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High resolution spherical and ellipsoidal harmonic expansions by FFT

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Abstract

High resolution transformations between regular geophysical data and harmonic model coefficients can be most efficiently computed by fast Fourier methods (FFT). However, a prerequisite is that the data grids are given in the appropriate geometrical domain.

For example, if the data are situated on the ellipsoid at equi-angular reduced latitudes, spherical harmonic analysis can be employed and the coefficients subsequently converted by Jekeli's transformation. This results in the spherical harmonic spectrum in the domain of geocentric latitudes.

However, the data are most likely given at geodetic (ellipsoidal) latitudes which means that the FFT base needs to be shifted by latitude dependent phase lags in order to obtain the correct spherical harmonic spectrum. This requires appropriate sample rate conversion about the shifted latitudes by means of Fourier summation and cannot be treated efficiently by an FFT algorithm.

In this article another solution is discussed instead.

Since the variable heights between the spherical and ellipsoidal surfaces can be accurately approximated by a series of Tschebyshev polynomials, they can be convolved into the spherical basis. It will be shown how this new type of fast Fourier transformation to and from the ellipsoid in combination with Jekeli's conversion of the spectra between the two surfaces allowes eventually the sample rate conversion to shifted latitudes. This avoids the inexpedient Fourier summation mentioned previously.

In this paper three applications for FFT in the domain of spherical and ellipsoidal surfaces, and using geocentric, reduced and geodetic latitudes are discussed. The Earth gravitational model EGM2008 of 5 arcminutes resolution has been used to demonstrate numerical results and computational advantages.

Key words: 2-D Fourier expansion · gravity anomalies · spherical harmonics · ellipsoidal harmonics · spectral transformation · Tschebyshev transformation · spherical and ellipsoidal harmonic analysis and synthesis · Jekeli transformation ·

1 Introduction

During the last decades the growing demand in Earth sciences to model geophysical processes in unprecedented resolution provides the need for reliable and efficient computational algorithms. Spherical harmonic transformation between space and frequency domains (both analysis and synthesis) are of high practical interest in geodesy when applied to globally distributed physical quantities such as gravity. By employing the FFT, the computational burden can nowadays be easily implemented on an ordinary desktop computer system, despite the improved resolution of latest and forthcomming gravitational models and their increased parameter extention.

The spherical harmonic series are based on the solution of the Laplace equation by the associated Legendre functions. Their general applicability is limited by the numerical stability of the algorithms that are used to generate these special functions. Stable computations can be achieved by either choosing extremely large floating point numbers with extended mantissa or by controlling them during computation, into the numerical range of double precision numbers, (e.g. Fukushima, 2012) [18]. Following Seljebotn (2012) [16] where the Legendre Transform for synthesis is discussed, underflow values for the associated Legendre Functions can be safely neglected (from $P_{l|m|} < 10^{-30}$) if the dynamic range of the input data is small enough. Other numerical libraries can be found (e.g. Mohlenkamp, 2000 [14]) but it is often not clear to what extend the expansions are rigorous or approximate. Some authors truncate for certain latitudes of higher spherical harmonic orders.

If analysis or synthesis of regular, equi-angular values is applied, the Fourier domain is useful not only for the transformation of the data but for the employed base functions as well. Mantissa under- and overflow for each frequency can be handled then individually, (see Gruber, 2011) [8]. This is impeded by an independent scaling of the constants during their computation.

Standard recursive algorithms for the associated Legendre functions pass the entire signal bandwidth to the successive computational step and are therefore difficult to scale in double precision.

The performance and stability of the FFT algorithm concerning spherical harmonic expansions are outstanding and numerical double precision accuracy can be maintained to highest resolution. Computer systems and compilers are generally optimized for this type of numbers and operations, and so the majority of written programs are compliant with them. Since an external rescaling factor can be conveniently introduced to the significand in case of an exponent under-/overflow we can easily avoid computation in the extended precision domain.

The computation of the respective Fourier coefficients belonging to a given spherical or ellipsoidal harmonic model is performed by the analysis (2D FFT) of the discrete data. In a subsequent step, the Fourier coefficients are transformed into spherical (or ellipsoidal) harmonic coefficients by least squares (e.g. Colombo, 1981 [3]) or numerical integration (Sneeuw and Bun, 1996 [17]). Of course, the point-wise, discretized data and the fast Fourier methods applied are only an approximate solution to the continuous spherical harmonic transformation that is defined by complete surface integrals, (e.g, Hwang et al., 2005 [13]) but with increasing resolution, this discretization error evenly diminishes.

By computing the synthesis (inverse 2D FFT) of linearly assembled spherical harmonic coefficients (known as: lumped harmonic coefficients), they are transformed into globally distributed equi-angular gravity data of high resolution onto the sphere or ellipsoid. Here, we consider the ellipsoid as a biaxial body or surface of revolution that resembles thus an oblate spheroid. In Gruber et. al (2011b)[7] it has been shown that this formulation remains stable even in very high resolution as well as efficient when using state-of-the-art shared memory and multicore architectures. Abrykosov and Förste (2012 [1]) showed that FFT can be used partially for the transformation of global gravity anomalies given in geodetic latitudes but the spectra has to be shifted to reduced latitude which requires direct synthesis by means of a Fourier summation.

In the sequel we discuss three different processing examples where emphasis is placed on the processing of data located in geocenteric and geodetic latitude. We will show how Jekeli's (1988) [15] transformation in combination with a Tschebyshev approximation of the continuation term can be applied for an implicit transformation of the grids.
For this purpose we will first commit an independent coefficient recovery using the $[5 \times 5]$ arcmin global gravity field model computed by external software (HARMONIC SYNTH, see Holmes & Pavlis, [12]) in geocenteric coordinates on the ellipsoid. Second, the same will be done in regular reduced latitude positions enabling the direct analysis on the ellipsoid by applying Jekeli's spectral transformation and third, this will be repeated for data centered in regular geodetic coordinates. The last recovery can be achieved through a combination of the two previous methods thereby offering new applications and transformations by FFT between regular data grids.

Fig. 1 shows the gravity anomaly field that will serve as a reference data set.

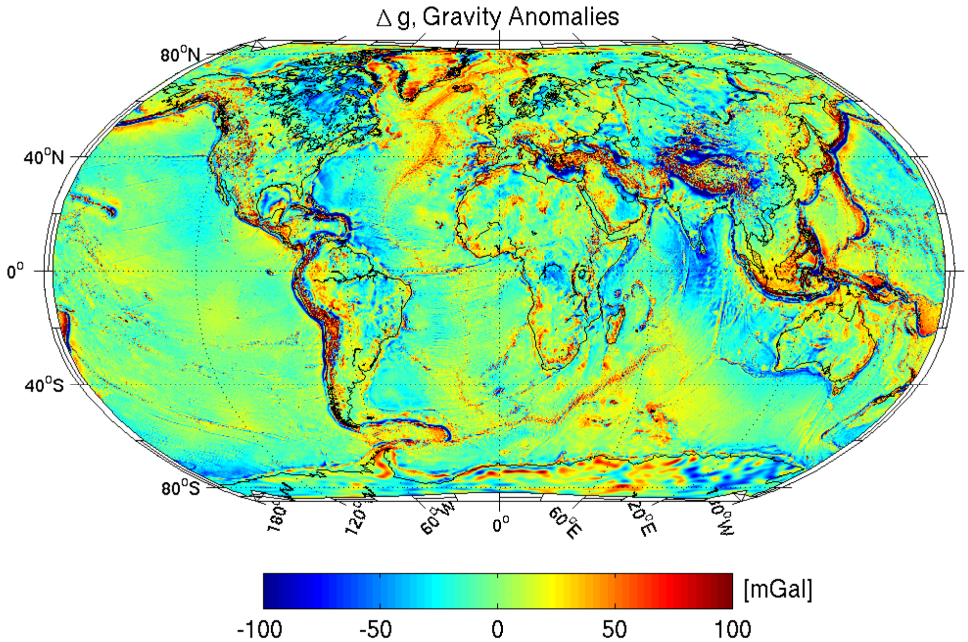

Figure 1: Test data that has been used: globally distributed [5 × 5] gravity anomalies according to EGM2008. Colorbar range has been limited to $\pm 100m$Gal for better visibility (peak values are 10 × higher).

2 Spherical transformation

Any harmonic function on the sphere, $f(\theta, \lambda, r)$, can be expanded into a series of solid spherical harmonics

$$f(\theta, \lambda, r) = \sum_{l=0}^{\infty} \frac{R^l}{r^{l+1}} \sum_{m=-l}^{l} \hat{c}_{l,m} \hat{Y}_{l,m} \tag{1}$$

where R is the radius to the reference sphere, the complex base functions $\hat{Y}_{l,m}$ are fully normalized surface spherical harmonics and $\hat{c}_{l,m}$ are the respective normalized spherical harmonic coefficients. The triplet (θ, λ, r) defines the spatial position by geocentric radius r, spherical co-latitude θ and longitude λ counted positive eastwards. The complex functions $\hat{Y}_{lm}$ are fully normalized surface spherical harmonics, obtained by

$$\hat{Y}_{l,m} = \sqrt{\frac{(2l+1)(2-\delta_m^0)}{(l-m)!(l+m)!}}\; P_{l\,|m|}(\cos\theta)\;\exp(im\;\lambda)\;, \quad \delta_m^0 = \begin{cases} 1 \quad \text{if} \quad m=0 \\ 0 \quad \text{else}\;, \end{cases} \tag{2}$$

with the associated Legendre Functions $P_{l\,|m|}(\cos\theta)$.

In Gruber et al. (2011) a compact expression for the spherical function

$$f(\theta,\lambda,r) \;=\; \sum_{l}^{\infty}\sum_{m=-l}^{l}\sum_{k=-l}^{l} \frac{R^l}{\bar{r}^{l+1}}\;\exp\left[\;i(k\;\theta+m\;\lambda)\;\right]\;\hat{q}_{lmk}\;,\quad \bar{r}=const, \tag{3}$$

has been introduced with the transformed coefficients

$$\hat{q}_{lmk} \;=\; \hat{A}_{lmk}\;\hat{c}_{lm}\;,\quad k=\;-l,\ldots,l\ \text{step}\ 2, \tag{4}$$

obtained from a Fourier expansion of the associated Legendre functions (Hofsommer and Potters 1960 [11]),

$$\sqrt{\frac{(2l+1)(2-\delta_m^0)}{(l-m)!(l+m)!}}\;P_{l\,|m|}(\cos\theta) \;=\; \sum_{k=-l\,(2)}^{l}\hat{A}_{lmk}\;\exp(ik\;\theta)\;. \tag{5}$$

For more details on the arithmetic for $\hat{A}_{lmk}$, refer to Gruber (2011)[8]. As we are now interested in those cases, where $\bar{r} \neq const.$ in Eq. (3), such as along the surface of an ellipsoid, we approximate the radial continuation by truncating a Tschebyshev expansion of the type

$$\left(\frac{R}{r}\right)^{l+1} = \sum_{n=0}^{l} t_n^{(l)}\;T_n(x),\quad T_n(x)=\cos(n\;\arccos x),\quad t_n^{(l)}=0\quad\forall\quad n-\text{odd} \tag{6}$$

and

$$x=\cos(\theta),\quad \left(\frac{R}{r}\right) = \sqrt{\frac{1-e^2\;\sin^2\theta}{1-e^2}}\;,\quad \theta\in[0,\pi]\;, \tag{7}$$

with $e^2 = (a^2-b^2)/a^2$ being the squared first eccentricity of the ellipsoid of revolution with semi -major and -minor axis a, b , and $t_n^{(l)}$ are the polynomial coefficients, differing for each degree. Substituting the transformed continuation into Eq. (3) leads to

$$f(\theta,\lambda,r) = \frac{1}{R}\sum_{l}^{\infty}\sum_{m=-l}^{l}\sum_{k=-l}^{l}\sum_{n=0}^{l}\left[t_n^{(l)}\cdot T_n(\cos\theta)\right]\;\hat{q}_{lmk}\;\exp\left[i(\mathrm{k}\;\theta+\mathrm{m}\;\lambda)\right]\;. \tag{8}$$

Our intention is then to first convolve the $t_n^{(l)}$ with the $\hat{A}_{lmk}$ since they belong to the same trigonometric function in the Fourier basis. This is achieved with $\bar{A}_{lmk}$'s, converted into real notation

$$\bar{A}_{lmk} \;=\; \hat{A}_{lmk} \cdot i^{\,\mathrm{mod}\,(l-m,2)} \cdot (2\;\delta^0_{[k/2]} - 1)\;, \tag{9}$$

$$2\delta^0_{[k/2]} - 1 \;=\; \begin{cases} 1 & \forall \quad \mathrm{mod}\,([k/2],2) = 0 \\ -1 & \mathrm{else} \quad , \end{cases} \tag{10}$$

leading to modified coefficients

$$\bar{A}^{\&}_{lmk} \;=\; \sum_{n=-l}^{l} \frac{1}{2} t^{(l)}_{|n|} (1 + \delta^0_n) \cdot \bar{A}_{l,m,k-n}\;, \tag{11}$$

where

$$\bar{A}_{l,m,k-n} = 0 \quad \forall \quad |k-n| > l \tag{12}$$

and

$$k \;=\; \begin{cases} -2l, \ldots, 2l & \text{step } 2 \quad \forall \quad l - \text{even} \\ -2l+1, \ldots, 2l-1 & \text{step } 2 \quad \text{else}\;. \end{cases} \tag{13}$$

After the reverse transformation

$$\hat{A}^{\&}_{lmk} \;=\; \bar{A}^{\&}_{lmk} \cdot (-i)^{\,\mathrm{mod}\,(l-m,2)} \cdot (2\;\delta^0_{[k/2]} - 1)\;, \tag{14}$$

we obtain *continued* normalized associated Legendre Functions,

$$\bar{P}^{\&}_{l\,|m|}(\cos\theta) \;=\; \sum_{k=-2l}^{2l} \hat{A}^{\&}_{lmk}\;\exp(ik\;\theta)\;. \tag{15}$$

We can then write Eq. (8) as

$$f(\theta,\lambda,r) = \frac{1}{R} \sum_{l}^{\infty} \sum_{m=-l}^{l} \hat{c}_{lm} \bar{P}^{\&}_{l|m|} \exp(im\;\lambda)\;. \tag{16}$$

The well known spherical approximation for gravity anomalies, (e.g. Heiskanen & Moritz 1967 [9]) limited to a specific bandwith $\{l \mid 0 < l \leq L\}$

$$\Delta g = \frac{\mathrm{GM}}{\mathrm{R}^2} \sum_{l=2}^{L} (l-1) \left(\frac{\mathrm{R}}{r}\right)^{l+2} \sum_{m=-l}^{l} \hat{c}_{l,m} \; \hat{Y}_{l,m}(\theta, \lambda) \; , \tag{17}$$

with GM the universal gravitational constant times mass of the Earth, is thus constructed by the adopted surface spherical harmonics

$$\begin{aligned} \Delta g &= \frac{\mathrm{GM}}{\mathrm{R}^2} \sum_{l=2}^{L} (l-1) \sum_{m=-l}^{l} \hat{c}_{l,m} \; \hat{Y}^{\&}_{l,m}(\theta, \lambda) & (18) \\ \hat{Y}^{\&}_{lm}(\theta, \lambda) &= \bar{P}^{\&}_{l|m|} \exp(im \; \lambda) & (19) \\ &= \sum_{k} \hat{A}^{\&}_{lmk} \; \exp[i \; (k\theta + m\lambda)] & (20) \\ &= \sum_{k} (\hat{A}_{l,m} * t)_k \cdot \exp[i \; (k\theta + m\lambda)] & (21) \\ t : \left(\frac{\mathrm{R}}{r}\right)^{l+2} &= \sum_{n=0}^{l} t_n^{(l)} \cdot T_n \; . & (22) \end{aligned}$$

that contain radial continuation and can be used for computations in the spectral domain in the same way as the solid spherical harmonics.

Next, we briefly outline the computational steps necessary to recover given data computed from the Earth gravitational model EGM2008 where gravity anomalies Δg are aligned in regular geocentric latitudes on the ellipsoid (or are to be computed):

▷ perform a degree-wise Tschebyshev transformation of the continuation

$$\sum_{n=0}^{l} t_n^{(l)} \; T_n(\cos\theta) = \left(\frac{1 - e^2 \sin^2\theta}{1 - e^2}\right)^{\frac{l}{2} + 1} \tag{23}$$

In practise only a few terms are necessary as convergence is quick. The following serves as a rule of thumb: $\max(n) \approx 2\sqrt{l}$.

- ▷ convolve with the Fourier coefficients of the associated Legendre functions (cf. Eq. 11) to obtain the adopted surface spherical harmonics $\hat{Y}^{\&}_{l,m}$. In Fig. 2 the spectral "fingerprints" for both Fourier coefficients of the associated Legendre Functions before and after convolution with the continuation term are illustrated. It can be observed that further frequencies arise and hence additional harmonic degrees have to be introduced in order to preserve a defined upper bandwith L for the solid spherical harmonic expansion. This is in full anology to the extension of the coefficients when Jekeli's transformation of the spectra is applied (see section 3 below).

- ▷ forward computation: assemble 2-D harmonic coefficients from the adopted base functions and apply an inverse FFT to obtain grid values. Refer to Gruber et al. (2011b) for details.

- ▷ perform FFT analysis of the given regular gravity data in order to obtain 2-D harmonic coefficients.

- ▷ complete the transformation by Least squares spherical harmonic coefficient estimates with the adopted surface spherical harmonics from the 2-D harmonic coefficients. Refer to Colombo (1981) for details.

After performing the described transformation steps the resulting spherical harmonic coefficient errors are compared in Fig. 3 on the level of individual coefficients $\left|\hat{c}_{l,m} - \hat{c}^{\circ}_{l,m}\right|$, degree variances $\nu_l = \sqrt{\frac{1}{2l+1}\sum_{m=-l}^{l}\left|\hat{c}_{l,m} - \hat{c}^{\circ}_{l,m}\right|^2}$, and order variances $\zeta_m = \sqrt{\frac{1}{2(L-m)+1}\sum_{l=m}^{L}\left|\hat{c}_{l,m} - \hat{c}^{\circ}_{l,m}\right|^2}$.

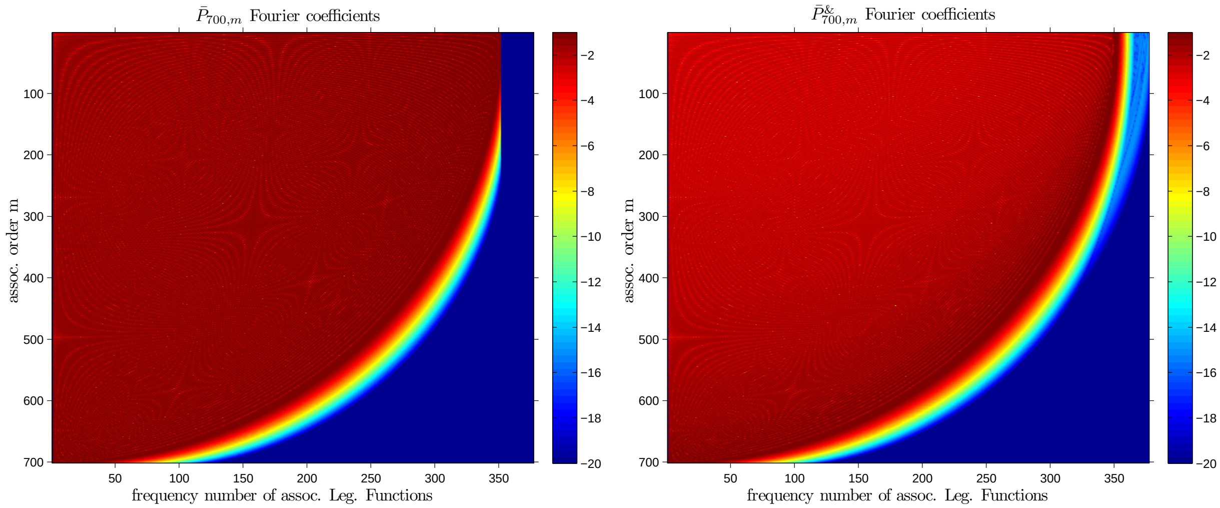

Figure 2: Amplitudes for Fourier coefficients of the associated Legendre Functions for $l = 700$, $0 \leq m \leq 700$, $0 \leq p \leq 350$. Left: original band of the positive spectra to an upper frequency of $k = 2p$. Right: continued positive spectra in the adopted (ellipsoidal) basis showing additional frequencies to arise, mainly in low orders. Substantially, only a small extension of the frequency range suffices. Figures are condensed to non-zero (positive) frequencies for better legibility.

3 Ellipsoidal transformation

Data are now given in equi-angular reduced latitudes (β) on the ellipsoid or shall be computed there. We confine our derivation to the case where data is located on the surface of the ellipsoid, with semi-major and -minor axes a, b

$$f(\frac{\pi}{2} - \beta, \lambda, b) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Q_{l\,|m|}(i\frac{b}{E})\, \hat{c}_{l,m} \hat{Y}_{l,m} \ , \tag{24}$$

where E is the linear eccentricity $E^2 = a^2 - b^2$. $Q_{l\,|m|}$ are the Legendre Functions of second kind. For more general details on the theory of spherical and ellipsoidal harmonics refer to Hobson (1955) [10]. In Jekeli (1988) the theory was elaborated to directly transform between ellipsoidal and spherical spectra. Note, that the transformation of a truncated series in one domain leads to an infinite number of degrees in the other domain, while the maximum order is preserved. In practice this concerns only few additional degrees and poses no numerical restriction concerning

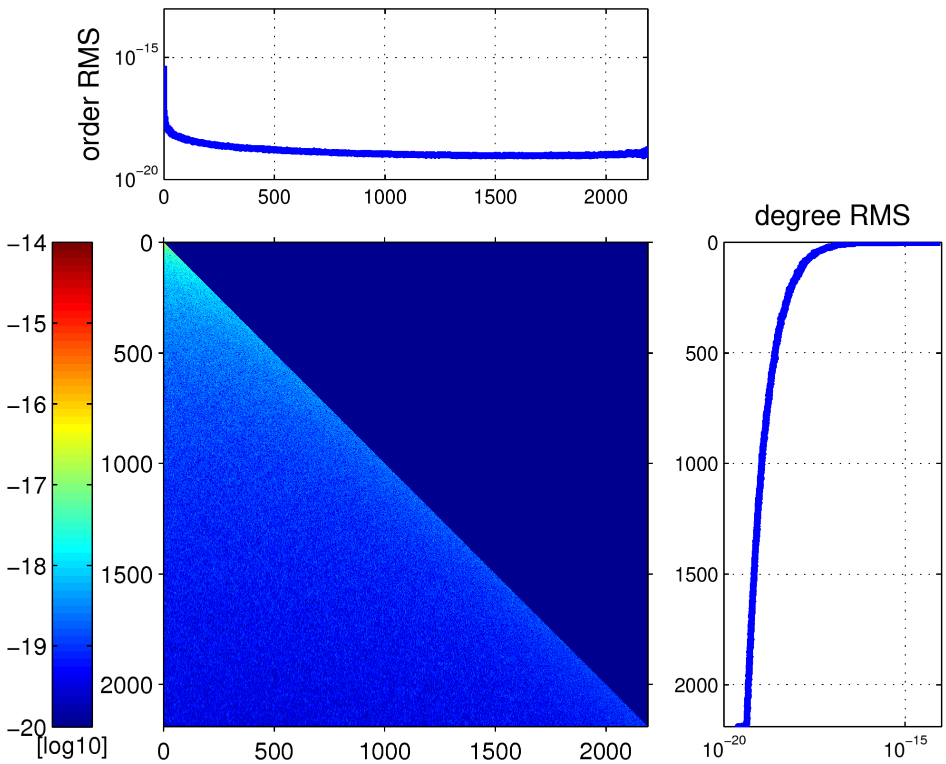

Figure 3: Closed loop comparison to EGM08 gravity field coefficients: FFT result from geocentric gravity anomalies computed on the surface of an Earth ellipsoid applying the Tschebyshev transformed continuation to the surface spherical harmonics. Degree and order variances are computed from dimensionless coefficients.

convergence and comparability of the solutions. As was done for EGM2008, the maximum degree for the computed output spectrum is 30 degrees (n) higher than that of the input spectrum. The computational steps to be considered are,

- ▷ FFT data analysis of the regular gravity grid leading to 2-D harmonic coefficients and subsequent spherical harmonic coefficient estimation (cf. section 2) resulting in $\hat{c}^{\varepsilon}_{l,m}$.
- ▷ Jekeli's (re-)transformation of the ellipsoidal spectrum to the spherical one $\hat{c}^{\varepsilon}_{l,m} \longrightarrow \hat{c}_{l,m}$.

For practical reasons, the surface spherical harmonic expansion for the ellipsoidal gravity anomalies in reduced latitude is either performed by

$$r\Delta g = \mathrm{GM} \sum_{l=0}^{L+30} (l-1) \sum_{m=-l}^{l} \hat{c}^{\varepsilon}_{l|m|} \hat{Y}_{l|m|}(\pi/2-\beta,\lambda) \ , \tag{25}$$

or by the corresponding solid spherical harmonic series, now in θ

$$r\Delta g = \mathrm{GM} \sum_{l=0}^{L+30} (l-1) \frac{R^l}{r^{l+1}} \sum_{m=-l}^{l} \hat{c}_{l|m|} \hat{Y}_{l|m|}(\theta,\lambda) \ , \tag{26}$$

rather than by the expansion in ellipsoidal harmonics themselves. Note that no continuation term in Eq. (25) anologuous to Eq. (24) is required, but the ellipsoidal radius r has been applied to the gravity data in order to convert them into a harmonic function. For the details, refer to (Gleason 1988 [5]). The relation

$$\phi = \arctan\left(\tan\beta \cdot \sqrt{1-e^2}\right) \ , \quad \phi = \pi/2 - \theta \tag{27}$$

connects reduced and geocentric coordinates.

Jekeli's (re-)transformation was implemented from Gleason's (ibid.) revised algorithm and shows that it should be processed with quad-precision numbers in order to recover machine epsilon (double) precision. This explains few missing digits in our recovery (Fig. 4).

Note that quad precision needs to be applied only to a small sequence of the algorithm and is not critical for this study. For higher resolution it can be easily considered and does not affect the general FFT concepts being discussed.

It needs to be pointed out that reduced latitudes have not changed during this transformation, although from a spherical harmonic perspective the solution on the ellipsoid now corresponds to geocentric latitudes which are not equi-angular any more, as shown by comparing Eqs. (25) and (26). It is important to keep this in mind in order to understand the combined approach following in the next section.

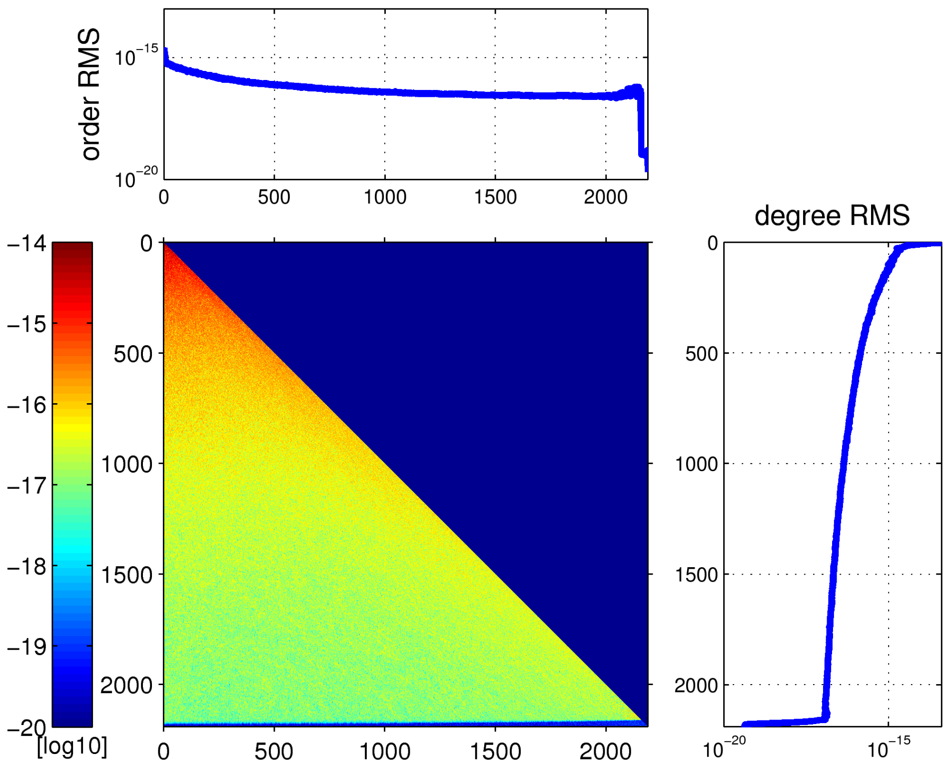

Figure 4: Misclosures in recovered spherical harmonic coefficients from reduced latitude spacing by applying Jekeli's transformation to the ellipsoidal spectra. Degree and order variances are computed from dimensionless coefficients.

4 The combined transformation

The main idea that is discussed in this paper is to rigorously transform data on graticules based in equi-angular geodetic latitudes on the ellipsoid to spherical harmonic coefficients. This is a very common case, concerning not only gravity anomalies but also spaceborn radar topography data, GPS/ Levelling data, and numerous others. It is shown how the combination of the previous two methods leads to appropriate results that are useful for practical applications, such as the derivation of topographic potential from global DEM models (Gruber 2014 [6]). There exist also methods for evaluating Fourier series at non-equispaced points, so that the time-consuming direct synthesis by Fourier summation can be avoided (cf. Dutt and Rokhlin, 1993[4], Beylkin 1995[2]) but the user has to adopt the filtering for the individual purpose and the dynamical range of the data.

The transformation sequence reads as follows: data given on the ellipsoid are transformed to the sphere by a first Jekeli transformation. Then, they are implicitly downward continued by Tschebyshev modified base functions and re-analyzed. Eventually a second Jekeli transformation completes the transformation. More in detail:

- ▷ initial FFT analysis of the given data to obtain 2-D harmonic coefficients and subsequent estimation of the ellipsoidal spectra.
- ▷ initial Jekeli transformation to the bounding sphere. As the reduced latitudes (section 3) now correspond to the equi-angular geodetic latitudes the subsequent downward continuation needs to be done in this domain.
- ▷ convolution of the spectral coefficients for the Legendre Functions , Eq. (11) to obtain the adopted surface harmonics (section 2).
- ▷ FFT synthesis (cf. forward, section 2) with implicit harmonic continuation to the ellipsoidal surface to obtain data on the ellipsoid in reduced latitude.
- ▷ second FFT analysis of the shifted data to estimate the ellipsoidal spectra (now in the domain of reduced latitudes).
- ▷ second Jekeli transformation to the bounding sphere, finalizing the coordinate/data transformation.

In Fig. 5 the resulting spectral coefficients are again compared to the initial coefficient set. The described procedure demonstrates how the individual steps accumulate the respective errors towards the final solution in Fig. 6. The largest error contribution to the combined solution (B) stems from the spectral transformation that is applied twice. See also comments on Gleason's algorithm in the previous section. However, the current solution meets precision requirements and can be used for the transformation of gravity anomalies at the given resolution. See also Fig. 7 for the global distribution of the spatial error budget for the combined solution with min$= -6.9 \times 10^{-3}$mGal and max$= 10.5 \times 10^{-3}$mGal.

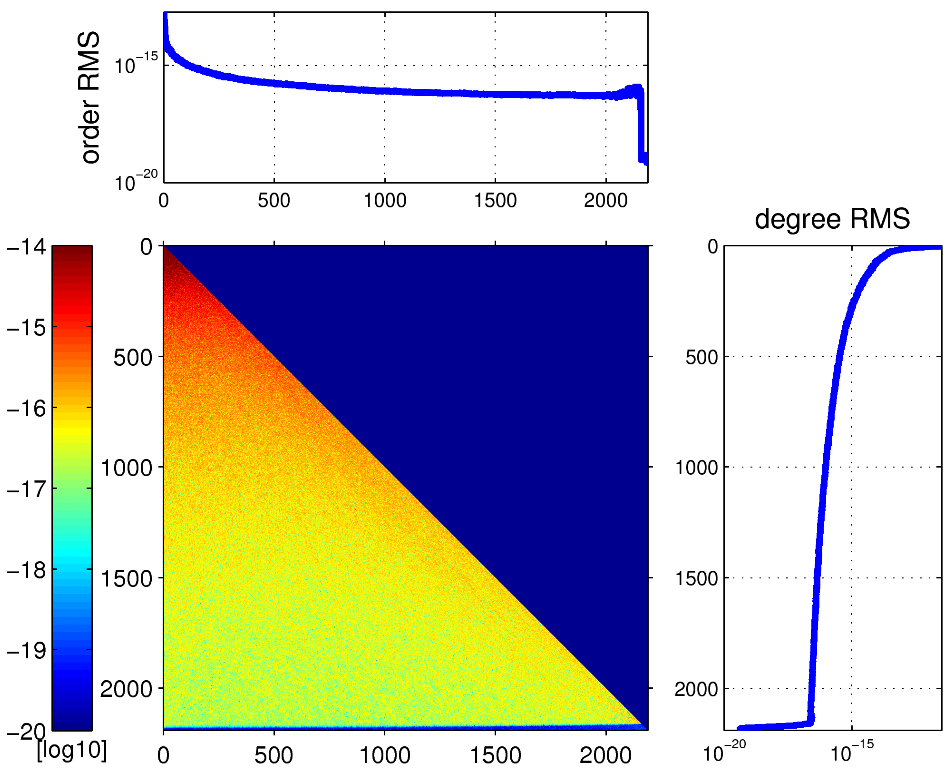

Figure 5: Residual coefficient results for the combined solution. Degree and order variances are computed from dimensionless coefficients.

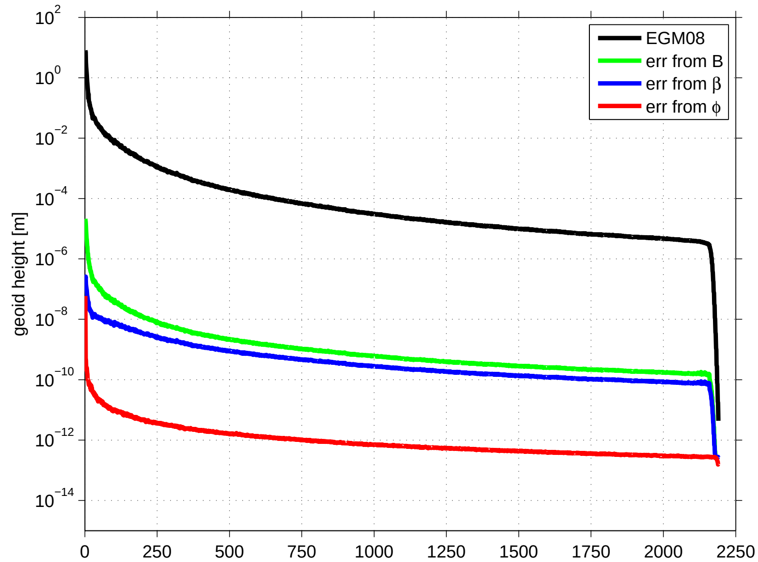

Figure 6: Spectral error budgets of gravity field recoveries

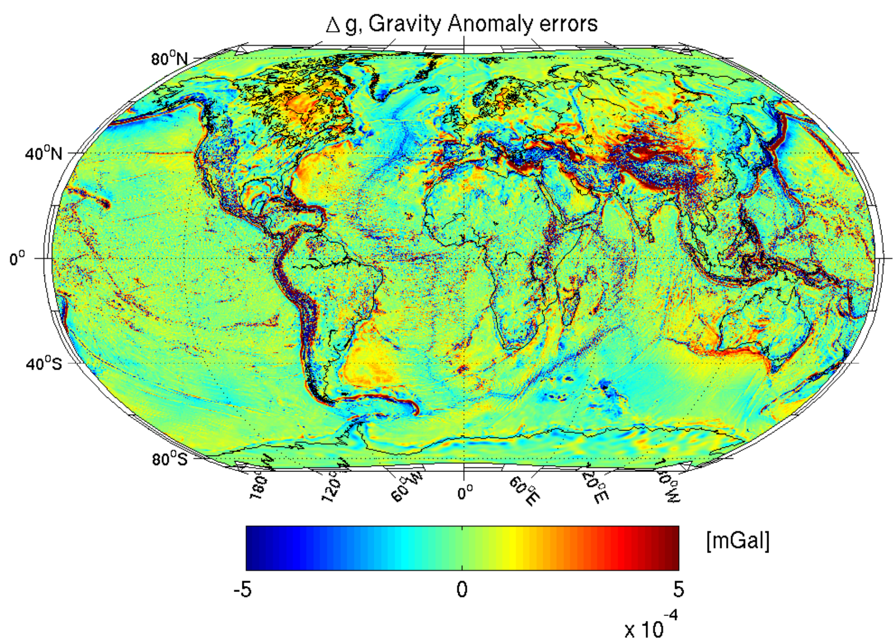

Figure 7: Spatial errors in mGal, global error $\sigma = 0.196\mu$Gal

5 Conclusions

A practical solution for the fast Fourier transformation of ellipsoidal data in geocenteric latitudes has been introduced. It can be used in conjunction with the well known transformation of the spectra between ellipsoidal and spherical surfaces introduced by Jekeli (1988). While on one hand Jekeli's transformation modifies the coefficients in the harmonic functional basis, the Tschebyshev tranformation of the continuation term modifies on the other hand the surface spherical harmonic functions themselves. The radial continuation to a concentric harmonic surface has then been included. In fact and for this special application, the surface spherical harmonics are transformed to solid spherical harmonics. As a consequence, 2D-FFT can be conveniently employed to these surfaces.

By combining the two methods of modifications to the spectral coefficients and to the basis functions, it is shown how Jekeli's transformation can be generalized to transformations between equi-angular coordinates in one domain and their reduced (or augmented) counterpart in the other. The relation between reduced and geocenteric latitudes has become a special realization of this transformation.

The problem of spherical harmonic transformation of geophysical data on the ellipsoid, that are not given in reduced latitudes can be solved by the combination of Jekeli's transformation and a Tschebyshev approximation of the continuation term. The inconvenient latitude shift for the transformation of data given in one equi-angular coordinate frame to another can then be successfully achieved by FFT methods, which is often required for geodetic and geophysical transformation purposes.

The stability and efficiency of the presented method are moreover a neccessary requirement for future spherical harmonic models with increased spatial resolution below 5×5 arcminutes. With the proposed approach, 1×1 arcminute fields with as many as $\mathcal{O}(10^8)$ distinct evaluation points become accessible on personal desktop computer resources, programmed by easy to administer, high level programming codes such as `python` or `matlab`$^{\mathrm{TM}}$.

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