitation of the decadal variations in the length of day (ΔLOD) by EM core–mantle coupling. The associated EM coupling torques are conventionally derived from the geomagnetic field in an electrically conducting mantle (e.g. Rochester 1960). Provided the mantle conductivity, σₘ, is known, the
magnetic field in the mantle can be determined solving the induction equation.

The poloidal part of the magnetic field is inferred from its values at the Earth’s surface by inverse solutions of this equation. On the other hand, the toroidal part of the geomagnetic field, which determines an important part of the EM torques, cannot be observed at the Earth’s surface. However, boundary values of the toroidal field due to field advection can be inferred from the velocity field of the fluid outer core and the poloidal geomagnetic field at the CMB (e.g. Stix & Roberts 1984). Therefore, the poloidal geomagnetic field at the CMB must be known there as accurately as possible.

In addition to the geophysical implications of the problems addressed here, the mathematical issues are of interest. The determination of the poloidal magnetic field in the mantle requires the solution of a parabolic partial differential equation that is derived from the vectorial induction equation. The necessary boundary conditions are connected with magnetic observations on the Earth’s surface. If fields with non-restricted time-behaviour are to be studied, then a full initial-boundary value problem has to be solved. However, determining the poloidal magnetic field is not a standard initial-boundary value problem as there is only data from one boundary (Earth’s surface), and not at any interior surface such as the CMB. Therefore, this task is classified as an inverse boundary value problem for a parabolic partial differential equation. In contrast to this problem, the term ‘forward problem’ is used when a standard boundary value problem is given, i.e. data at both boundaries is available.

To calculate the non-potential field at the CMB, perturbation methods have been mainly applied up to now. They replace the unknown time derivative by a given time function. The unperturbed field is then the time-variable geomagnetic potential field continued to the CMB, while the perturbed field must be derived from the mantle induction equation according to the given time variations of the unperturbed field. The perturbation method is described by Braginsky & Fishman (1977), Benton & Whaler (1983) and Stix & Roberts (1984) and was applied to electromagnetic core–mantle coupling by Stix & Roberts (1984), Greiner-Mai (1993) and Holme (1998). In this method, the magnetic field is developed into a series of fields, the convergence of which depends upon some scaling arguments. The perturbation solution is a useful approximation of the geomagnetic field if it varies smoothly (e.g. in the decadal time scale) in a weakly conducting mantle (i.e. \( \sigma_M \) is so low that the core can be approximated by a perfect conductor). It has not yet been tested for rapid field changes, subdecadal periods and spatially small-scale magnetic fields. Here, we develop another solution algorithm where the geomagnetic field is not approached by a series and the solution does not depend a priori on scaling arguments.

Because the inverse boundary value problem is severely ill-posed, theoretical and numerical instabilities should be taken into account. Therefore, considering the noise of the data and its spectral structure, the selection of a regularization strategy that forms the basis for a stable solution procedure is necessary. At this point the close analogy this problem has with the inverse heat conduction problem (often referred to as the sideways heat equation or the non-characteristic Cauchy problem) is helpful. Benton & Whaler (1983) described the analogy between magnetic diffusion and heat conduction in great detail. For the geomagnetic downward continuation, Bloxham (1989) used a stochastic inversion approach. Some basic approaches to these inverse (heat conduction) problems and numerous references are presented in Dinh Nho Hao & Gorenflo (1991) and Reinhardt & Sciffarth (1993).

In this paper the non-harmonic downward continuation of the poloidal magnetic field to the CMB is solved numerically using a procedure outlined in Ballani et al. (1995). This solution is based on an inversion approach used in geothermics (Stromeyer 1983, 1984) and includes some of the theoretical elements presented in Eldén (1983). After introducing the basic physical and mathematical problems (Subsections 2.1 and 2.2), we formulate the inverse boundary value problem for the downward continuation. Some basic properties of the inversion are discussed (Subsection 3.1) to be followed by a mathematically supported description of the single steps of the solution method (Subsection 3.2). Subsection 3.3 contains a short description of the perturbation approach for the downward continuation developed by Benton & Whaler (1983). The non-harmonic downward continuation method is then numerically verified and together with the perturbation theory approach is tested using two models of mantle conductivity connected with a supposed layering near the core–mantle boundary. The results are single time functions (spherical harmonic modes) and the radial component of the global \((8,8)\) poloidal field (shown for two epochs) calculated for the core–mantle boundary and a passive outer core layer 50 km beneath the CMB (Subsections 4.1 and 4.2). The study concludes with a discussion of the properties of the procedures used and the results obtained (Subsections 5.1 and 5.2), including a comparison between the different possibilities for the downward continuation of single time functions and of the global field (Subsection 5.3). Some proposals for future work are included (Subsection 5.4).

2 BASIC ASSUMPTIONS AND EQUATIONS

2.1 Basic assumptions

The Earth’s mantle is represented by a rigid shell (Fig. 1) with an outer radius \( R_E \) (mean Earth radius) and an inner radius \( R_c \) (core radius).

The mantle conductivity, \( \sigma_M(r) \), is assumed to be a function of the radial distance \( r \) with \( \sigma_M \neq 0 \) for \( R_c < r < R_e \) and \( \sigma_M = 0 \) for \( R_e \leq r \leq R_E \). The two analytical formulae that describe \( \sigma_M(r) \) dependence on \( r \) considered for \( R_c < r < R_e \) are given by the semiconductor formula and its first Taylor term,

\[
\sigma_M(r) = \sigma_0 \exp \left( -\alpha \left( \frac{r - R_c}{R_e} \right) \right) \tag{1}
\]

![Figure 1. Geometrical assumptions in a spherically symmetric conducting Earth model. \( \sigma(r) \) is the electrical conductivity, \( R_E = 6370 \) km, \( R_o = 5480 \) km, \( R_c = 3480 \) km.](image)
\[ \sigma_M(r) = \sigma_0 \left( \frac{R_c}{r} \right)^\alpha \]  

(2)

where \( \sigma_0 \) is the mantle conductivity at the CMB \((r = R_c) \) and \( \alpha \) is the scaling factor that determines the decrease of the conductivity with increasing distance from the CMB, with the ratio \( \sigma_0/\alpha \) determining the magnitude of the electromagnetic torques. Estimates of these parameters can be obtained from geomagnetic induction studies. Rotanova et al. (1985) used the 30 and 60 yr periods of the secular variation to determine \( \alpha \) and \( \sigma_0 \) in (1) and obtained values of \( \alpha \) of approx. 6–8 and of \( \sigma_0 \) between 1500 and 4000 Sm\(^{-1}\).

Stix & Roberts (1984) and Greiner-Mai (1987, 1993) found that the magnitudes of the electromagnetic torques are consistent with LOD variations if \( \sigma_0 = 3000 \text{ Sm}^{-1} \) and \( \alpha = 30 \), i.e. \( \sigma_0/\alpha = 100 \). For this model, the resulting conductance, \( \int \sigma_M(r) dr \), is \( 3.6 \times 10^8 \) S, a value that produces electromagnetic coupling consistent with LOD estimate (Holme 1998). However, the value of \( \sigma_M \) in the lower mantle is still under discussion (see Subsection 4.1) and the necessary conductance can also be obtained by other spatial distributions of conducting material (e.g. a thin layer at the bottom of the mantle).

The solutions of the mantle induction equation are derived for the poloidal magnetic field, \( \mathbf{B}_p \), the part of the magnetic field that can be observed outside the conductor. As mentioned in the introduction, the toroidal field, which cannot be observed, is not considered in this paper. However, the equations for the poloidal and toroidal parts are decoupled for a radially distributed mantle conductivity, and therefore we can study the behaviour of the poloidal field independently of the assumptions made about the toroidal field. We represent the field by poloidal and toroidal scalars, the poloidal scalar field, \( S \), being represented by a spherical harmonic expansion whose associated coefficients, \( S^{\alpha,i}(r, t) \), are called the (poloidal) harmonic modes.

### 2.2 Basic equations

The magnetic induction equations can be derived from the Maxwell equations by substituting the current density and electric field strength by the magnetic flux density, \( \mathbf{B} \) and its time derivative, \( \mathbf{B}_t \). The vector induction equations are then given by

\[ \text{curl} \left[ (\mu_0 \sigma_M)^{-1} \text{curl} \mathbf{B} \right] = -\mathbf{B}_t, \quad R_c < r < R_o \]  

(3)

\[ \text{curl} \mathbf{B} = 0, \quad r > R_o \]  

(4)

\[ \text{div} \mathbf{B} = 0, \quad \forall r \]  

(5)

where \( \mu_0 \) is the permeability of free space. The boundary conditions at \( r = R_c, r = R_o \) and \( r = R_E \) are the continuity of the flux density, \( \mathbf{B}^+ = \mathbf{B}^- \), where the signs + and − denote the outer and inner sides of any given boundary. As a result of the spherical symmetry of \( \sigma_M \) and the solenoidal field (div \( \mathbf{B} = 0 \)), the flux density can be decomposed orthogonally into toroidal and poloidal parts by

\[ \mathbf{B} = \mathbf{B}_t + \mathbf{B}_p \]  

(6)

where

\[ \mathbf{B}_t = \text{curl} (r \mathbf{T}) \]  

(7)

\[ \mathbf{B}_p = \text{curl} (r \mathbf{S}) \]  

(8)

The scalar functions \( T \) and \( S \) are normalized on the sphere by

\[ \iint T \sin \vartheta d\vartheta d\varphi = 0 \quad \text{and} \quad \iint S \sin \vartheta d\vartheta d\varphi = 0 \]  

(9)

For the poloidal scalar, the induction equations thus have the forms of the diffusion equation and the Laplace equation such that (Krause & Rädler 1980):

\[ (\mu_0 \sigma_M)^{-1} \Delta S = \dot{S}, \quad R_c < r < R_o \]  

(10)

\[ \Delta S = 0, \quad r > R_o \]  

(11)

where \( \Delta \) is the Laplace Operator. The boundary (interface) conditions are given by the continuity of \( S \) and its radial gradient according to the continuity of the radial and tangential components of the flux density (see Eq. 16).

The scalar function \( S \) is described by spherical harmonics \( (P_{nm}) \)

[excluded here for brevity]

which corresponds to a separation according to the pairs of variables \((r, \vartheta, \varphi) \) and \((\theta, \varphi) \). Using the orthogonality of the spherical harmonics, we obtain from eqs (10) and (11) the fully decoupled 1-D induction equations for the harmonic modes \( S^{\alpha,i}(r, t) \):

\[ D_v S^{\alpha,i}_n = \mu_0 \sigma_M S^{\alpha,i}_n, \quad R_c < r < R_o \]  

(13)

\[ D_v S^{\alpha,i}_n = 0, \quad r > R_o \]  

(14)

with the operator \( D_v \) defined as

\[ D_v = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{n(n+1)}{r^2} \]  

(15)

The 1-D parabolic partial differential eq. (13) describe the outward diffusion of the harmonic modes \( S^{\alpha,i}_n(r, t) \) of \( S(r, t) \) through the electrically conducting spherical shell that is the lower part of the Earth’s mantle. The interface conditions at \( r = R_E \) are

\[ (S^{\alpha,i}_n) = (S^{\alpha,i}_n)^{-}, \quad \left( \frac{\partial}{\partial r} S^{\alpha,i}_n \right)^{+} = \left( \frac{\partial}{\partial r} S^{\alpha,i}_n \right)^{-} \]  

(16)

Eq. (14) is fulfilled with the usual potential solution for a field outside a conductor that is regular at infinity. From this solution, the associated modes \( C^{\alpha,i}_n(t) \) are defined by

\[ S^{\alpha,i}_n(r, t) = C^{\alpha,i}_n(t) r^{-\alpha-1}, \quad r > R_o \]  

(17)

The geomagnetic potential \( V \) and its secular variation on the Earth’s surface \( r = R_E \) are related to the scalar \( S \) by

\[ V = -\frac{\partial}{\partial r} (r S) \]  

(18)

However, \( V \) is conventionally described by its expansion into spherical harmonics, whose coefficients are known as Gauss coefficients \( g_{nm} \) and \( h_{nm} \) (Mauersberger et al. 1959). Therefore, we have on the Earth’s surface the following relations

\[ C^{\alpha,i}_n(t) = \frac{1}{r} g_{nm}(t) \lambda_{nm} R_E^{\alpha+2} \]  

(19)

\[ C^{\alpha,i}_n(t) = \frac{1}{r} h_{nm}(t) \lambda_{nm} R_E^{\alpha+2} \]  

(20)

where \( \lambda_{nm} \) are the Schmidt’s normalization coefficients defined by

\[ \lambda_{nm} = \left( 2 - \delta_{nm} \right) \left( \frac{n-m}{n+m} \right)^{1/2} \]  

(21)

with \( \delta_{ik} \) the Kronecker’s symbol (\( \delta_{ik} = 0 \) if \( i \neq k \) and \( \delta_{ik} = 1 \) if \( i = k \)). Eq. (17), together with (19), then gives the poloidal harmonic modes
\[ S^\alpha_{nm}(r, t) = \frac{1}{n} g_{nm}(t) \lambda_{nm} R_E^{n+2} r^{-n-1} \]

\[ S^\nu_{nm}(r, t) = \frac{1}{n} h_{nm}(t) \lambda_{nm} R_E^{n+2} r^{-n-1} \]  

outside the conducting part of the mantle dependent upon the Gauss coefficients, hereafter sometimes referred to as ‘data’. The conventional harmonic downward continuation of the radial field component \( B_r(r, t) \) of the poloidal field \( B_p \) is then obtained by substituting eq. (21) into eq. (47) below (Subsection 4.2).

It follows also from eq. (14) that the secular variation field can be obtained via the time derivatives, i.e. \( S^\alpha_{nm} = \frac{\partial}{\partial t} C_{nm}^\alpha r^{-n-1} \) with \( C_{nm}^\alpha \) calculated using the secular variation coefficients, \( g_{nm} \) and \( h_{nm} \), instead of the Gauss coefficients. Therefore, the boundary values of the harmonic modes and their time derivatives at \( r = R_o \) can be derived from the solution (17) dependent upon the Gauss coefficients and their time derivatives (e.g. Mauersberger et al. 1959; Barraclough 1978).

3 Solution of the Inverse Boundary Value Problem

3.1 Basic properties of inversion

The determination of the CMB function \( S^\alpha_{nm}(R_c, t) \) in eq. (13) can be formulated as an inverse boundary value problem if the conditions described by the eqs (16) are included. To simplify the notation, we substitute \( S^\alpha_{nm} \) for \( u \) in (13). Therefore the inverse boundary value problem for determining \( u(R_c, t) \) is given as

\[
\frac{\partial^2 u}{\partial t^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{n(n + 1)}{r^2} u = \mu_0 \sigma_M(r) \frac{\partial^2 u}{\partial t^2}, \quad R_c < r < R_o, \quad 0 \leq t \leq T
\]

(22)

with the boundary conditions

\[ u(R_o, t) = \phi(t), \quad \frac{\partial u}{\partial r}(R_c, t) + \frac{n + 1}{R_o} u(R_c, t) = 0 \]

and an initial condition

\[ u(r, 0) = \psi(r) \]

The first boundary condition in (22) is directly connected with the geomagnetic data on the Earth’s surface, as described by eq. (21). The second condition, which is a boundary condition of the third kind, concerns the transition between the non-conducting and conducting mantle shell and is derived by the second continuity condition in relation (16). Both boundary conditions in (22) are given for \( R_o \), i.e. only on one side of the radial interval \((R_c, R_o)\) in contrast to the usually studied stable two-side boundary-value problems in the field of parabolic differential equations.

Previous theoretical results, (e.g. Tsutsumi 1965; Knabner & Vessella 1987; Dinh Nho Hao 1995) mean that the existence and uniqueness of the solutions of (22) can be assumed. However, there exists an instability such that higher oscillatory parts of the signal are amplified exponentially (at least) in the solution with increasing frequency and depth (Engl et al. 1996). Moreover, the unknown function \( u(R_c, t) \) cannot be reconstructed reliably for times near the end of the epoch under consideration. A third effect is that the instability grows with the magnitude of the electrical conductivity coefficient (Eldén 1983).

Thus, because of the temporal filtering imposed by the conducting mantle on the signal originating from the core, it is obvious that any high frequency noise introduced into the observed data from the Earth’s surface will tend to produce strong instabilities in the recovered field at the CMB. Dealing with this issue either requires a delicate mathematical treatment (see e.g. Knabner & Vessella 1988; Engl & Manselli 1989), or to simply introduce some regularization in the inverse approach we use here.

To solve the inverse boundary value problem (22) it is equivalently written as an operator equation using the linear (integral) operator \( A \) such that

\[ Af = \phi \]

where \( f = u(R_c, t) \) is an unknown function and \( \phi = \phi(t) \) is an input (data) function. The solution method of regularization to be used here, which is a special case of Tikhonov regularization (cf. Engl et al. 1996), consists in constructing a solution \( f \) by the minimization

\[
\min \| f \|_\beta \text{ subject to } \| Af - \phi \|_\alpha \leq \epsilon
\]

via norms \( \| \cdot \|_\alpha, \| \cdot \|_\beta \) to be specified (Hansen 1998). That means a function \( f \) with optimal smoothness, corresponding to \( \| f \|_\beta \), on the core–mantle boundary, has to be found by approximating the data \( \phi \) within a fixed error range \( \epsilon \) at \( R_c \).

The norms used in the procedure (24) are defined by

\[
\| u(r, t) \|_2 = \left( \int_0^T |u(r, t)|^2 \, dt \right)^{1/2} \quad (L_2 \text{ norm})
\]

(25)

and

\[
\| u(r, t) \|_{W^1_2} = \| u(r, t) \|_2 + \left\| \frac{\partial u}{\partial t}(r, t) \right\|_2 \quad (W^1_2 \text{ norm})
\]

(26)

The initial condition \( u(r, 0) = \psi(r) \), which is included in the inverse problem (22), introduces some degree of arbitrariness in the solution. This effect (see also Subsections 4.2 and 5.2) is strongest at the start of the reconstruction time interval, but disappears after some time, such that the solution afterwards is only controlled by the boundary values and the basic model (geometry and conductivity).

3.2 Construction of the solving algorithm

Recently, several regularizing procedures have been developed and applied to solve unstable inverse heat conduction related problems (e.g. Dinh Nho Hao & Gorenflo 1991; Reinhardt & Seifarth 1993). In this work, we have developed a method that solves the inverse problem for the differential equation (22) in the equivalent integral form of a Volterra integral equation of the first kind. A finite dimensional approximation of the unknown boundary function \( f(t) = u(R_c, t), 0 \leq t \leq T \) with a certain smoothness is searched by means of the regularization algorithm (24). The quantity \( \| \phi(t) - u(R_c, t) \|_2 \) (where \( \| \cdot \|_2 \) means any norm) is a measure of the fit of the solution \( u \) to the boundary data \( \phi(t) \) on \( R_c \), which is fixed at a predefined error level. The selected inverse approximation technique is universal, i.e. it is independent of any analytical possibilities and allows arbitrary coefficient functions in the differential equation (22), such as the radially variable electrical conductivity \( \sigma_M(r) \).

3.2.1 Preparation step

The algorithm starts with a preparation step: the partitioning of the original problem (22) for \( u \) into two problems with the decomposition ansatz \( u(r, t) = v(r, t) + w(r, t) \). This step aims to eliminate

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(to ‘shift’) the initial condition $\psi(r)$, so that the more important and more difficult problem dealing with $w(r, t)$ can then work with an homogeneous initial condition.

Let $v(r, t)$ be the solution of the stable initial boundary value problem (e.g. Cannon 1984; Carslaw & Jaeger 1992)

$$\frac{\partial^2 v}{\partial r^2} + 2 \frac{\partial v}{\partial r} \frac{n(n+1)}{r^2} v = \mu_0 \sigma_{M}(r) \frac{\partial v}{\partial t}$$

$R_c < r < R_g, \ 0 \leq t \leq T$ \hspace{1cm} (27)

with the boundary conditions at $R_c$ and $R_g$

$$v(R_c, t) = 0, \ \frac{\partial v}{\partial r}(R_g, t) + \frac{n+1}{R_g} v(R_g, t) = 0$$

and an initial condition

$$v(r, 0) = \psi(r)$$

Obviously, the difference $w = u - v$, where $u$ is from eq. (22), satisfies

$$\frac{\partial^2 w}{\partial r^2} + 2 \frac{\partial w}{\partial r} \frac{n(n+1)}{r^2} w = \mu_0 \sigma_{M}(r) \frac{\partial w}{\partial t}$$

$R_c < r < R_g, \ 0 \leq t \leq T$ \hspace{1cm} (28)

with the boundary conditions

$$w(R_c, t) = \phi(t) - v(R_c, t), \ \frac{\partial w}{\partial r}(R_g, t) + \frac{n+1}{R_g} w(R_g, t) = 0$$

and a zero initial condition

$$w(r, 0) = 0.$$  

While the problem (27) is solved for $v$ by a standard algorithm (e.g. Ciarlet & Lions 1992) using any prescribed initial condition $\psi(r)$ (e.g. it can be taken as the harmonic downward continuation), the second problem presented in eq. (28) of determining $u(R_c, t)$ has the same type and structure as the original unstable one (22), but now with an homogeneous initial condition $w(r, 0) = 0$.

The solution of this second problem is described in the following. For simplicity we retain for (28) the old symbols of (22): i.e. $u$ instead of $w, \phi(t)$ for the boundary data and $f(t)$ $= u(R_c, t)$ for the unknown function, respectively.

Because of the homogeneous initial condition in (28), the existing linear relationship (cf. 23) between $f$ and $\phi$ can be derived from (28) via an integral (e.g. Laplace) transform technique applied to a two-point boundary value problem. It has the structure of a Volterra integral equation of the first kind (Smylie 1965; Eldén 1983; Poularikas 1996)

$$\phi(t) = Af(t) = \int_0^t k(t - \tau) f(\tau) d\tau$$ \hspace{1cm} (29)

However, the convolution kernel $k(t)$ can be explicitly determined only for very simple cases. E.g. $\sigma_{M}(r)$ a constant, gives an analytical description for $k(t)$ by an infinite series (Eldén 1983).

For the general case $\sigma_{M}(r) \neq$ constant, the situation is more difficult and requires a method without presupposing knowledge of the integral kernel $k(t)$ in order to invert (29). Therefore, to overcome this difficulty, our inversion algorithm proceeds in two steps:

1. the identification of an approximating matrix $(a_k)$ for the operator $A$ in (29)

2. the inversion of (29) by a regularization corresponding to (24).

3.2.2 Identification of approximating matrix

We represent the (unknown) time function $f(t), 0 \leq t \leq T$ by an abstract basis $(e_k(t), k = 1, \ldots, N, 0 \leq t \leq T)$ (being specified numerically in Subsection 4.2.)

$$f(t) = \sum_{k} f_k e_k(t)$$ \hspace{1cm} (30)

Using the first part of relation (29) this leads by the linearity of $A$ to

$$\phi(t) = Af(t) = A \left( \sum_{k} f_k e_k(t) \right) = \sum_{k} f_k A(e_k(t))$$ \hspace{1cm} (31)

It is now important to determine the functions $A(e_k(t))$ that will produce the $k$th column of the (approximating) matrix $(a_k)$. Defining $\phi_k(t)$ by

$$\phi_k(t) = A(e_k(t))$$ \hspace{1cm} (32)

this relation means a reversal of (23) or (28) with respect to the location of the first boundary condition.

For a given $e_k(t)$ at $R_c$ the stable upward continuation problem has to be solved for $u_k(r, t)$

$$\frac{\partial^2 u_k}{\partial r^2} + 2 \frac{\partial u_k}{\partial r} \frac{n(n+1)}{r^2} u_k = \mu_0 \sigma_{M}(r) \frac{\partial u_k}{\partial t}$$

$R_c < r < R_g, \ 0 \leq t \leq T$ \hspace{1cm} (33)

with the boundary conditions prescribed at $R_c$ and at $R_g$

$$u_k(R_c, t) = e_k(t), \ \frac{\partial u_k}{\partial r}(R_g, t) + \frac{n+1}{R_g} u_k(R_g, t) = 0$$

and the initial condition

$$u_k(r, 0) = 0.$$  

The important result is that $\phi_k(t)$ is the upper boundary value $\phi_k(t) = u_k(R_g, t)$ that is the searched function in (32) leading to the $k$th column in the matrix for $A$. (For the numerical solution of such stable initial-boundary value problems see e.g. Ciarlet & Lions 1992.)

With a time discretization $t = t_i, 0 \leq t_i \leq T, i = 1, \ldots, N$, in accordance with the offered data, and an adapted limitation of the number of the approximating bases functions $(e_k(t), k = 1, \ldots, N)$, eq. (31) reads as

$$\phi(t_i) = \sum_{k=1}^{N} f_k \phi_k(t_i) = \sum_{k=1}^{N} f_k u_k(R_g, t_i), \ i = 1, \ldots, N$$ \hspace{1cm} (34)

This relation shows the approximative structure of the Volterra integral equation (29) which can also be written as

$$\phi(t_i) = \sum_{k=1}^{N} f_k a_k$$ \hspace{1cm} (35)

if the calculated matrix $(a_k)$ is marked by

$$(a_k) = u_k(R_g, t_i), \ i = 1, \ldots, N, \ k = 1, \ldots, N$$ \hspace{1cm} (36)

Therefore, the coefficients $f_k, k = 1, \ldots, N$ remain the real unknowns of our problem.
Due to the convolution kernel in the integral equation (29), the matrix \((a_{ij})\) has a special structure: each diagonal (top left to bottom right) contains identical elements (Toeplitz matrix) and for \(k > i\) holds \(a_{ki} = 0\). This means, it is only necessary to calculate the first matrix column \((k = 1)\). The others (i.e. \(k = 2, \ldots, N\)) are then generated by shifting their elements downward iteratively. Therefore, implementing this step (3.2.2) only requires the numerical solution of one stable problem that is eq. (33).

### 3.2.3 Inversion step

Having now the matrix \((a_{ij})\) determined by step (3.2.2), the approximative solution of the inverse boundary value problem (22) can be determined by regularization. For our purposes, it is the proper inversion procedure with some degrees of freedom left that may be used to account for the geomagnetic and numerical aspects of the problem: choice of norm \(\| \cdot \|_\alpha, \| \cdot \|_\beta\) (smoothness), data and approximation error \(\epsilon\). It solves the matrix equation (35) in a stable way, by using an optimization algorithm for the discrete equivalent of eq. (24):

\[
\min \| (f_i) \|_\beta \quad \text{subject to} \quad \| (a_{ij}) (f_i) - (\phi(t_j)) \|_\alpha \leq \epsilon
\]  

where the data are \((\phi(t_j))\) on \(R_e\) and \((f_i)\) is the vector of unknowns, representing the time function \(u(R_e, t)\) (CMB) of interest.

Eq. (37) is a quadratically constrained least-squares problem. We solve it using the algorithm DISCREP of the well-established public domain Regularization Toolbox by Hansen (1998). It reduces the problem (37) to a sequence of classical Tikhonov regularization steps that are nonlinearly connected with the approximation error \(\epsilon\).

A final step that reverses the partitioning of the original problem into (27) and (28) is not necessary since in (27), \(\nu(R_e, t) = 0\) has been chosen and therefore the functions \(u(R_e, t)\) and \(w(R_e, t)\) agree on \(R_e\), i.e. with \(w(R_e, t)\) the real solution \(u(R_e, t)\) has been already reached.

### 3.3 Downward continuation by the perturbation method

The method of perturbation solution for the magnetic field calculation in the Earth’s mantle has been applied with certain modifications by different authors (Braginsky & Fishman 1977; Benton & Whaler 1983; Sist & Roberts 1984). Without loss of generality, the approach of Benton & Whaler (1983) will be presented since it is nearest to this paper with regards to the topic and the formalism.

The solution of the problem (22), given here only for \(S_{nm}^t(r, t)\), is searched by a series of the separated form

\[
n S_{nm}^t(r, t) = p_{nm}^{(0)}(r)g_{nm}(t) + p_{nm}^{(1)}(r)\tilde{g}_{nm}(t) + p_{nm}^{(2)}(r)\tilde{g}_{nm}(t) + \cdots
\]

with the Gauss coefficients \(g_{nm}(t)\) and its time derivatives \(\tilde{g}_{nm}(t)\), \(\tilde{g}_{nm}(t)\), ... as input data and the radially dependent functions \(p_{nm}^{(k)}(r)\), \(k = 0, 1, \ldots\) to be determined iteratively. The first term \(p_{nm}^{(0)}(r)g_{nm}(t)\) represents the unperturbed (inverse) solution, which is the harmonic downward continuation, i.e. the function \(p_{nm}^{(0)}(r)\) satisfies the initial equation

\[
(p_{nm}^{(0)})'' + \frac{2}{r}(p_{nm}^{(0)})' - \frac{n(n + 1)}{r^2}p_{nm}^{(0)} = 0
\]

(’ means the differentiation by \((d/dr)\) with two boundary conditions in accordance with (22)

\[
S_{nm}^t(r, t) = \lambda_{nm}\frac{R_E^{n+2}}{R_n^{n+1}}
\]

\[
(p_{nm}^{(0)})'(r_e) = -\lambda_{nm}(n + 1)\frac{R_E^n}{R_n^{n+1}}
\]

For \(k \geq 1\) the functions \(p_{nm}^{(k)}(r)\) of the perturbation terms are calculated by the iteration equations

\[
(p_{nm}^{(k)})'' + \frac{2}{r}(p_{nm}^{(k)})' - \frac{n(n + 1)}{r^2}p_{nm}^{(k)} = \mu_0\sigma_{\alpha}(r)p_{nm}^{(k-1)}; \quad k \geq 1
\]

with boundary values (function and radial derivative) taken as zero.

The field with \(k = 0\) corresponds to the field continuation through an insulating mantle (potential solution) whereas the perturbation terms \(k \geq 1\) describe the influence of the mantle conductivity.

Truncating the series (39) at \(k = 1\), the solution is formulated in the typical structure

\[
S_{nm}^t(r, t) = \lambda_{nm}\frac{R_E}{n} \left( R_n r \right)^{n+1} g_{nm}(t) \left[ 1 + \tau_0(r) \frac{\tilde{g}_{nm}(t)}{g_{nm}(t)} \right]
\]

with

\[
\tau_0(r) = \frac{\mu_0 R_E}{2(n + 1)} \int_r^{R_E} \left[ 1 - \left( \frac{r}{\xi} \right)^{2n+1} \right] \left( \frac{\xi}{R_E} \right) \sigma_{\alpha}(\xi) d\xi
\]

\(S_{nm}^t(r, t)\) is analogously formulated with \(h_{nm}, \tilde{h}_{nm}\). It shows that the first-order perturbation scheme combines the action of two time-scales, \(\tau_0 = \tau_0(R_e)\) and \(\tau_1 = \frac{\sigma_{\alpha}}{\sigma_{\alpha}}\) (\(\mu_0\) means any Gauss coefficient), that are related to the conductivity model and the temporal variation of the data connected with their accuracy and their spacing, respectively. Truncating the series at \(k = 2\), the temporal scale \(g(t)/\tilde{g}(t)\) is also included in an enlarged solution formula (42) etc. (see Benton & Whaler 1983).

### 4 RESULTS

#### 4.1 Used magnetic data and electrical conductivity models

The input data that describes the geomagnetic field at the Earth’s surface are the Gauss coefficients \(g_{nm}(t), h_{nm}(t)\) (see eq. 19). With the geomagnetic potential \(V\) given on the Earth’s surface by eq. (18), the link to our parabolic differential equation is given via eq. (17). By this relation, the first boundary condition \(\phi(t)\) in (22) can simply be calculated by means of the Gauss coefficients presented in eq. (21) while the second boundary condition as an internal one requires only the order \(n\) as input.

A comparison between Gauss coefficients for a time interval beginning in 1550 as determined by different authors is given by Mauersberger (1952). However, for the downward continuation problem we only use time series beginning in the middle of the 19th century for reasons of contradictions of earlier data, accuracy and data density. Because of non-regular and wide spacing, it is impossible to obtain significant spectral estimates (see e.g. Barracough 1978).

Another aspect of the data to be addressed here concerns the uncertainties involved which are difficult to estimate. Comparing different references it is reasonable to assume that there is a decrease in the uncertainty of probably an order of magnitude, from 100 nT to 10 nT for the epoch between 1900 and 1990. This order of magnitude decrease by a factor of 10 has also been found in the Gauss
coefficients \( g_{00} \) (Bloxham & Jackson 1992). The incompletely known error of the data on the Earth’s surface is an important reason to use regularizing methods in downward continuation, i.e. fitting the data between certain bounds only.

The data interval of 150 years (1840–1990) that is considered in this paper is compatible with the spectral range of the decadal variations. For this we use the Bloxham & Jackson (1992) data that can be generated via given knots using cubic B-splines in a self-selected time discretization of two years. In the same way, time derivatives are also available that are needed for the perturbation method. As the data are relatively smooth, only the lowest order Gauss coefficients are suitable for our comparison studies. Therefore, the methods discussed in this work were further tested with an artificial data series \( g_{55yn} \) with a range corresponding to real data (Gauss coefficient for \( n = 5 \), containing as spectral parts periods of 100, 60, 35, 15 and 6 yr.

The electrical conductivity of the mantle, one of the input quantities in our algorithm, is not well resolved, especially for the lower mantle (Achache et al. 1981; Constable 1993; McLeod 1994; Honkura & Matsushima 1998). In Honkura & Matsushima (1998) a comparative summary is given that lists two groups of derived deep mantle conductivities: 1–3 Sm\(^{-1}\) and >10 Sm\(^{-1}\). The traditional analytic forms (power law or exponential law formulae, cf. (1) and (2), Subsection 2.1) are largely consistent with the deep Earth interior material investigations (semiconductor property of the material, high-pressure experiments). During the last decade, such experimental studies resulted in curves monotonically increasing with depth in the mantle and ending with values of only some Sm\(^{-1}\) at the core–mantle boundary (e.g. Poirier & Le Mouël 1992; Shankland et al. 1993).

But with these values, the electromagnetic coupling torques are not sufficiently high to explain the observed \( \Delta \) LOD variations (e.g. Holme 1998). However, the D\(^{0}\) layer at the bottom of the mantle supposedly consisting of core-infiltrated material with a thickness of between 100 km to 200 km and a conductivity of up to 4000 Sm\(^{-1}\) could generate the lacking torque.

The \( \sigma_M \) models used for these calculations are given in Table 1. Besides the case of zero-conductivity (model 3, harmonic downward continuation) we examine results from two other models: For conductivity model 1, the parameters (conductivity and thickness) for the D\(^{0}\) layer were selected in such a way that the corresponding conductance (being here simply the product of conductivity and thickness) has the value \( 6 \times 10^4 \) S corresponding to the required magnitude of the related coupling torques. Conductivity model 2 has a low conducting mantle (8 Sm\(^{-1}\) over 2000 km) which is compatible with the observation of jerks (Alexandrescu et al. 1999) resulting from jerk analyses as optimum. This conductivity model can also be considered as a compromise between the two groups of models (see above) assessed by Honkura & Matsushima (1998).

An experiment was also carried out to examine the effectiveness of the downward continuation algorithms when a passive upper core layer (‘PUL’) in the fluid outer core is included. This layer is assumed to be locked to the mantle with a conductivity of the order of \( 10^5 \) Sm\(^{-1}\) (Table 1). Such a layer is considered also in different (moving and non-moving) variants (e.g. Braginsky 1999; Lister & Buffett 1998).

### 4.2 Downward continuation of magnetic field components

In this section we present results from our non-harmonic downward continuation algorithm, what modifications and criteria are possible and what mathematical or numerical features are obtained. The consistency of the method is numerically verified by combining the downward and upward continuation. The perturbation approach for downward continuation as well as the purely harmonic field continuation are shown as alternative methods with examples given in parallel. Of particular interest are the influence of the electrical conductivity, the geometry (depth) as well as some approximation and spectral properties of the methods.

The 1-D ‘model’ component and boundary functions \( \phi(t) \) in (22) is the data function \( g_{55}(t) \) (Gauss coefficient) from the spherical harmonic expansion of the geomagnetic potential \( V \) at the Earth’s surface (18). It has been chosen because of its sufficiently modulated temporal behaviour. In addition, we use the artificial data series \( g_{55yn}(t) \) containing higher spectral portions to show special effects. For the global field calculations, all Gauss coefficients \( g_{55}(t), h_{yn}(t) \) with degree and order up to 8 are used.

To present our downward continuation results, we define the time functions \( g_{55}^{\text{CMB}}(t) \) and \( g_{55}^{\text{PUL}}(t) \) (see also eq. 21) as

\[
g_{55}^{\text{CMB}}(t) = \frac{1}{R_E} \frac{n}{\lambda_{yn}} S_{yn}^{\text{CMB}}(R_c, t)
\]

and

\[
g_{55}^{\text{PUL}}(t) = \frac{1}{R_E} \frac{n}{\lambda_{yn}} S_{yn}^{\text{PUL}}(R_c - 50 \text{ km}, t)
\]

respectively, and \( h_{yn}^{\text{CMB}}(t), h_{yn}^{\text{PUL}}(t) \) analogously by \( S_{yn}^{\text{CMB}} \).

The first main step in the solution procedure (Subsection 3.2.2) is the numerical integration of the stable two-side boundary value problems, described by eq. (33) to determine the matrix \( (a_{ij}) \) in (35), as defined by (36). We adopt here the Kronecker’s symbol (definition see Subsection 2.2) as the simplest choice for the base functions \( e_i(t) \) for the unknown boundary function (30):

\[
e_i(t) = \delta_{ni}
\]

### Table 1. Conductivity models \( \sigma_M(r) \) used for downward and upward field continuation. Conductivity model 3 is related to the harmonic field continuation. The radii are \( R_a = 5480 \) km and \( R_c = 3480 \) km (see Fig. 1).

<table>
<thead>
<tr>
<th>Upper mantle</th>
<th>Main lower mantle</th>
<th>Lower mantle’s D(^{0}) layer</th>
<th>Passive upper core layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r &gt; R_a )</td>
<td>( R_a \geq r &gt; R_c + 200 \text{ km} )</td>
<td>( R_c + 200 \text{ km} \geq r \geq R_c )</td>
<td>( R_c &gt; r \geq R_c - 50 \text{ km} )</td>
</tr>
<tr>
<td>( \sigma_M = 0 )</td>
<td>( \sigma_M(r) = 10 \text{ Sm}^{-1} \left( \frac{r}{R_c} \right)^3 )</td>
<td>( \sigma_M(r) = 3 \times 10^3 \text{ Sm}^{-1} )</td>
<td>( \sigma_{\text{PUL}}(r) = 2 \times 10^5 \text{ Sm}^{-1} )</td>
</tr>
</tbody>
</table>

**Conductivity Model 1**

<table>
<thead>
<tr>
<th>Upper mantle</th>
<th>Main lower mantle</th>
<th>Lower mantle’s D(^{0}) layer</th>
<th>Passive upper core layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r &gt; R_a )</td>
<td>( R_a \geq r \geq R_c )</td>
<td>( \sigma_M(r) = 8 \text{ Sm}^{-1} )</td>
<td>( \sigma_M = 0 )</td>
</tr>
</tbody>
</table>

**Conductivity Model 2**

<table>
<thead>
<tr>
<th>Upper mantle</th>
<th>Main lower mantle</th>
<th>Lower mantle’s D(^{0}) layer</th>
<th>Passive upper core layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r &gt; R_a )</td>
<td>( \sigma_M = 0 )</td>
<td>( \sigma_M = 0 )</td>
<td>( \sigma_M = 0 )</td>
</tr>
</tbody>
</table>

**Conductivity Model 3**

\( g_{55}^{\text{PUL}}(t) \) 1998. © 2002 RAS, **GJI**, 149, 374–389
and approximate this setup by triangle-like peak-functions in the time points $t_i$.

As an initial condition ($\psi(r)$ in (22) and (27)) the solution of the harmonic downward continuation described by the eqs (21) is not assumed here how it would be done normally. Instead, we use for comparison the initial condition of the perturbation theory. The perturbation solution starts with an initial value of the form (see eqs 42 and 43),

$$
S_{lm}(r, 0) = \lambda_{nm} \frac{R_c}{n} \left( \frac{R_c}{r} \right)^{n+1} g_{nm}(0)
+ \lambda_{nm} \frac{R_c}{n} \left( \frac{R_c}{r} \right)^{n+1} \tau_\epsilon(r) g_{nm}(0)
$$

which differs from the harmonic initial condition $\psi(r)$ by the perturbation term (the second item), i.e. (46) presents no harmonic function in general.

The numerical integration itself is implemented by the conventional Crank–Nicholson algorithm with a radial discretization of 1 km and time steps of about 0.01 yr.

For the regularization according to the procedure (24) and (37) (Subsection 3.2.3) the norms used are as defined by eqs (25) ($L_2$ norm) and (26) ($W_1^2$ norm). It should be noted that the derivative considered here in the regularization by means of the $W_1^2$ norm refers to the time variable and not to the spatial variable, since the inversion approach discussed here only applies to temporal functions.

For the data approximation at $R_c$ in the $L_2$ norm according to the first term in the regularization functional (37) normally a matching bound $\epsilon$ has to be fixed a priori considering the (only weakly known time-dependent) real uncertainties of the past data sets. To become independently of the individual data series a normalized matching bound $\epsilon^*$ is introduced by $\epsilon^* = \epsilon / ||S_{lm}^{\text{obs}}(R_c, t)||_2$, $S_{lm}^{\text{obs}}(R_c, t)$ poloidal harmonic mode (data) (see eqs 21). For comparison of the non-harmonic downward continuation with the perturbation approach, at first the downward continued perturbation solution is stable upward continued. The $L_2$ norm difference between the resulting function and the original data then gives an $\epsilon^*$ that is selected as matching bound for the comparative non-harmonic downward continuation solution.

The numerical verification of the constructed method of non-harmonic downward continuation can be shown by combining its results with a complementary upward continuation procedure to determine the corresponding Gauss coefficient at the Earth’s surface for a data series given at the CMB depth, for example. (This procedure means to solve a boundary value problem of the same type as described by eq. (33) and can be done in the same manner as the stable direct problems solved in Subsection 3.2.2. The data input is given via the lower boundary condition of the first kind.) A possibility also exists to assess the perturbation solutions and their properties. Examples of the combination of (non-harmonic) downward to the CMB and upward continuation calculation to the Earth’s surface that reproduce the original data at the Earth’s surface are shown in Figs 2(a) and 3(a). Comparisons with the perturbation theory are also presented there. The downward continuation results are shown in Figs 2(b) and 3(b) while a finer comparison for the CMB is given in Figs 2(c) and 3(c) revealing the differences of both methods relatively to the harmonic downward continuation.

Fig. 4 compares the components $g_{32}$ and $g_{53}$ for each of three different data approximation levels $\epsilon^*$ at $R_c$ showing their influence on the downward continuation solution at $R_c$.

The significance of the solution in the whole time interval has to be studied in connection with the influence of the (artificial) initial condition. Fig. 5 shows the decreasing importance of the initial condition, especially its effect on the upper boundary function at $R_c$, with respect to different degrees $n$ and to the two conductivity models $\sigma_{M_1}$, $\sigma_{M_2}$. The presented quantity is the solution of a stable boundary
Figure 3. Downward continuation of the synthetic Gauss coefficient $g_{5\text{syn}}$ onto the CMB ($r = R_c$) and following upward continuation. (a) ($R_c$): Upward continuations of the different downward continued functions presented below: using the non-harmonic downward continuation (black full line), using the perturbation approach (dashed line). Normalized approximation error $\epsilon^* = 5.51 \cdot 10^{-3}$. (b) ($R_c$): Downward continuation of $g_{5\text{syn}}$ using the method of non-harmonic downward continuation (black full line), the perturbation approach (dashed line) and the harmonic downward continuation (grey line). (c) ($R_c$): Downward continuation of $g_{5\text{syn}}$ using the method of non-harmonic downward continuation (black full line) and the perturbation approach (dashed line) after subtracting the harmonic downward continuation.

value problem $u(R_\sigma, t)$ with lower zero-boundary-condition (at $R_\sigma$), upper boundary condition of the third kind (see 27) and a normalized harmonic initial condition (with $u(R_\sigma, 0) = 1$, initial time $t = 0$).

Some general properties of the connection between the data and solutions can be found if the operator $A$ (eq. 29), i.e. the approximating matrix $(a_{ik})$ (eqs 35 and 36), is traced back numerically to its kernel $k(t)$ via the solution of forward problems with the form of eq. (33) specifying the lower boundary function ($\delta$ function-like) with sufficient temporal resolution. Fig. 6 gives some kernel

Figure 4. Harmonic downward continuation (grey lines) and non-harmonic downward continuation onto $R_c$ (black lines) of the Gauss coefficients $g_{32}$ (a) and $g_{5\text{syn}}$ (b) using different data approximation levels in the regularization (the normalized approximation error $\epsilon^*$).

Figure 5. Decreasing influence of a normalized harmonic initial condition on the upper boundary function for different degrees $n$ and different conductivity models from Table 1: Model 1 (black lines), model 2 (grey lines).
approximations for alternative conductivity models, different degrees \( n \) for the downward continuation from \( R_E \) to \( R_c \).

Fig. 7 shows the downward continuation of the two Gauss coefficients \( g_{32} \), \( g_{5 \text{syn}} \) onto the CMB, comparing the influence of the conductivity models, \( \sigma_{M1}, \sigma_{M2} \) (Table 1). For comparison, the harmonic downward continuation is also included (\( \sigma_{H}(r) \equiv 0 \)).

The downward continuation behaviour of amplitudes and phases of harmonic oscillations for the stronger conductivity model \( \sigma_{M1} \) used and different degrees \( n \) to compare both downward continuation procedures is presented in Fig. 8 for periods between 0.1 yr and 200 yr.

In addition to the downward continuation onto the CMB with a relatively weak mantle conductivity, solutions were calculated for a second depth level as an experiment. The downward continuation of the field components \( g_{32}(t) \) and \( g_{5 \text{syn}}(t) \) is presented for 50 km beneath the CMB for a proposed passive layer on top of the fluid outer core (with the high core conductivity \( \sigma_{\text{PUL}} \), Table 1). A numerical verification and comparison with the perturbation solution and the harmonic downward continuation as in the CMB case is also given here (Figs 9 and 10).

In addition to examining the temporal character of the non-harmonic downward continuation by means of single Gauss coefficients, the spatial behaviour and global effects can be examined by analysing the radial component \( B_r \) of the geomagnetic field. Using the Gauss coefficients \( g_{nm}(t) \) and \( h_{nm}(t) \) as boundary values taken up to degree and order 8, \( B_r(r, t) \) is expressed as

\[
B_r(r, t) = \frac{1}{r} \sum_{n,m} \left( S_{nm}(r, t) \cos m\varphi + S'_{nm}(r, t) \sin m\varphi \right) \times n(n+1)P_{nm}(\cos \theta)
\]  

(47)
Figure 9. Downward continuation of the Gauss coefficient \( g_{32} \) in the passive upper layer of the outer core \((r = R_c - 50 \text{ km})\) and following upward continuation. (a) \((R_E)\): Data (grey line) and upward continuations of the different downward continued functions presented below: Using the non-harmonic downward continuation (black full line), using the perturbation approach (dashed line). Normalized approximation error \( \epsilon^* = 5.49 \cdot 10^{-3} \). (b) \((R_c - 50 \text{ km})\): Downward continuation of \( g_{32} \) using the method of non-harmonic downward continuation (black full line), the perturbation approach (dashed line) and the harmonic downward continuation (grey line).

This has been calculated at the core–mantle boundary \( r = R_c \) and at a depth of 50 km beneath the core–mantle boundary for two time points—1910 and 1960—situated in the central part of the epoch 1840 to 1990 (see Figs 11a and 12a) using the conductivity models \( \sigma_M \) and \( \sigma_{\text{PUL}} \) in the case of the 50 km layer beneath the CMB (see Table 1). The solutions presented here were calculated by means of regularization with the \( W_1^2 \) norm and a constant normalized approximation error \( \epsilon^* = 0.01 \) for each data component. Also, these global fields were calculated for the same time points and conductivity models by the downward continuation perturbation approach (see Figs 11b and 12b) using the relations (42) and (43) while for the pure harmonic downward continuation (see Figs 11c and 12c) the eqs (21) were inserted in (47). For comparison the difference fields are shown in Figs 11b,c and 12b,c.

5 CONCLUDING REMARKS

5.1 Non-harmonic downward continuation

The development of the non-harmonic downward continuation method presented here is motivated by the desire to resolve the strength of the magnetic field at the core–mantle boundary zone and within the Earth’s lower mantle over the decadal timescale and if possible for higher frequency magnetic features originating in the Earth’s core. One reason is the necessity to calculate the electromagnetic core–mantle coupling, one mechanism that explains the correlation between Earth rotation and geomagnetic field variations over this timescale. Our results suggest that calculating coupling torques by the regularization method in the downward continuation process using a sufficiently high conductivity may lead to new aspects. In particular, knowing the magnetic field strength at the CMB will allow velocity fields in the fluid outer core (frozen-field theory) to be determined since these also play an important role in the calculation of toroidal coupling torques.

The problem of downward continuation is itself an inverse problem while at the same time being one part of the coupled (joint) inverse problems that apply to other phenomena (e.g. Earth rotation, fluid dynamics). The calculated magnetic field is dependent upon the assumed conductivity model of the lower mantle here described as a radially dependent function. Because geomagnetic data is only available on one (the outer) side of the radial interval, the
corresponding mathematical problem can be characterized as an inverse boundary value problem.

There is a formal relationship between the magnetic field continuation and the inverse heat conduction problem, one which has been intensively studied because of the possible applications for industry and laboratories. This has led us to develop a regularizing numerical algorithm for this unstable, while uniquely solvable, inverse problem. It accounts for arbitrary conductivity functions, for the decomposition of the solving time function by any base functions as well as for different solution strategies, i.e. the assumption of norm bounds on the solution to reach a certain type of smoothness or to account for the degree of approximation to the data. The downward continuation method can be numerically verified by the complementary stable upward continuation and has been tested using different criteria.

When solving the unstable non-harmonic downward continuation, a regularization algorithm has to be used. We applied a modification of the Tikhonov regularization (Hansen 1998) while at the same time using special norms for the data approximation \( L_2 \) and for the smoothness of the solution \( W_2 \). The method was also tested for a conductivity model including a more conductive layer slightly below the CMB. This showed a stronger...
amplitude amplification and phase shifting of the geomagnetic field components.

5.2 Comparison of methods

The methods of purely harmonic downward continuation (i.e. assuming vanishing mantle conductivity), the downward continuation based on a perturbation approach and the non-harmonic downward continuation can be considered as alternative downward continuation methods. Some common properties and differing aspects are discussed in the following.

A theoretical classification of the three methods can be considered if the original inverse boundary value problem (22) is modified in the following way:

Using (13) through to (17) and (19), the function \( R_{n,m}^{c,\phi} (r, t) \) defined as

\[
R_{n,m}^{c,\phi} (r, t) = S_{n,m}^{c,\phi} (r, t) - C_{n,m}^{c,\phi} (t) r^{-n-1}
\]

(48)
describes the difference between non-harmonic and harmonic downward continuation. It is a solution of the inverse boundary value problem, now with an inhomogeneous differential equation (for \( D_n \) see Subsection 2.2 (15)),

\[ \copyright \ 2002 \mathrm{RAS}, \mathrm{GJI}, \ 149, \ 374-389 \]
\( D_k R_{n,m}^k(r, t) = \mu_0 \sigma_M(r) R_{n,m}^{k+1}(r, t) \)
\[ + \mu_0 \sigma_M(r) \left[ \frac{g_{n,m}(t)}{h_{n,m}(t)} \right] \frac{1}{n} \lambda_{nm} R_{E}^{n+2} r^{-n-1} \]  

(49)

but indifference to (22) with homogeneous initial and boundary conditions

\[ R_{n,m}^{c,j}(r, 0) = 0, \quad R_0 > r > R_c \]
\[ R_{n,m}^{c,j}(R_0, t) = 0, \quad \frac{\partial}{\partial r} R_{n,m}^{c,j}(R_0, t) = 0 \]  

(50)

The type of this problem compared with the original (eq. 22) remains unchanged. However, as can be seen from the right hand side of eq. (49), the clearly controlled inhomogeneity of the differential equation containing the time derivatives of the data \( g_{n,m} \), \( h_{n,m} \) together with the conductivity function \( \sigma_M(r) \) are now revealed.

Considering the proper perturbation terms \((k \geq 1)\) of the perturbation approach (39) as separation ansatz in (49), here confining to \( R_{n,m}^{k} \),

\[ R_{n,m}^{k}(r, t) = \frac{1}{n} \left( p_{n,m}^{(k)}(r) g_{n,m}(t) + p_{n,m}^{(k)}(r) \frac{\partial}{\partial r} g_{n,m}(t) + \cdots \right) \]  

(51)

it follows from (49) that e.g. the first-order term \((k = 1)\) fulfills this differential equation (49) only if \( g_{n,m}(t) = 0, \frac{\partial}{\partial r} g_{n,m}(t) = 0, \ldots \) holds, i.e. the perturbation solution appears as a special approximative solution of our original boundary value problem (eq. 22).

Another point of view when comparing the methods concerns the kind of approximation in which the solution function is generated. The perturbation method approximates with respect to two separated kinds of approximation in which the solution function is generated. The type of this problem compared with the original (eq. 22) remains unchanged. However, as can be seen from the right hand side of eq. (49), the clearly controlled inhomogeneity of the differential equation containing the time derivatives of the data \( g_{n,m} \), \( h_{n,m} \) together with the conductivity function \( \sigma_M(r) \) are now revealed.

5.3 Discussion of results

Apart from comparing the methodical aspects of the magnetic downward continuation, several results using real data will be discussed here that are in some cases of basic and exemplary importance.

Using these results with the single Gauss coefficients as examples (Figs 2 to 8, Subsection 4.2) some features of the downward continuation methods are presented (see Subsections 5.1 and 5.2). Varying the parameters, quantities and influences (different sigma models, different time structure of the data, including the data error, comparison of numerical algorithms) allows an assessment to be made of what influence these factors would have on the interpreting of results. The examples of global (8,8) field (Figs 11 and 12) are time slices being partly interpretable.

One important outcome is the influence of data uncertainties have on the downward continuation results. This issue should have more attention devoted to it. The figures in Fig. 4 demonstrate this by varying the degree of data fitting, therefore simulating different assumed \( \epsilon^* \) ranges.

It should be stressed that the middle time section of each solution provides the best information about the downward continued input data (Subsection 5.2) while estimating the interpretable interval (diffusion equation, convolution integral) requires additional calculations (to solve direct initial-boundary value problems for alternative conductivity models). Fig. 5 showing the dependence of an harmonic initial value function on the upper boundary function (‘data’) gives the result that a time interval of one to two years.

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(conductivity model 2) or of the order 5 to 10 yr (conductivity model 1) at the beginning is sufficient to allow the influence of the initial condition on the solution to be neglected. This can also be estimated by the kernels, which reflect the (inverse) local and temporal transfer properties (amplitude change and phase shift) between the data on the Earth’s surface and the deep Earth interior field components (Fig. 6). In particular, the effect of the different conductivity models connected with the geometry can be studied this way, where the positive part of the axis of ordinates is identified with the kernel of harmonic downward continuation (case of zero-conductivity). Under these aspects, also the epochs (1910, 1960) of the radial component of the presented (8,8) core–mantle boundary fields were selected (Figs 11 and 12).

Of relevance on the downward continuation results is the choice of the conductivity model, itself a topic of debate. This can be seen from the curves of the kernels that characterize the transitional properties (Fig. 6) but also directly in the examples of Fig. 7 where periodicities and other specific modulated parts are more obvious. The main feature observed are different phase shifts for different conductivity models. With greater depth and sufficiently high conductivity values the solution becomes stronger modulated. The effects of phase shift and amplitude amplification are very large for high conductivity values, e.g. below the CMB. To restrict the discussion, we assume a hypothetically passive, non-moving layer (being under discussion). While the harmonic downward continuation reproduces only the surface time structure and thus shows no phase shift, the two other downward continuation methods give different results (Figs 9b and 10b) which becomes more significant for data containing higher frequency parts. The differences in the spectral behaviour between these methods, especially the sensitivity of the phase shift, can be seen in Fig. 8. This topic should be investigated in greater detail when denser data sets become available.

The temporal properties, that can be derived from the couples presented in Figs 11 and 12 confirm the results discussed previously for the downward continuation of the single Gauss coefficients (temporal modulation with increasing depth, phase shifting and amplitude amplification).

If we consider those figures that present the same time for different depths (e.g. the left hand figures in Figs 11a and 12a for 1910) the spatial features of regional anomalies can be better resolved, especially those around the south pole and beneath Siberia. Analogous observations concerning the spatial structure of the geomagnetic field can also be made for 1960. From these significant spatial modulations, it may be concluded that the velocity field in the top layer of the fluid outer core must have finer structures with increasing depth. Figs 11(c) and 12(c), respectively, show the global (8,8) residual fields, i.e. the differences between non-harmonic and harmonic downward continuation for both downward continuation levels and two time points. The temporal and local variability (up to 15000 nT at the CMB) reveals some new structures that are not directly correlated with those known from the harmonic downward continuation. Therefore, we find that highly conducting layers near the CMB can change the continuation results relative to the harmonic downward continuation considerably. This would have consequences for the physical quantities that are associated with the magnetic field at the CMB.

The comparisons between methods may be summarized as follows: For the weak mantle conductivities, especially model 2, and the long wave, smoothed data of Bloxham & Jackson (1992), the results from both downward continuation methods display very small differences. The results derived from the non-harmonic downward continuation procedure can differ from those based only on the pure harmonic downward continuation of the magnetic field components. As seen from the additional higher frequency synthetic example and those distinctly using the higher conductivity below the CMB, clear differences between all methods are observed.

Therefore, we conclude that the solution technique of the non-harmonic downward continuation presented here is an alternative of the perturbation method, particularly for those cases, first, where data manipulations lead to a loss of spectral information required in the subdecadal range (e.g. for jerks) and second, where conductivity models with conductance higher than $10^8$ S are assumed. The residual fields (Figs 11b and 12b) given as differences between non-harmonic and perturbation approach continuations show a variability—not only in the outer core but also on the CMB (in the order of 5000 nT)—that is significantly different from that of the dominant harmonic part of the field. Thus, the application of this method is justified by allowing regional and local structures of the magnetic field and the outer core fluid motion to be revealed, especially in the presence of higher conductivity zones.

5.4 Future work

Further improvements of existing results depends upon the quality of the data, especially, since the largely unknown noise of the older data or if the data had been preparatively smoothed means that the results of the downward continuation process have only a limited validity. To obtain better error estimates of the magnetic field at the core–mantle boundary, it is necessary to find more adequate time-dependent uncertainties for the Gauss coefficients $g_{mn}$ and $h_{mn}$ and to generate estimates of the effect of time-dependent errors on the construction of solutions of the inverse boundary value problem. With regards to the methodology and theory connected, these should be developed in different directions. A 2- or 3-D framework would enable the study of regional and local effects, however this will need the inclusion of the toroidal field and therefore further theoretical elements. Another issue is the treatment of the near-core magnetic field in moving media so as to find an approximation not only for the core-surface but to some extent for a limited depth within the core. Such an approach should include the coupled modelling and inversion of different global and local phenomena (Earth rotation, coupling mechanisms, balance, magneto-fluid motion etc.) to come to a clearer interpretation of the magnetic data and other proxy quantities. In connection with this, it would be useful to decrease the required computing resources via e.g. selecting adapted bases structures and other types of approximations and algorithms.

In the near future, the downward continuation method will be applied in the modelling of highly resolved geomagnetic data, e.g. from the Danish Ørsted and German CHAMP satellite missions (Olsen et al. 2000; Reigber et al. 2000) and in the inverse investigation of the (moving) upper layers of the fluid outer core. These applications of our formalism will be a challenge to the numerical processing.

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