Topic	Theoretical source representation
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1 Introduction

In seismology the problem of understanding and describing the seismic source consists in relating observed seismic waves (i.e., seismograms) generated by this source to suitably conceived geometric, kinematic and dynamic parameters of a mechanical source model that represents the physical phenomenon of a brittle fracture in the Earth's lithosphere. Representations of the source are defined by parameters whose number depends on the complexity of the source models (e.g., Aki and Richards, 1980; Ben-Menahem and Singh, 1981; Das and Kostrov, 1988; Lay and Wallace, 1995; Udías, 1999). In the direct problem, theoretical seismic wave displacements are determined from source models and in the inverse problem parameters of source models are derived from observed wave displacements. In the following we will consider only source models related to earthquakes and explosions (see Chapter 3), volcanic tremors (see Chapter 13) and rock bursts. Here we will not discuss sources of seismic noise (see Chapter 4).

Strong non-linear and non-elastic processes take place in a seismic source volume. Parts of it may crack, phase transitions may take place, the temperature may increase, and so on. These kinds of processes are not described by most seismic source theories; however,there are special theories to model such processes, e.g., the time-dependent pressure within an explosion cavity, the rupture propagation on an earthquake fault, and the material behavior on a crack tip (crack criteria). We limit ourselves to the phenomenological description of a seismic source. The aforementioned complicated processes need not to be considered when looking only for their integral effect on a surface surrounding the seismic source, i.e., by replacing a volume integral by a surface integral (see, e.g., Aki and Richards, 1980).

2 Continuum mechanics

The description of the source mechanism is based on the solution of the equation of motion. In a deformable solid medium this equation is derived from classical Newtonian mechanics. The linearized equation of motion (i.e., by neglecting density changes and other second order effects) is

$$\rho \ddot{u}_i(x_s,t) - \sigma_{ij,j}(x_s,t) = f_i^b(x_s,t) \quad . \tag{1}$$

In this equation ρ is the density of the solid body, u_i are the components (i = 1, 2, 3) of the displacement field that describe the deformation of the body, σ_{ik} is the stress tensor, f_i^b is the body force density acting per unit volume, $\ddot{u}_i$ is the second time derivative $\partial^2/\partial t^2$ of the displacement and the comma between two subscripts, e.g., in $\sigma_{ik,k}$ indicates the spatial derivative of the considered quantity. We generally use the *summation convention* which requires that one has to sum when a subscript appears twice, e.g.,

$$\sigma_{ik,k} = \frac{\partial}{\partial x_1}\sigma_{i1} + \frac{\partial}{\partial x_2}\sigma_{i2} + \frac{\partial}{\partial x_3}\sigma_{i3}.$$

The displacement is a function of the spatial co-ordinates x_i and the infinitesimal deformation is defined as

$$du_i = u_{i,k}\, dx_k \tag{2}$$

with

$$u_{i,k} = \beta_{ik} \tag{3}$$

as the distortion tensor. We now consider the location of a particle before and after it is deformed, described by the vectors a_i and x_i, respectively. Accordingly, an infinitesimal vector da_i at the point a_i is moved (i.e., deformed) to the vector dx_i, as shown in Figure 1.

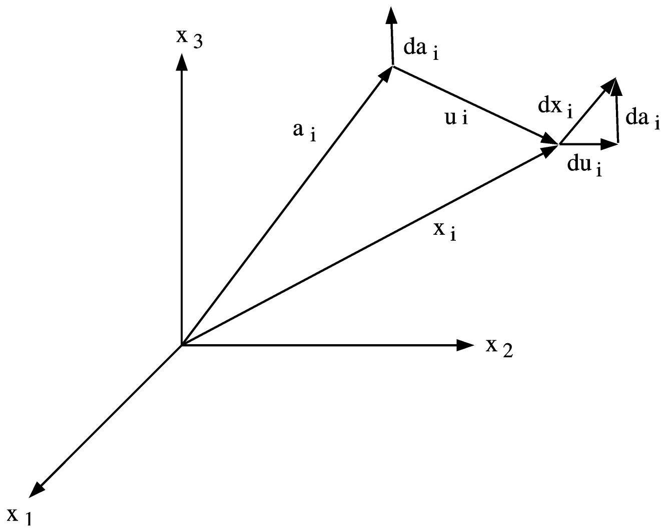

Figure 1 Coordinates and vectors describing the displacement field (see text).

Introducing ds^2, which is the difference between the square of the length of the vectors dx_i and da_i, and, thus, a measure of the deformation of the body, i.e., $ds^2 = dx_i\, dx_i - (dx_i - du_i)(dx_i - du_i)$, we get with (2) and (3)

$$ds^2 = (\beta_{ij} + \beta_{ji} - \beta_{ki}\beta_{kj})\, dx_i\, dx_j = 2\varepsilon_{ij}\, dx_i\, dx_j. \tag{4}$$

Equation (4) is the definition of the *strain tensor* ε_{ij}. It is a symmetric tensor. For small deformations it can be approximated by its linear terms

$$\varepsilon_{ij} = \frac{1}{2}(\beta_{ij} + \beta_{ji}) = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{5}$$

Thus, the strain tensor ε_{ij} is the symmetric part of β_{ij}. Any symmetric tensor can be transformed into a co-ordinate system such that

$$\varepsilon_{ij} = \varepsilon^{(i)} \delta_{ij} \tag{6}$$

where δ_{ij} is the dimensionless *Kronecker symbol*, defined by

$$\delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases} \tag{7}$$

and $\varepsilon^{(i)}$ are the eigenvalues of the strain tensor. The co-ordinate system where ε_{ij} is of the form (6) is called system of principle axes. The three eigenvalues describe the relative deformation in direction of the principle axes.

In continuum mechanics one distinguishes between body forces and surface forces. The body forces are sometimes also termed volume forces because they act on volume elements dV of the body. In Equation (1) we consider infinitesimal masses $\rho\, dV$ where dV is the infinitesimal volume of the mass element. Accordingly, an infinitesimal body force (Aki and Richards, 1980) can be written as $dF_i = f_i^b (x_s,t)\, dV$. Typical examples of body forces are the gravity field and the centrifugal force.

In contrast, surface forces such as cohesion, the sliding friction, or the internal stress during the deformation of the body, act on surface elements dS of the volume dV. The stress is a tensor of second order, i.e., it has two subscripts, because it is characterized both by the orientation of the force and by the orientation of the surface on which the force acts. A second-order tensor has generally 9 independent components which can be written explicitly as

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}.$$

In general, σ_{ij} depends on position and time. It acts only between adjacent particles. Because of the conservation law of angular momentum this tensor has to be symmetric, i.e.,

$$\sigma_{ij} = \sigma_{ji}. \tag{8}$$

The relation between the incremental body force density df_i^s which acts on an internal surface element dS and the stress is

$$df_i^s = \sigma_{ij}\, n_j\, dS \tag{9}$$

where n_j is the normal vector of the surface elements (see Figure 2). $\sigma_{ij}\, n_j$ is called the *traction* of the stress tensor. The pressure and the surface tension in fluids are special examples of internal surface forces. Figure 3 shows the different components of σ_{ij} which act on the surfaces of an infinitesimal cube.

In the linear theory of elasticity, the strain and the stress tensor are linearly coupled. A relatively simple stress-strain relation is the generalized *Hook`s law*

$$\sigma_{ij} = c_{ijkl}\, \varepsilon_{kl}. \tag{10}$$

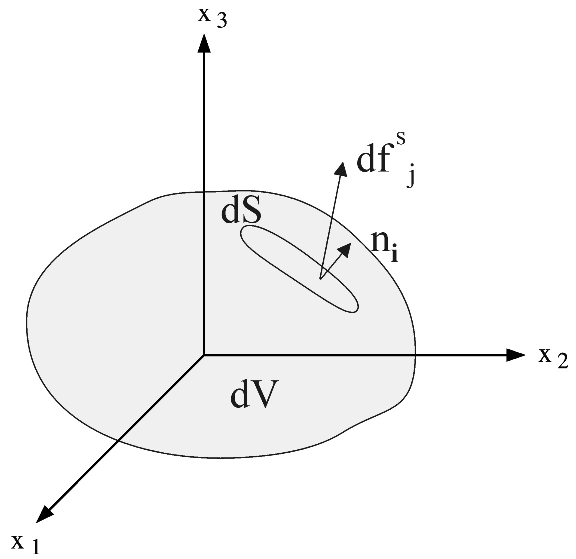

Figure 2 Schematic depiction of the considered source volume *dV* , a surface element *dS* (with its normal vector n_i) on which the force df_j^S acts.

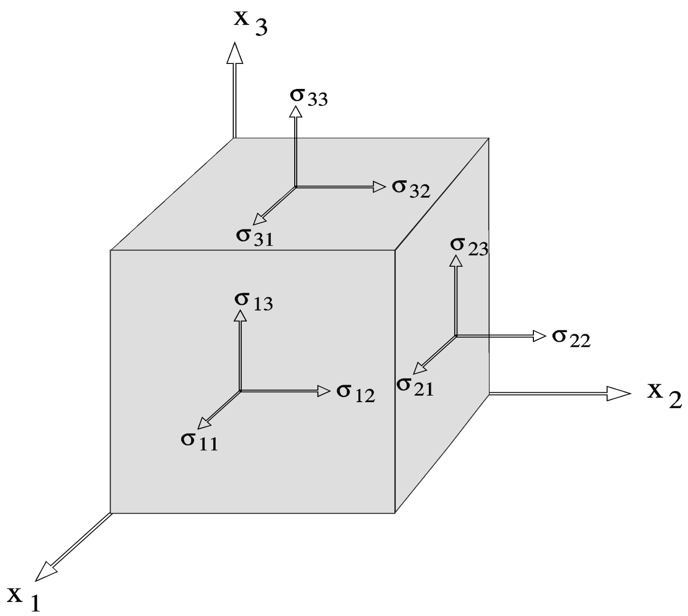

Figure 3 The nine components of the stress tensor. σ_{ij} are the components of the stress tensor parallel to x_j on planes having n_i as their normals.

The body that obeys the relation (10) is said to be *linearly elastic*. The c_{ijkl} are called *elastic constants* because they are independent of strain, however, in the case of an inhomogeneous medium, they depend on the position in the body. Due to the symmetry of strain (see Equation (5)) and stress tensor (see Equation (8)) and because of the energy balance in the body, the fourth-order tensor c_{ijkl} has the following three symmetries:

$$c_{ijkl} = c_{jikl}, \qquad c_{ijkl} = c_{ijlk}, \qquad \text{and} \qquad c_{ijkl} = c_{klij}. \tag{11}$$

These symmetries reduce the independent components in c_{ijkl} from 81 to 21. In the case of an *isotropic medium*, i.e., when the elastic properties are independent of the orientation in the body, the elastic constants reduce to just two. Then c_{ijkl} has the form

$$c_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}). \tag{12}$$

The two parameter λ and μ are known as the *Lamé constants*.

If attenuation has to be included the relatively general *Boltzmann law*

$$\sigma_{ij}(t) = \int_{-\infty}^{t} b_{ijkl}(t-\tau)\, \varepsilon_{kl}(\tau)\, d\tau \tag{13}$$

can be used.

It is advantageous to introduce now the *Fourier transform* $f(\omega)$ of a time dependent function $f(t)$. Here, ω *is* the angular frequency 2π f, where f is frequency in units of Hz.. We use the definitions

$$f(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt \qquad \text{and} \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega \tag{14}$$

where $i = \sqrt{-1}$ is the imaginary unit, and $f(\omega)$ is a complex function, called the complex spectrum of $f(t)$. It can be represented by

$$f(\omega) = a(\omega) + i\, b(\omega) = A(\omega)\, e^{i\Phi(\omega)}$$

where $A(\omega)$ is the amplitude spectrum and $\Phi(\omega)$ the phase spectrum. $a(\omega)$ and $b(\omega)$ are the real and the imaginary parts of $f(\omega)$, respectively. When applying the Fourier transformation to Equation (13) the integral is replaced by the product of $b_{ijkl}(\omega)$ and $\varepsilon_{kl}(\omega)$. The imaginary part of b_{ijkl} describes a linear attenuation for a propagating displacement field.

With Eqs. (5), (10), and (14) the equation of motion (1) becomes (Udías, 1999)

$$\rho\, \omega^2 u_i\, (x_s, \omega) + \sigma_{ij,j}(x_s, \omega) = -\, f_i^b(x_s, \omega) \tag{15}$$

and in a linear elastic but inhomogeneous medium

$$\rho\, \omega^2 u_i\, (x_s, \omega) + (c_{ijklj}\, u_{k,l}\, (x_s, \omega))_{,j} = -\, f_i^b(x_s, \omega). \tag{16}$$

The second term on the left side is the stress due to the displacement u_k. In order to specify u_i in a unique way, the initial conditions have to be fixed for the displacement u_i and the related velocity $\dot{u}_i$ as well as the boundary conditions for the displacement or the traction. The homogeneous initial condition, that both u_i and $\dot{u}_i$ are zero before the beginning of the seismic event, is the precondition for the existence of the related Fourier transform $u_i(x_s, \omega)$. Boundary conditions can be specified for the displacement u_i or the traction $\sigma_{ij}\, n_j$ on internal surfaces S (or *external surfaces* such as the Earth's free surface) (see Figure 4), namely

$$u_i(\xi_s, \omega) \quad \text{or} \quad \sigma_{ij}\, (\xi_s, \omega)\, n_j \quad \text{on the internal surface } S(\xi_s) \tag{17}$$

where S(ξ_s) may consist of several unconnected surfaces. The Greek letter ξ_s used as co-ordinates should indicate that the quantities u_i and σ_{ij} are lying on the surface S(ξ_s) which is generally curved. These boundary conditions are indispensable for modeling seismic sources and computing the wave propagation through a layered medium.

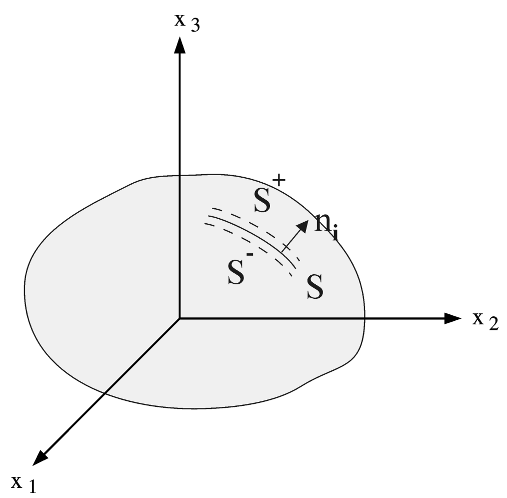

Figure 4 Illustrating the definition of boundary conditions for seismic faults representation.

3 Kinematic source models

The first mathematical formulation of the mechanism of earthquakes used the representation of the processes at the source by a distribution of the body force density $f_i^b(\xi_s, t)$ acting inside the source volume V_0 . Since these forces must represent the phenomenon of fracture, they are called equivalent forces. If it is assumed that no other body forces are present (gravity, etc.), and that on its surface S displacements and tractions are zero, we can use the representation theorem in terms of the *Green's function* to write the elastic displacements in an infinite medium in the time domain as

$$u_i(x_s,t) = \int_{-\infty}^{\infty} d\tau \int_{V_o} f_k^b(\xi_s,t)\, G_{ik}(x_s,t,\xi_s,\tau)\, dV \qquad (18)$$

or in the frequency domain by

$$u_i(x_s,\omega) = \int_{V_o} f_k^b(\xi_s,\omega)\, G_{ik}(x_s,\xi_s,\omega)\, dV\ . \qquad (19)$$

The Green's function G_{ki} is the solution of the equation of motion (16) for special impulsive single point forces, termed *Dirac* or *needle impulses*, which act inside the body. The spectrum of the Dirac impulse is 1 for all frequencies and, thus, does not appear in Equation (20) below. According to Ben-Menahem and Singh (1981) and Udías (1999), the following equation holds for the Green's function

$$\rho\,\omega^2\, G_{in}(x_r,\xi_r,\omega) + (c_{ijkl}\, G_{kn,l}(x_r,\xi_r,\omega))_{,j} = -\delta_{in}\, \delta(x_r - \xi_r) \qquad (20)$$

where $\delta(x_r - \xi_r)$ is the three-dimensional Dirac delta function which is the product of three one-dimensional Dirac delta functions, i.e., $\delta(x_r - \xi_r) = \delta(x_1 - \xi_1)\ \delta(x_2 - \xi_2)\ \delta(x_3 - \xi_3)$. Note that $\delta(x_r - \xi_r)$ has the dimension of 1/(unit volume). The three one-dimensional Dirac functions define the point in space where the three perpendicular point forces, as described by the Kronecker symbol in Equation (7), act.

The Green's function acts as a "propagator" of the effects of forces f_i^b, from the points where they are acting (ξ_i inside V_0) to points x_i outside V_0, where the elastic displacement u_i produces the seismogram. A simplification, often used in the practice, is made by applying the point source approximation. It is valid if the source dimension is much smaller than the considered wavelength and the distance of the observation point from the source. For a point source at x_s^o we develop the Green's function in Equation(19) in a Taylor series at this point:

$$u_i(x_s,\omega) = \int_{Vo}\left[f_k^b(x_s^o + s_s,\omega)G_{ik}(x_s,x_s^o,\omega) + s_j f_k^b(x_s^o + s_s,\omega)\frac{\partial}{\partial x_j^o}G_{ik}(x_s,x_s^o,\omega) + \ldots\right] dV(s_s)$$

$$= F_k(x_s^o,\omega)G_{ik}(x_s,x_s^o,\omega) + M_{jk}^f(x_s^o,\omega)G_{ik,j}(x_s,x_s^o,\omega) + \ldots \qquad (21)$$

If the source volume is small the Taylor series can be finished after the second term with the first derivative to the source co-ordinates x_l^o. Then (21) defines the force F_k and a seismic moment tensor M_{kl}^f for which the following relations hold:

$$F_k(x_s^o,\omega) = \int_{Vo} f_k^b(x_s^o + s_s,\omega)\, dV(s_s) \qquad (22)$$

and

$$M_{jk}^f(x_s^o,\omega) = \int_{Vo} s_j\ f_k^b(x_s^o + s_s,\omega)\, dV(s_s)\,. \qquad (23)$$

If f_k^b is a single point force then M_{kl}^f as a whole describes a force couple (see Figure 5).

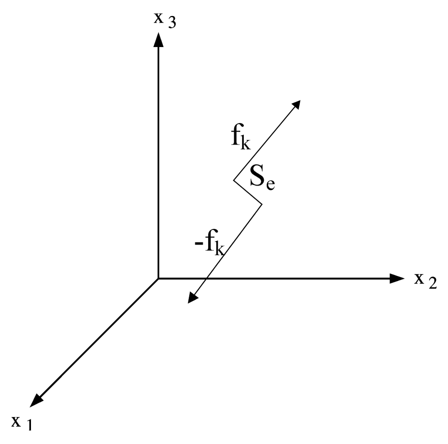

Figure 5 Schematic presentation of a general force couple $f_i\ s_j$

Equation (21) contains the partial spatial derivatives of the Green's function. In a homogeneous infinite body they can be written as

$$G_{ki,j} = \frac{1}{r^4\omega^2}\sum_{n=0}^{3} \overset{P}{A}{}_{ijk}^{(n)} \left(\frac{\omega r}{v_P}\right)^n + \frac{1}{r^4\omega^2}\sum_{n=0}^{3} \overset{S}{A}{}_{ijk}^{(n)} \left(\frac{\omega r}{v_S}\right)^n \qquad (24)$$

where the $A_{ijk}^{(n)}$ are complex coefficients proportional to the amplitudes and phases of the P and S waves (see 2.2). The term of $G_{ij,k}$ with n = 3 is called the *far-field term* because it can still be observed at rather large distances r between the point source and the point of observation (seismic recording). In contrast, the terms with n = 0, 1 and 2 are called the *near field terms* because they decay with distance more rapidly than the far-field term, namely proportional to r^{-2}, r^{-3}, and r^{-4}, respectively.

Elastic displacements are given now by the time convolution of the forces acting at the focus with the Green's function for the medium. The simplest Green's function is that corresponding to an homogeneous infinite medium (full space). Internal sources must be in equilibrium, thus satisfying the condition that their resulting total force and moment are zero. Therefore, we consider as a seismic source only the symmetric part of M_{kl}^{f} as a seismic moment tensor, i.e.,

$$M_{jk} = M_{jk}^{f} + M_{kj}^{f} . \qquad (25)$$

Fig. 3.34 shows all possible 6 couples and three dipoles of the seismic moment tensor M_{jk}.

If we want to represent the shear motion on a fault, the equivalent system of forces is that of two couples with no resulting moment, called a double-couple model (DC) (see Figure 8). If the couples are oriented in the direction of the two perpendicular unit vectors e_i and l_i, respectively, with $e_i\, l_i = 0$, and if their scalar seismic moment is $M_0(\omega) = \lim_{|s_i|\to 0} |s_i||F_k|$, where $|s_i|$ is the length of the arm of the couple and $|F_k|$ the amount of the force, the displacement caused by the double-couple source is given by

$$u_i^{DC}(x_s,\omega) = M_0(\omega)(e_k l_j + e_j l_k)\, G_{ik,j}(x_s, x_s^o, \omega) . \qquad (26)$$

Note that in the given case the comma in the subscripts of G represents the partial derivative with respect to the source co-ordinates.

If an earthquake is produced by a fault in the Earth's crust, a mechanical representation of its source can be given in terms of fractures or dislocations in an elastic medium. A displacement dislocation consists of an internal surface S with two sides (S^+ and S^-) inside of the elastic medium (see Figure 5) across which there exists a discontinuity of displacement; however, stress is continuous. Thus, S is a model of a seismic fault. Coordinates on this surface are ξ_k and the normal at each point is n_i. From one side to the other of this surface there is a discontinuity in displacement D_i, which is termed the slip or dislocation on the fault:

$$D_i(\xi_k,\omega) = u_i^+(\xi_k,\omega) - u_i^-(\xi_k,\omega) . \qquad (27)$$

The plus and minus signs refer to the displacement at each side of the surface S. If there are no body forces (F_i = 0), and the stresses are continuous through S, then, for an infinite medium, the equation relating the displacement to the dislocation D_i, results in

$$u_n(x_s,\omega) = \int_S D_i(\xi_s,\omega)\, c_{ijkl}\, n_j(\xi_s)\, G_{nk,l}(x_s,\xi_s,\omega)\, dS(\xi_s)\,. \qquad (28)$$

Equation (27) corresponds to a kinematic model of the source, that is a model in which elastic displacements u_i are derived from slip vector D_i. The latter represents a non-elastic displacement of the two sides of a fault (i.e., of the model surface S). In a kinematic model slip is assumed to be known. It is not derived from stress conditions in the focal region as it is in dynamic models. Equation (28) contains the Green's function discussed in conjunction with Equation (24). When seismic waves, generated by the source, are observed in the far-field, i.e., at distances r much larger than the wavelength and the linear source dimension, than the Green's function is proportional to ω. Accordingly, the dominant term of the integrant in Equation (28) is ωD_i which is, in the time domain, proportional to the slip velocity. Thus, the elastic displacement observed in the far-field does not depend on the slip in the source but on the slip velocity and, similarly, on the seismic moment rate $\partial M_{ik}(t)/\partial \mathrm{t}$ $= \dot{M}_{ik}$ (see Fig. 2.4). Or, in the frequency domain, the displacement is proportional to $i\omega$ $M_{kl}(\omega)$. This means that the source radiates elastic energy only while it is moving; when motion at the source stops it ceases to radiate energy.

The most common model for the source of an earthquake is a shear fracture, that is, a fracture in which the slip D_i is perpendicular to the normal of the fault. For a fault plane S of area A and normal n_i, the slip $D_i(\xi_s,t)$ is in the direction of the unit vector l_i contained in the plane. Accordingly. l_i and n_i are perpendicular and the scalar product $n_i{\cdot}l_i$ = 0. For an infinite, homogeneous isotropic medium, displacement according to Equation (28) is given by

$$u_i(x_s,\omega) = \int_S \mu \left|D_l(\xi_s,\omega)\right| (l_k\, n_j + l_j\, n_k)\, G_{ik,j}(x_s,\xi_s,\omega)\, dS(\xi_s) \qquad (29)$$

For modeling a shear dislocation source, the parameters on the right-hand side of Equation (29) have to be known. Implicitly these parameters include information about the rupture propagation, i.e., on the shape of the crack front, its propagation direction and propagation velocity (crack velocity), and shape of the final ruptured surface S.

The circular fault and the rectangular fault are the most important approximations. In the first case the rupture begins at the center and the crack front is described by an outward propagating circle. However, the direction of the dislocation is not necessarily radiallysymmetric. This circular model, described by Brune (1970) and Madariaga (1976), should be valid for small earthquakes with magnitudes smaller than about 4 to 5. Another approximation, for large earthquakes in the Earth's crust in particular, is a rectangular fault model, also called Haskell-model (Haskell, 1964). The length of the fault, generally assumed to be horizontal, is larger than its width (depth) by a factor of 2 to 10 or even more for very large earthquakes. This is due to the limited thickness of the seismogenic zone of the upper lithosphere, usually ranging between about 10 and 25 km, where brittle fracturing is possible. On the other hand, large crustal earthquakes may have a rupture length of 200 km or even more, e.g., about 450 km for the Alaska earthquake of 1964 and about 1000 km for the Chile earthquake of 1960. This rectangular model is also useful for describing deeper earthquakes in subduction zones.

When the Haskell-model is used the behavior of the rupture front must be known. The first approximation is that the rupture starts along a line and propagates unilaterally or bilaterally over the rectangular fault plane (see Figure 6). This approximation is useful for long ruptures with small width (the line-source approximation). It is also suitable for distinguishing between an in-plane and an anti-plane fault geometry. In the case of an in-plane fault the rupture moves into the direction of the slip whereas in the anti-plane case the direction of slip is parallel to the rupture front (see Figure 6).

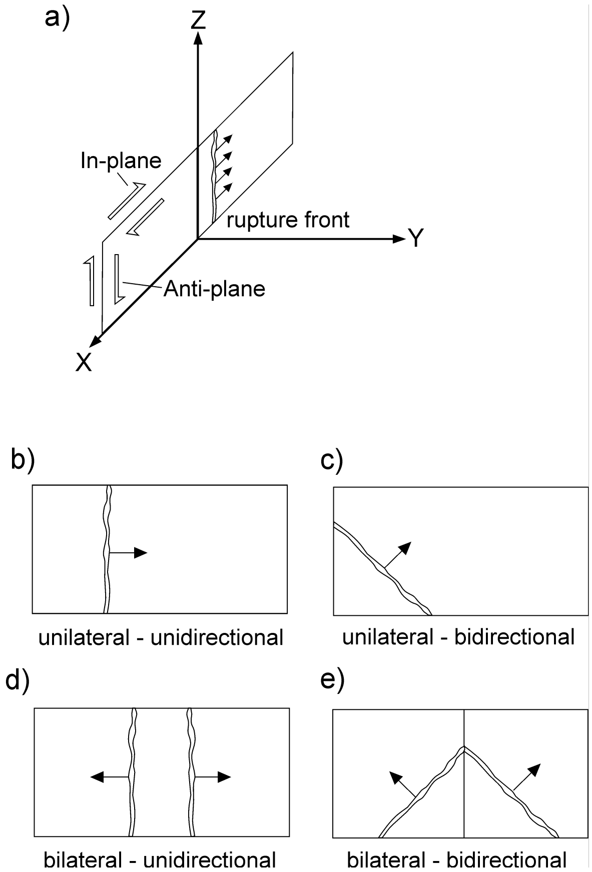

Figure 6 Several models of rupture propagation

For describing the rupture propagation in the case of a rectangular fault the following four terms and definitions, shown in Figure 6, are important:

- unilateral rupture propagation – one rupture front propagates over the entire fault plane;
- bilateral rupture propagation – two rupture fronts with different directions propagate over the rupture plane;
- unidirectional rupture propagation – the direction of rupture propagation is parallel to the length of the fault plane; and
- bidirectional rupture propagation – the rupture starts at a point and propagates across the fault plane.

Other models for describing the shape of the fault plane, the shape of the rupture front, and the mode of the rupture propagation are possible.

With respect to the velocity of rupture propagation the most common models assume values between about 0.6 to 0.9 of the shear-wave velocity v_s (see 2.2) in the source region; however, detailed field and laboratory investigations have shown that both slower (so-called "silent earthquakes") and supersonic (> v_s) rupture propagation velocities are possible (e.g., Tibi et al., 2001). Rupture velocity depends on the material properties, the internal friction of the unbroken material, the frictional conditions along the fractured surface and the stress conditions (ambient and on the crack tip) in the given case.

For the point source approximation Equation (29) takes the simpler form

$$u_i(\omega) = \mu A |D_l(\omega)| (l_k n_j + l_j n_k) G_{ik,j}(\omega) \quad (30)$$

or, in the time domain,

$$u_i(t) = \mu A (l_k n_j + l_j n_k) \int_{-\infty}^{\infty} |D_l(\tau)| G_{ik,j}(t-\tau) d\tau . \quad (31)$$

Displacements are given by temporal convolution of slip with the derivatives of the Green's function. The geometry of the source is now defined by the orientation of the two unit vectors n_i and l_i. These two vectors, which refer to the geophysical co-ordinate system of axes (North, East, Nadir), define the orientation of the source, namely n_i the orientation of the fault plane and l_i the direction of slip. These two vectors can be written in terms of the three angles that define the motion on a fault, namely, azimuth ϕ, dip δ and rake λ. The shear fracture itself is equivalent to a DC source in terms of forces (see Figure 7).

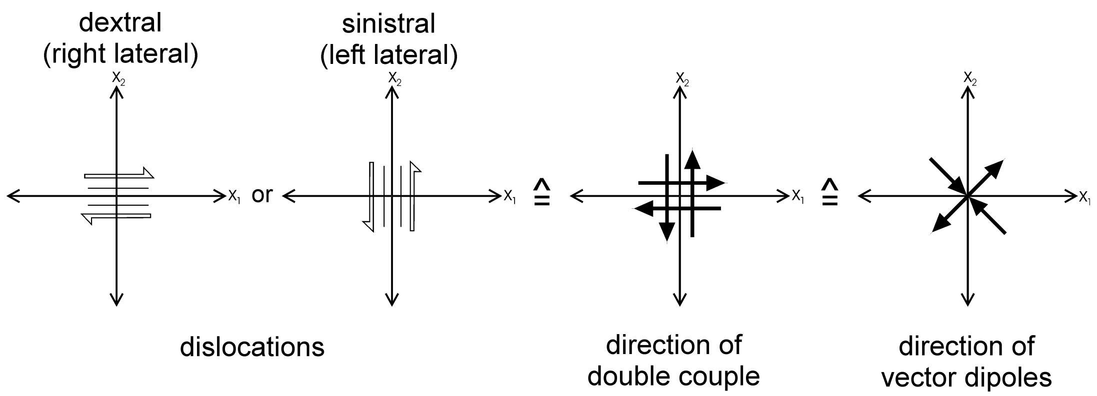

Figure 7 Depiction of the equivalence of a shear dislocation with the force double couple and the vector dipole models.

In the case that l_i and n_i are not perpendicular, Equation (29) has to be replaced by

$$u_k(x_s,\omega) = \int_S [\lambda\, \delta_{jk} n_l l_l + \mu (l_k n_j + l_j n_k) G_{ik,j}(x_s, \xi_s, \omega)] \, |D_n(\xi_s,\omega)| \, dS(\xi_s) . \quad (32)$$

The special case when l_i and n_i are parallel is often used to model tensional volcanic earthquakes (Figure 8).

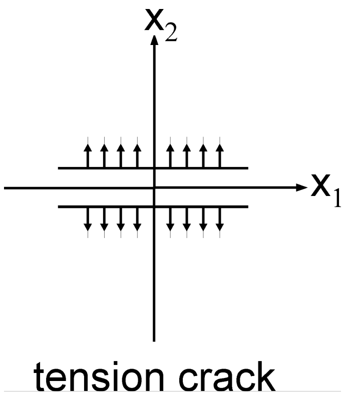

Figure 8 Illustration of a tension crack which is often used in modeling volcanic earthquakes.

Another more general representation of seismic source is given by the seismic moment tensor density m_{ij}. The moment tensor density represents that part of the internal strain drop which is dissipated in non-elastic deformations at the source. So far we have modeled the seismic source by means of the forces in the equation of motion (see Equation (1)) or by boundary conditions for the displacement (see Eqs. (17) and (28)). Now we take another approach and divide the true strain tensor ε_{ij}^{true} into an elastic and inelastic part, i.e.,

$$\varepsilon_{ij}^{true} = \varepsilon_{ij}^{ek} - \varepsilon_{ij}^{inel} . \qquad (33)$$

With this we define the true stress

$$\sigma_{ij}^{true} = \sigma_{ij} - m_{ij}^{V} \qquad (34)$$

where σ_{ij} is the elastic stress related to the strain by Equation (10) or (13) and m_{ij}^{V} is given by

$$m_{ij}^{V} = c_{ijkl}\varepsilon_{kl}^{inel} . \qquad (35)$$

Equation (35) defines the seismic moment tensor density m_{ij}^{V}. The superscript V indicates that it is a volumetric density. Rice (1980) and Madariaga (1983) denote ε_{ij}^{inel} as the stress-free strain or transformation strain, and m_{ij}^{V} as the stress glut. The seismic moment tensor M_{ij} is, thus, defined by

$$M_{ij}(\omega) = \int_{V_o} m_{ij}^{V}(x_k, \omega)\, dV(x_k) . \qquad (36)$$

The quantities m_{ij}^{V} and M_{ij} play a fundamental role in the theory of seismic sources. The relations between the different kinds of stress are shown in Figure 9. When σ_{ij} in (15) is substituted by σ_{ij}^{true} an additional force term appears on the right side. It can be interpreted as an equivalent force density f_i^{eq} or as an equivalent force F_i^{eq}

$$f_i^{eq}(x_k,\omega) = -m_{ij,j}^{V}(x_k, \omega) \quad \text{and} \quad F_i^{eq} = -\int_{V_o} m_{ij,j}^{V}(x_k, \omega)\, dV(x_k) . \qquad (37)$$

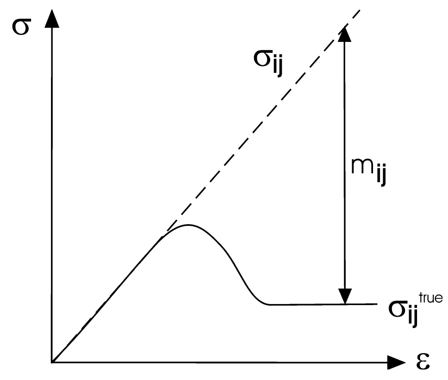

Figure 9 Relationship between the elastic stress σ_{ij}, related to the strain ε, the true stress σ_{ij}^{true} and the seismic moment density tensor m_{ij}^{V}.

In replacing the body force in Equation (19) by the equivalent force density in Equation (37) an additional volume integral $-\int_{V_o} G_{ij}\, m_{jk,k}^{V}\, dV$ appear. After an integration by parts and assuming that m_{jk}^{V} vanishes on S, i.e., the inelastic volume is bordered by *S*, the displacement produced by m_{ij}^{V} is

$$u_i(x_i, \omega) = \int_{V_o} G_{ik,j}(x_s, \xi_s, \omega)\, m_{jk}^{V}(\xi_s, \omega)\, dV(\xi_s)\,. \tag{38}$$

When comparing Eqs. (38) and (28) one realizes that the integrants have the same form but the integration in (38) is over a volume while it is over a surface in Equation (28). Accordingly, the stress glut m_{ij}^{V} is equivalent to a dislocation when the inelastic volume can be approximated by an inelastic internal surface. Naming this stress glut by m_{ij}^{S} from Equation (28) we see that

$$m_{kl}^{S} = c_{ijkl}\, D_i\, n_j \tag{39}$$

for the general linear elastic case and for the shear crack in an isotropic medium holds

$$m_{ij}^{S} = \mu (D_i n_j + D_j n_i)\,. \tag{40}$$

For the spatially averaged dislocation $\overline{D_i}(\omega)$, the seismic moment tensors M_{ij} in these two cases become

$$M_{ij}(\omega) = c_{ijkl}\, \overline{D_k}(\omega)\, n_l\, A \quad \text{and} \quad M_{ij}(\omega) = \mu\, [\overline{D_i}(\omega)\, n_j + \overline{D_j}(\omega)\, n_i]\, A\ , \tag{41}$$

respectively. In the latter case, when $\overline{D_i}$ and n_j are perpendicular, the scalar seismic moment is

$$M_0(\omega) = \mu \left| \overline{D_i}(\omega) \right| A\,. \tag{42}$$

In the general case of an arbitrary moment tensor the scalar seismic moment is defined by

$$M_0 = \sqrt{\frac{1}{2} M_{ik} M_{ik}} \, . \tag{43}$$

4 Dynamic Source Models

Dynamic source models, or crack models, use a given stress on an internal surface (fault) to describe a seismic source. In this case Equation (19) is not valid. In general, two terms must be added to the right side of Equation (19). These terms include boundary conditions for the displacement and the stress. Note that only one of these conditions can be freely chosen while the other one has to be calculated. The computation of the Green's function requires boundary conditions as well, either for the Green's function itself or for the stress produced by it. These boundary conditions do not influence the result of the computation of the displacement $u_i(x_s,\omega)$. Therefore, we can freely select any suitable boundary conditions. When selecting a Green's function which produces a vanishing stress on the internal surface S this Green's function is called G_{ij}^{free} because the related internal surface behaves like a free surface. The advantage is, that this kind of source representation does not require a knowledge of the displacement produced by the given stress on the internal surface. When no body force acts it holds that

$$u_i(x_i,\omega) = \int_S G_{ik}^{free}(x_s,\xi_s,\omega)\, n_j\, \sigma_{kj}(\xi_s,\omega)\, dS(\xi_s) \, . \tag{44}$$

Equation (44) simplifies the computation of the displacement or the dislocation on the fault when the stress on the fault is given. When using other kinds of representations an inhomogeneous integral equation for $u_i(\xi_s,\omega)$ on the fault has to be solved.

In the dynamic models the static stress drop $\Delta\sigma_{ij}$ plays an important role. It is defined as the difference between the stress distribution σ_{ij}^o on the fault plane before the occurrence of the earthquake and the stress σ_{ij}^1 after the earthquake. This static stress drop is

$$\Delta\sigma_{ij}(\xi_s) = \sigma_{ij}^o(\xi_s) - \sigma_{ij}^1(\xi_s) \tag{45}$$

with $\sigma_{ij}^1(\xi_s) = \lim_{t\to\infty} \sigma_{ij}(\xi_s,t) = \lim_{\omega\to 0} i\omega\, \sigma_{ij}(\xi_s,\omega)$. A more general time dependent stress on the fault is shown in Figure 10 (Yamashita, 1976).

A case of practical importance is that of a circular shear fault. It is probably a good approximation for small earthquakes in the Earth´s crust with magnitudes smaller than 4 as long as only frequencies < 5-10 Hz are considered. If a homogeneous shear stress drop $\Delta\sigma_{12}$ in the x_1-x_2 plane is assumed, the static dislocation on the fault is

$$D_1 = \frac{8}{\mu\pi} \frac{\lambda + 2\mu}{3\lambda + 4\mu} \Delta\sigma_{12} (R_0^2 - r^2)^{1/2} \tag{46}$$

where R_0 is the final radius of the broken fault and r the radial co-ordinate. If $r > R_0$ the dislocation in Equation (46) is zero. By inserting Equation (46) in (41) we get for the static seismic moment

$$M_0 = \frac{16}{3}\frac{\lambda + 2\mu}{3\lambda + 4\mu}\Delta\sigma_{12}\, R_0^3 \tag{47}$$

and for $\lambda = \mu$ the well known result derived by Keilis-Borok (1959) is given by

$$M_0 = \frac{16}{7}\Delta\sigma_{12}\, R_0^3 . \tag{48}$$

Similar relations hold for rectangular shear cracks of the length L and a width W:

$$M_0 = C\, L^2 W\, \Delta\sigma_{12} \tag{49}$$

where C is a model-dependent constant in the order of **1** and $\Delta\sigma_{12}$ is uniform over the fault. In the case of a buried in-plane shear crack holds

$$C = \frac{\pi}{8}\frac{\lambda + 2\mu}{\lambda + \mu} \tag{50}$$

and for a buried anti-plane case

$$C = \frac{\pi}{4} . \tag{51}$$

When the fault is perpendicular to the Earth's surface and outcropping then C in the Eqs. (50) and (51) is twice as large.

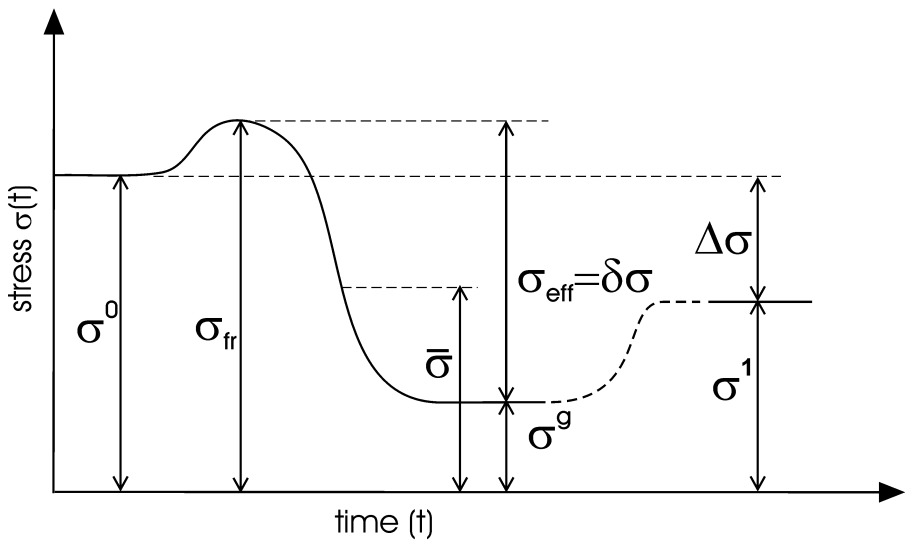

Figure 10 Time dependence of stress at a point on the fault surface during an earthquake. σ^o and σ^1 – stress before and after the earthquake, σ_{fr} – fracture strength, $\overline{\sigma}$ - mean stress,

σ^g – friction stress, σ_{eff} – effective stress = dynamical stress drop $\delta\sigma$ and $\Delta\sigma$ - static stress drop.

The dynamic relation between the shear stress drop $\Delta\sigma$ and the dislocation can be calculated numerically. An example is shown in Figure 11. The rupture starts at t=0 and r=0 and expands with constant velocity. The time t and the dislocation $|D_1(r,t)|$ are normalized to R_0/V_P and $\Delta\sigma R_0/\mu$ where V_p is the velocity of the P wave ($V_p^2 = (\lambda + 2\mu)/\rho$ with ρ as the density of the medium).

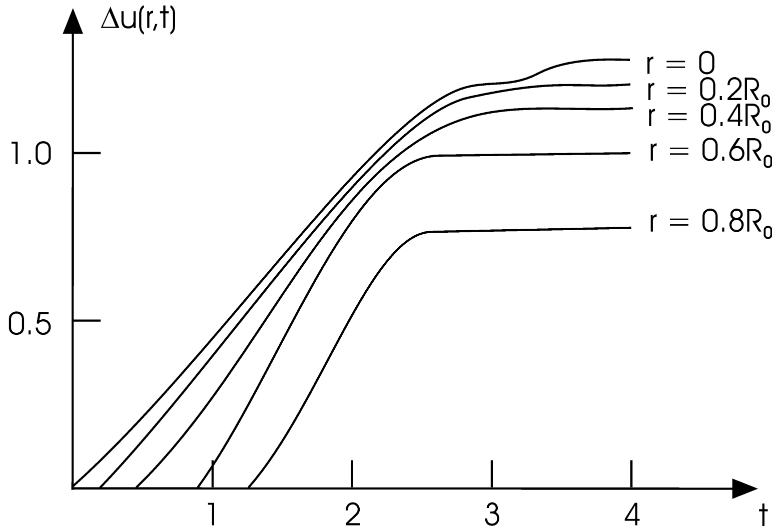

Figure 11 Dislocation function D(r, t) at several distances from the center on the circular crack plotted against the normalized time t. For explanation of symbols see text (according to Madariaga, 1976; modified from Aki and Richards, 1980).

5 Energy, Moment, Dislocation and Stress drop

The radiated energy of an earthquake can be computed assuming a specific source model and its source parameters. We describe the earthquake as a shear rupture on a surface. In a relatively general form Kostrov (1975) writes for the radiated seismic energy E_S

$$E_S = \int_0^{t_{\max}} dt \int_{S(t)} dS(\xi_k)\,(\sigma_{ij}^0 - \sigma_{ij})\,\dot{D}_i\, n_j - \frac{1}{2}\int_A dS\,\Delta\sigma_{ij}\, D_i^0 n_j - \int_A g\, dS \tag{52}$$

where t_{max} is the maximum duration of the motion on the fault plane, $S(t)$ the rupture plane developing during the rupture, A the final rupture plane with $A = lim_{t\to\infty}\ S(t)$, $\sigma_{ij}^0(\xi_s)$ the stress before the earthquake occurred, $\sigma_{ij}(\xi,t)$ the stress on the broken fault surface, $\dot{D}_i(\xi_s,t) = \partial/\partial t\, D_i(\xi_s,t)$ the dislocation velocity, $\Delta\sigma_{ij}(\xi)$ the static stress drop (see Figure 10), n_j the normal vector of the fault surface, $D_i^O(\xi_s)$ the static dislocation, and g the specific energy required to generate a new surface. Equivalent to Equation (52) is the often used form

$$E_S = \int_0^{t_{\max}} dt \int_{S(t)} dS\, \dot{D}_j\, n_j(\overline{\sigma}_{ij} - \sigma_{ij}) - \int_A g\, dS \tag{53}$$

where $\overline{\sigma_{ij}} = (\sigma_{ij}^0 + \sigma_{ij}^1)/2$ denotes the mean stress, σ_{ij}^0 is the stress before the earthquake, and σ_{ij}^1 is the final stress, which may be equal to the frictional stress. When taking into account the grow of the rupture area during the earthquake in the formulation of the dislocation (source time) function $D_i(t)$, Equation (53) becomes

$$E_s = \int_A dS \int_0^{D_i^f} dD_i\, n_j\, [\overline{\sigma}_{ij} - \sigma_{ij}(D_k)] - \int_A g\, dS \tag{54}$$

where $\sigma_{ij}(D_k)$ is the stress-dislocation relation on the fault plane and D_i^f the final dislocation.. In the Eqs. (52) to (54) the seismic energy E_S is composed of released deformation energy E_{tot}, frictional energy E_f, and rupture (crack) energy E_r

$$E_s = E_{tot} - E_f - E_r \tag{55}$$

with

$$\begin{aligned} E_{tot} &= \int dS\, \overline{\sigma}_{ij} D_i\, n_j \\ E_f &= \int dS \int dD_i\, n_j \\ E_r &= \int g\, dS \end{aligned} \tag{56}$$

With this we define the seismic efficiency η

$$\eta = \frac{E_S}{E_{tot}} \tag{57}$$

and the apparent stress σ_{app}

$$\sigma_{app} = \frac{1}{A\left|\overline{D_i}\right|} \int e\, dS \tag{58}$$

where $\left|\overline{D_i}\right|$ is the spatial averaged absolute value of the dislocation. The energy density e is identical with the integrant of the surface integral. Therefore the following relation between the seismic energy and the scalar seismic moment holds:

$$E_S = \sigma_{app} M_0 / \mu \tag{59}$$

Further special cases are:

a)

σ_{ij}^0, σ_{ij}, σ_{ij}^1 are homogeneous and σ_{ij} equal to the time-independent friction stress σ_{ij}^g Eqs. (3), (6) and (7) yield

$$E_S = (\overline{\sigma}_{ij} - \sigma_{ij}^g)\, D_i^f\, n_j\, S_0 - E_r \tag{60}$$

$$\eta = \frac{(\sigma_{ij}^0 + \sigma_{ij}^1 - 2\sigma_{ij}^g) e_i n_j - 2\bar{g}}{(\sigma_i^0 + \sigma_{ij}^1) e_i n_j} \tag{61}$$

$$\sigma_{app} = \frac{1}{2}(\sigma_{ij}^0 + \sigma_{ij}^1 - 2\sigma_{ij}^g) e_i n_j - \frac{\bar{g}}{|\bar{D}_i|} \tag{62}$$

where $\bar{g}$ is the averaged specific rupture energy and e_i a unit vector in the direction of the dislocation. With this we get

$$\sigma_{app} = \eta \quad \bar{\sigma}_{ij} \;\; e_i \; n_j. \tag{63}$$

b)

For a shear fracture, and $\sigma_{ij}^g = \sigma_{ij}^1$ with $g \approx 0$ as an approximation or $g = 0$ in the case of an *anti-plane* brittle rupture propagating with shear-wave velocity or of an *in-plane* brittle rupture propagating with Rayleigh-wave velocity, respectively, we get

$$E_s = \frac{1}{2} D_i^f n_j \Delta\sigma_{ij} S_o = \frac{1}{2\mu} \Delta\sigma_{ij} M_{ij} \tag{64}$$

with $$\Delta\sigma_{ij} = \sigma_{ij}^0 - \sigma_{ij}^1. \tag{65}$$

Ohnaka (1978) gives the following relationship for the seismic energy of a circular shear fracture propagating with the crack velocity $v_c = 0.8\ v_s$:

$$E_s = \frac{M_0 \overline{D_0}}{2R} \tag{66}$$

with M_o – scalar seismic moment, $\overline{D_0}$ - static averaged dislocation and R – source radius. For rectangular shear fractures of length L and with unilateral fracture propagation a similar approximate relationship holds:

$$E_s \approx \frac{M_o \overline{D_0}}{3L} \tag{67}$$

and in case of partial incoherence

$$E_s \approx \frac{M_o \bar{D}}{L}. \tag{68}$$

Further, E_s can be determined directly by integrating over the displacement field. It holds

$$E_s = \sum_k \int_{-\infty}^{\infty} dt \int_S dS\, \rho\, v^{(k)} \dot{u}_i^{(k)} \dot{u}_i^{(k)} \quad (69)$$

with S – a surface surrounding the source, ρ – density distribution on this surface, $\dot{u}_i^{(k)}$ – velocity of ground motion. The sum is over all kinds of waves which leave the volume enclosed by the surface S with the velocity $v^{(k)}$. However, one has to take into account that on the way from the source to S part of the energy has already been transformed into heat by inelastic effects of wave propagation.

Equation (69) forms the theoretical background for the simple relationship between seismic energy and magnitude M

$$\log E_s = \mathrm{a\ M + b} \quad (70)$$

which is based on rather simple assumptions. Nevertheless, the corresponding relationship given by Gutenberg and Richter (1956) is

$$\log E_s[\mathrm{J}] = 1.5\ \mathrm{M_s} + 4.8 \quad (71)$$

with M_s – surface wave magnitude (see 3.2.5.1). Equation (71) has proven to yield rather good estimates of E_s. More details on direct energy determination based on digital broadband recordings is outlined in 3.3.

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References (see References under Miscellaneous in Volume 2)